(June 2023) [Background QED for P22607]

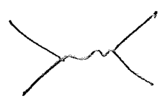
QED scattering process

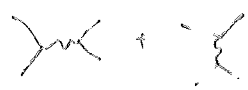
[we'll only try to do amplitude w/one diagram, so no interference : (1) & (2)]

(Identical particles →  Moller)

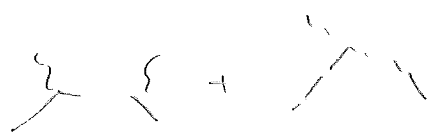
(1) Coulomb: 

(2) pair annihilation into other pairs: $e^+e^- \to \left\{ \begin{array}{l} \mu^+\mu^- \\ q\bar{q} \end{array} \right.$

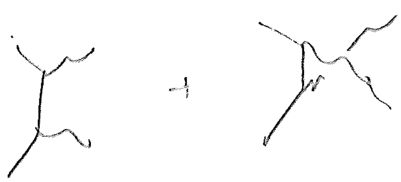


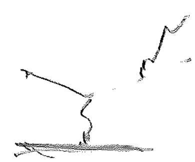
(Bhabha ⇒ )

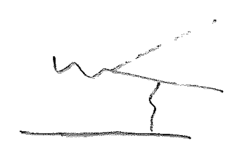
(3) Compton / Thomson: $e\gamma \to e\gamma$



(4) pair annihilation: $e^+e^- \to \gamma\gamma$



(5) bremsstrahlung: $eX \to eX\gamma$ 

(6) pair production $\gamma X \to X e^+e^-$ 

(7) photoelectric γ (atom) → (ion)$^+$ e^-

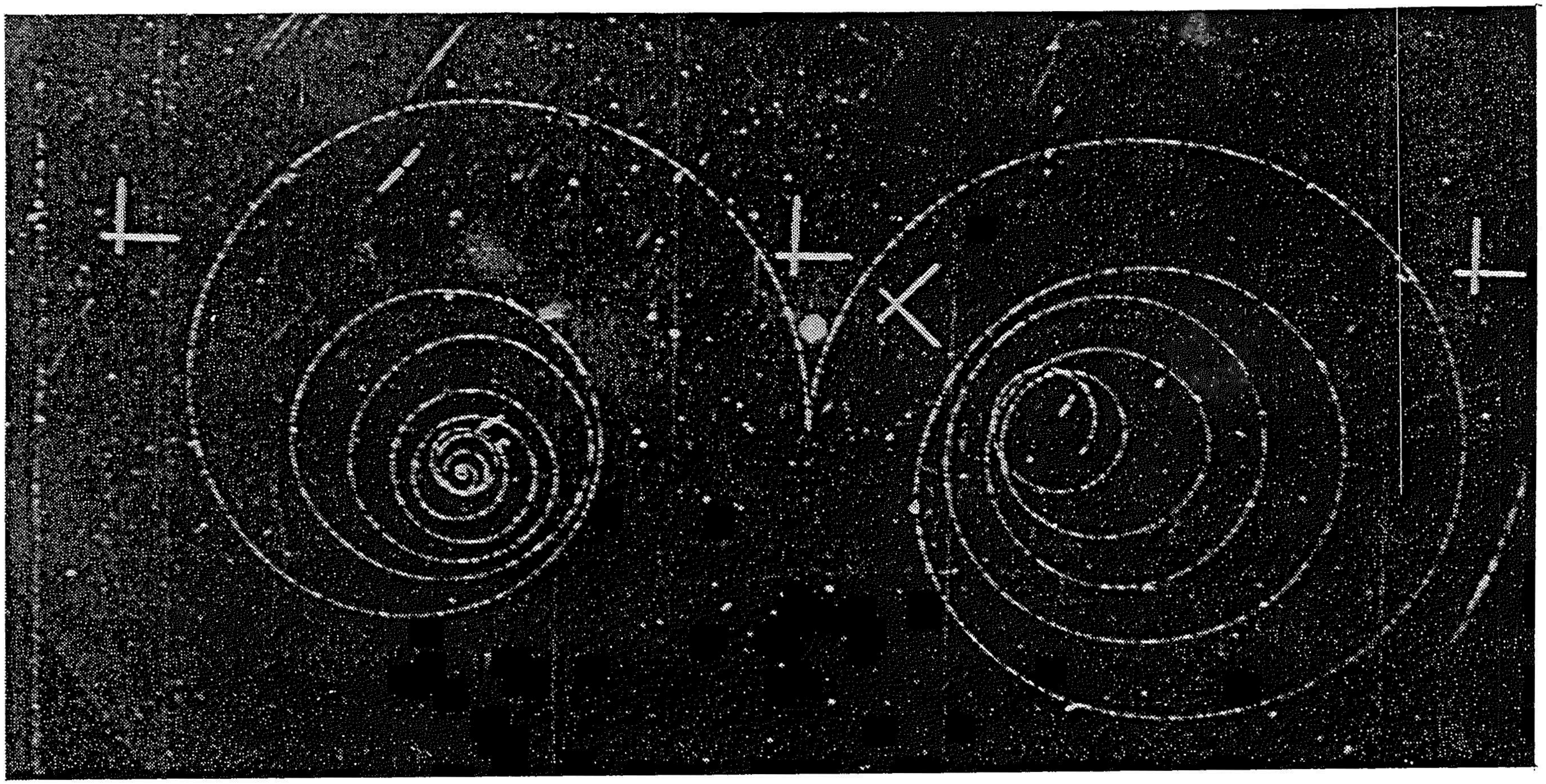
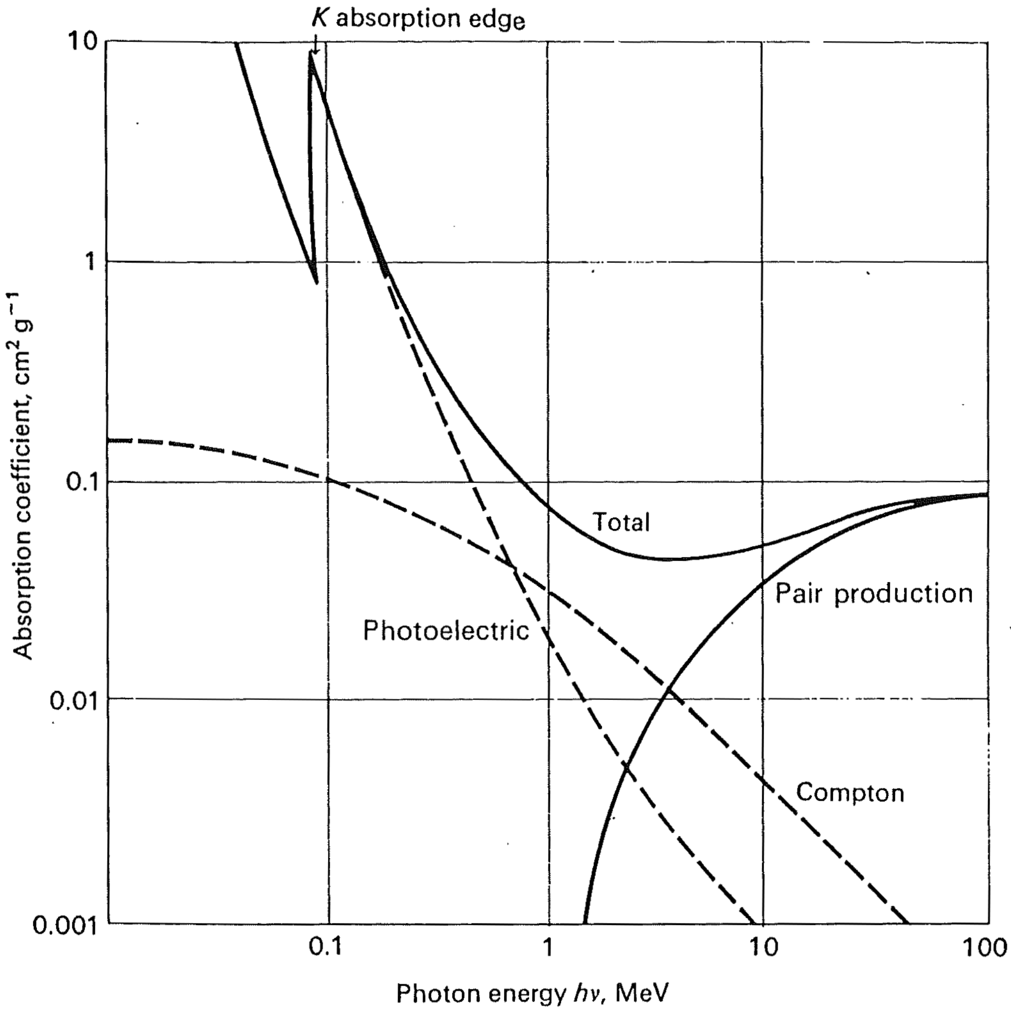


Fig. 6–7 Production of an electron-positron pair in a liquid hydrogen bubble chamber in a magnetic field.



The absorption coefficient per g cm^{-2} of lead for γ-rays as a function of energy.

Perkins

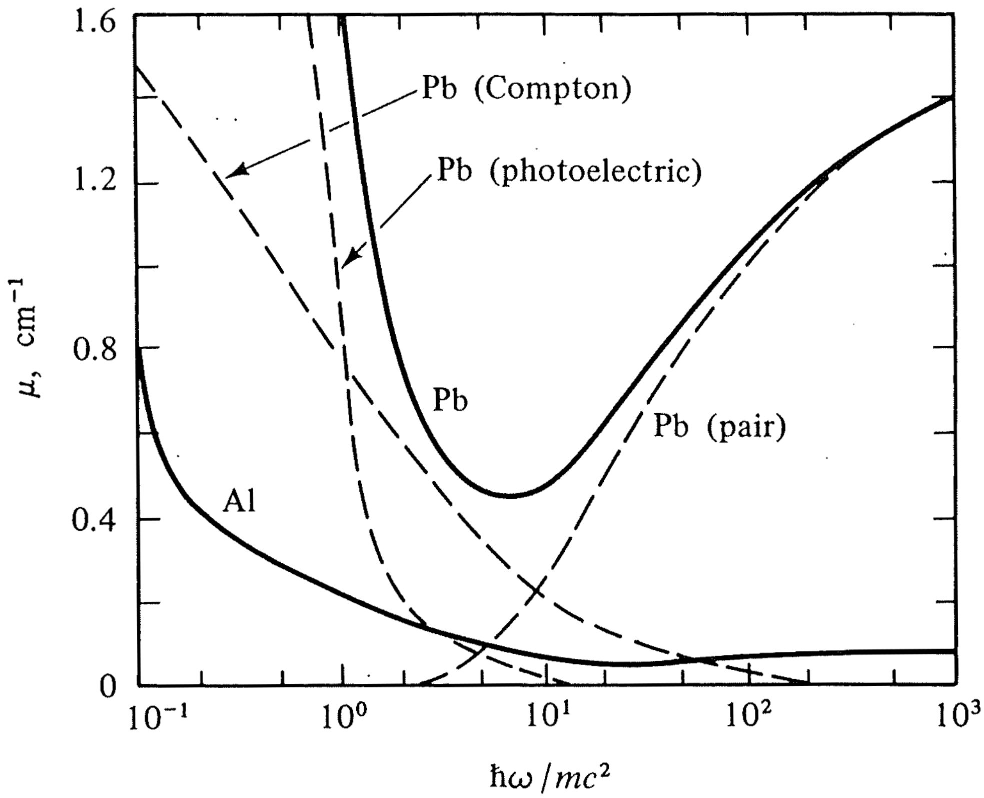
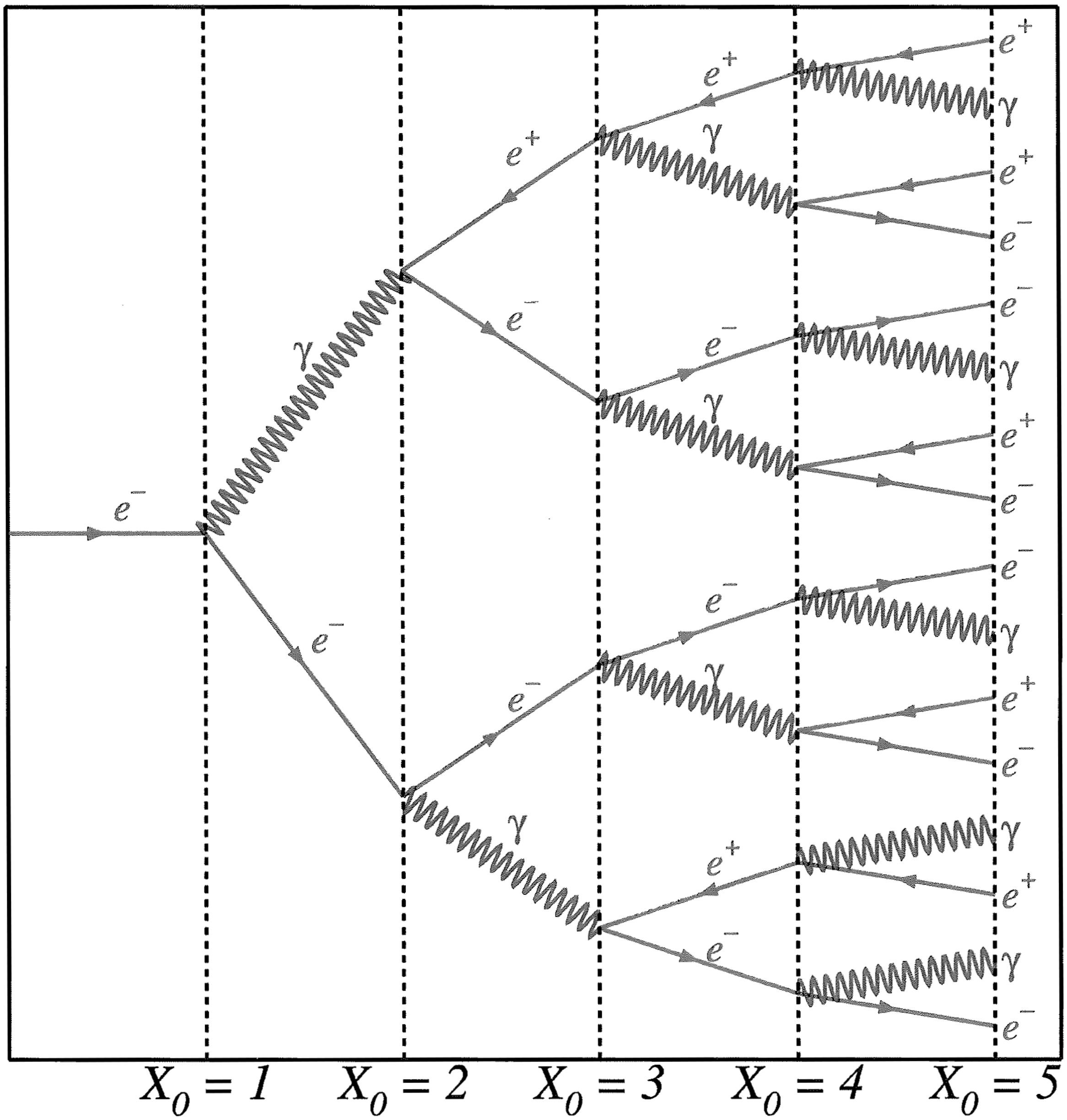
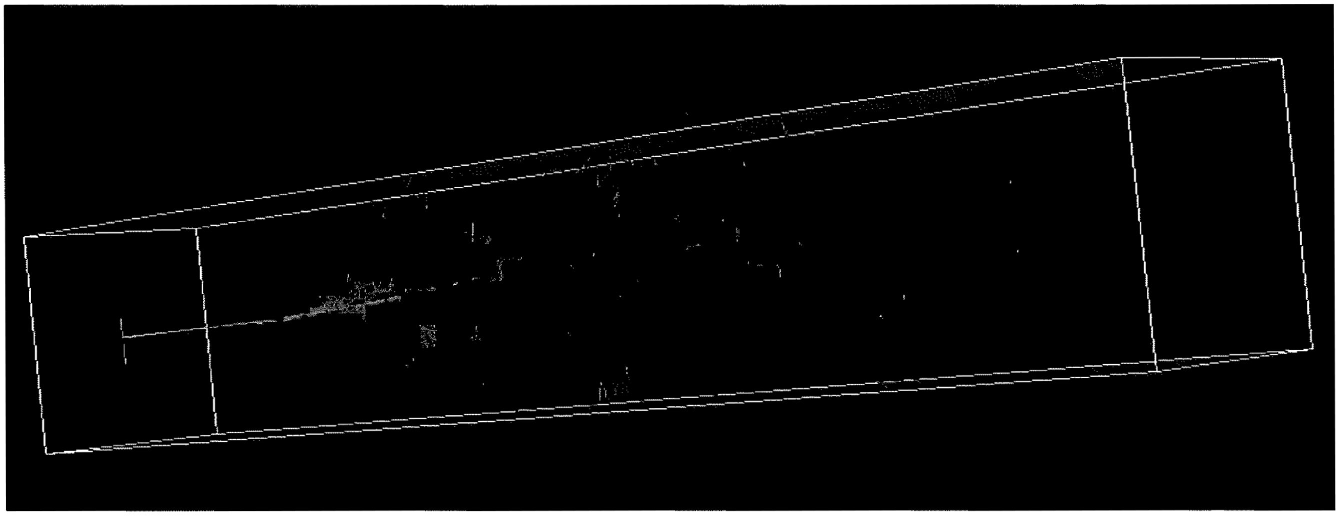


Fig. 3.7. Total absorption coefficients of γ rays by lead and aluminum as a function of energy (solid lines). Photoelectric absorption of aluminum is negligible at the energies considered here. Dashed lines show separately the contributions of photoelectric effect, Compton scattering, and pair production for Pb. Abscissa, logarithmic energy scale; $\hbar\omega/mc^2 = 1$ corresponds to 511 keV. (From W. Heitler, *The Quantum Theory of Radiation*, The Clarendon Press, Oxford, 1936, p. 216.)

Frauenfelder & Henley



e^-
γ
e^-
e^+
e^-
γ
e^-
e^+
γ
e^-
γ
e^-
γ
e^+
e^-
e^+
e^-
e^+
γ
e^+
e^-
e^-
γ
e^+
e^-
e^-
γ
e^+
e^-
γ
e^+
γ
e^-
$X_0 = 1$
$X_0 = 2$
$X_0 = 3$
$X_0 = 4$
$X_0 = 5$



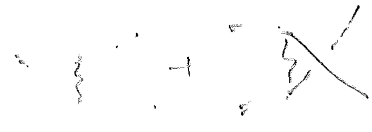
QED scattering processes

① Coulomb scattering

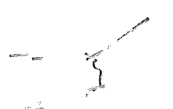
(a) Moller: $e^-e^- \to e^-e^-$ 

(b) $e^-\mu^- \to e^-\mu^-$  (avoids interference)

(c) $e^-p \to e^-p$  (proton structure) (Rosenbluth ...)

(d) $pp \to pp$  (proton structure + interference)

(e) $pX \to pX$  (proton + nuclear structure) (Mott?)

(f) $\alpha X \to \alpha X$  (spin 0: Rutherford)

- All considered in cm frame
- For $m_1 \ll m_2$, cm frame $\approx$ lab frame

Coulomb scattering

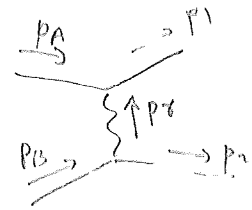
- CM frame

- since identities and charge, no [illegible], no [illegible].

  $p_f = p_i = p$ (CM momenta)

- $\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\hbar^2}{(8\pi E_{cm})^2}|A|^2$ $E_{cm} = E_A + E_B$

- A :



$p_A = (E_A, 0, 0, p)$

$p_B = (E_B, 0, 0, -p)$

$p_1 = (E_A, p\sin\theta, 0, p\cos\theta)$

$p_2 = (E_B, -p\sin\theta, 0, -p\cos\theta)$

$E_A^2 = p^2 + m_A^2$

$E_B^2 = p^2 + m_B^2$

$p_\gamma = (0, p\sin\theta, 0, p(\cos\theta - 1))$

$\therefore p_\gamma^2 = -4p^2\sin^2\frac{\theta}{2}$

crude approximation:

$$A = \frac{(Z_A e)(Z_B e)}{p_\gamma^2}\sqrt{(2E_A)(2E_B)(2E_1)(2E_2)} = \frac{Z_A Z_B(4\pi\alpha)\cdot 2(E_A E_B)}{-p^2\sin^2\left(\frac{\theta}{2}\right)}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \left[\frac{\hbar Z_1 Z_2\alpha}{2p^2\sin^2\frac{\theta}{2}}\left(\frac{E_A E_B}{E_A + E_B}\right)\right]^2$$

Our crude approximation gives $A = \dfrac{Z_A Z_B (4\pi\alpha) E_A E_B}{p^2 \sin^2\frac{\theta}{2}}$

and therefore

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \left[\frac{\hbar Z_A Z_B \alpha}{2p^2 \sin^2\frac{\theta}{2}}\left(\frac{E_A E_B}{E_A + E_B}\right)\right]^2$$

First we take the limit where both particles are nonrelativistic: $E_A \to m_A$, $E_B \to m_B$

$$\frac{E_A E_B}{E_A + E_B} \to \frac{m_A m_B}{m_A + m_B} = \mu$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \left[\frac{\hbar Z_A Z_B \alpha \, \mu}{2p^2 \sin^2\frac{\theta}{2}}\right]^2 \qquad p = \mu v$$

$$= \left[\frac{\hbar Z_A Z_B \alpha}{2\mu v^2 \sin^2\frac{\theta}{2}}\right]^2$$

the correct classical cross section
& also correct nonrel QM cross section

Next, we allow incident particle to be relativistic but w/ $E_A \ll m_B$ (so target is infinitely massive) and cm frame = lab frame (w/ target at rest)

$$\frac{E_A E_B}{E_A + E_B} \to E_A = E, \quad \text{so} \quad A \to \frac{Z_A Z_B (4\pi\alpha) E_A m_B}{-p^2 \sin^2\frac{\theta}{2}}$$

and

$$\left(\frac{d\sigma}{d\Omega}\right)_{\substack{cm \\ = lab}} = \left[\frac{Z_A Z_B \hbar\alpha c}{2p^2 \left(\sin^2\frac{\theta}{2}\right)}\right]^2 = \left[\frac{Z_A Z_B \hbar\alpha}{2mv^2 \sin^2\frac{\theta}{2}}\right]^2 (1-v^2)$$

$$\frac{1}{p^2} = \frac{\gamma m}{\gamma^2 m^2 v^2} = \frac{1}{\gamma m v^2}$$

We'll see that this is correct for relativistic scalar scattering provided $E_1 \ll m_2 \Rightarrow A = \frac{Z_1 Z_2 e^2 (s-u)}{t} \to \frac{Z_1 Z_2 (4\pi\alpha) E_1 m_2}{-p^2 \sin^2\frac{\theta}{2}}$

For relativistic spin $\frac{1}{2}$ particle incident on a [illegible] spinless target we have the Mott formula [B&D, [illegible], p 106] [Landau, [illegible], omits $1-v^2$] [[illegible] (7.131)] [P&S, prob 5.1]

$$\left(\frac{d\sigma}{d\Omega}\right)_{Mott} = \left[\frac{Z_A Z_B \hbar\alpha}{2mv^2 \sin^2\frac{\theta}{2}}\right]^2 (1-v^2)\left(1 - v^2 \sin^2\frac{\theta}{2}\right)$$

No. [illegible] [high energy limits]

In cm frame $p_A = p_B$ so $E_A = E_B = \ldots$, $p = \frac{1}{2} E_{cm}$

Our crude approximation gives $\mathcal{A} \approx \frac{Z_A Z_B (4\pi\alpha)}{- \sin^2\frac{\theta}{2}}$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \left[\frac{\hbar Z_A Z_B \alpha c}{4 p^2 \sin^2\frac{\theta}{2}}\right]^2 = \left[\frac{\hbar c Z_A Z_B \alpha}{2 E_{cm} \sin^2\frac{\theta}{2}}\right]^2$$

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scalar scattering at high energy gives $\mathcal{A} \approx \frac{Z_A Z_B (4\pi\alpha)}{\sin^2\frac{\theta}{2}} (1+\cos^2\frac{\theta}{2})$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \left[\frac{\hbar Z_A Z_B \alpha}{2E_{cm} \sin^2\frac{\theta}{2}}\right]^2 \left(1 + \cos^2\frac{\theta}{2}\right)^2$$

$e^-\mu^-$ scatt at high energy $\frac{d\sigma}{d\Omega} = \frac{\alpha^2[4 + (1+\cos\theta)^2]}{2E_{cm}^2(1-\cos\theta)^2} = \frac{\alpha^2(1+\cos^4\frac{\theta}{2})}{2 \ldots \sin^4\frac{\theta}{2}}$
[P+S, 5.65]
[Griffiths, 2e, prob 7.38]

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \left[\frac{\hbar Z_A Z_B \alpha}{2E_{cm} \sin^2\frac{\theta}{2}}\right]^2 2\left(1+\cos^4\frac{\theta}{2}\right)$$

see Mandl & Shaw prob 8.2 for $\left(\frac{d\sigma}{d\Omega}\right)_{lab}$

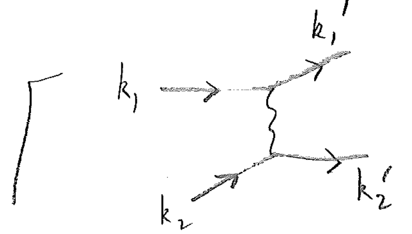
e^-e^- scatt at high energy

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \left[\frac{\hbar Z_A Z_B \alpha}{2E_{cm}}\right]^2 \left[\frac{1+\cos^4\frac{\theta}{2}}{2\sin^4\frac{\theta}{2}} + \frac{1+\sin^4\frac{\theta}{2}}{2\cos^4\frac{\theta}{2}} + \frac{1}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\right]$$

$\ldots \frac{16}{\ldots}$

[B & D, 7.84]
[Griffiths, 2e, prob 7.39 disagrees?]
[Mandl & Shaw, prob 8.6 gives
same formula but [illegible] ...]

(1-21-19) Coulomb scattering in cm frame of spin 0 particles



$$Z_1 Z_2 e^2 \frac{(k_1+k_1')\cdot(k_2+k_2')}{(k_1-k_1')^2}$$

Treat particles as charged scalars

[e^2 here means $\frac{(4\pi K)e^2}{\hbar c} = 4\pi\alpha$]

$k_1 + k_2 = k_1' + k_2'$

$$\Rightarrow (k_1+k_1')\cdot(k_2+k_2') = (k_1+k_1')\cdot(k_1-k_1'+2k_2)$$

$$= \underbrace{k_1^2 - k_1'^2}_{0} + \underbrace{2\,k_1\cdot k_2}_{s-m_1^2-m_2^2} + \underbrace{2\,k_1'\cdot k_2}_{m_1^2+m_2^2-u} = \underline{s-u}$$

$s = (k_1+k_2)^2 = m_1^2 + m_2^2 + 2k_1\cdot k_2$

$u = (k_1'-k_2)^2 = m_1^2 + m_2^2 - 2k_1'\cdot k_2 = m_1^2 + m_2^2 - 2k_1\cdot k_2'$

$t = (k_1-k_1')^2 = 2m_1^2 - 2k_1\cdot k_1'$

$s+t+u = 4m_1^2 + 2m_2^2 + 2k_1\cdot(\underbrace{k_2-k_2'}_{-k_1}-k_1') = 2m_1^2+2m_2^2$ ✓

$$\mathcal{M} = Z_1 Z_2 e^2 \frac{s-u}{t}$$

$$= \frac{Z_1Z_2e^2}{t}\left[2s - 2m_1^2 - 2m_2^2 + t\right]$$

k_2

$k_1 = (E_1, 0, 0, p_1)$

$k_2 = (E_2, 0, 0, -p_1)$

$k_1' = (E_1, p_1\sin\theta, 0, p_1\cos\theta)$

$k_1 \longrightarrow$ k_1' θ

$s = (E_1+E_2)^2$

$u = (E_1-E_2)^2 - p_1^2\sin^2\theta - p_1^2(\cos\theta+1)^2$

$= (E_1-E_2)^2 - p_1^2 - 2p_1^2\cos\theta - p_1^2$

$= (E_1-E_2)^2 - 2p_1^2(\cos\theta+1)$

$t = -p_1^2\sin^2\theta - p_1^2(\cos\theta-1)^2$

$= -2p_1^2 + 2p_1^2\cos\theta$

$= 2p_1^2(\cos\theta-1) = -4p^2\sin^2\frac{\theta}{2}$

$$\frac{s-u}{t} = \frac{(E_1+E_2)^2 - (E_1-E_2)^2 + 2p_1^2(\cos\theta+1)}{2p_1^2(\cos\theta-1)}$$

$$= \frac{4E_1E_2 + 2p_1^2(\overbrace{\cos\theta+1}^{2\cos^2\frac{\theta}{2}})}{2p^2(\cos\theta-1)}$$

cancels the $\frac{1}{2E_1}\frac{1}{2E_2}$ normalization factors!

$$\frac{s-u}{t} = -\frac{E_1E_2 + p^2\cos^2\frac{\theta}{2}}{p^2\sin^2\frac{\theta}{2}}$$

(16-2023)

Spin 0 coulomb scattering

$$A = z_A z_B e^2 \left(\frac{s-u}{t} \right)$$

crude approx just gives the

$$\left[A = \frac{z_A z_B (4\pi\alpha)}{-p^2 \sin^2 \frac{\theta}{2}} \left(E_A E_B + p^2 \cos^2 \frac{\theta}{2} \right) \right]$$

If $E_A \ll E_B$ then $p \leq E_A \ll E_B$

and $p^2 \leq p E_A \ll E_A E_B$ so 2nd term

is negligible, so

get same result as crude approx.

$$\Rightarrow \left(\frac{d\sigma}{d\Omega} \right)_{cm} = \left(\frac{\hbar z_A z_B \alpha E_A}{2p^2 \sin^2 \frac{\theta}{2}} \right)^2 = \left(\frac{\hbar z_A z_B \alpha}{2 m v^2 \sin^2 \frac{\theta}{2}} \right)^2 (1 - v^2)$$

High energy limit: $E_A \approx E_B \approx p$

$$A \longrightarrow \frac{z_A z_B (4\pi\alpha)}{\sin^2 \frac{\theta}{2}} \left[1 + \cos^2 \frac{\theta}{2} \right]$$

$$\left(\frac{d\sigma}{d\Omega} \right)_{cm} = \left(\frac{\hbar \alpha z_1 z_2 \left(1 + \cos^2 \frac{\theta}{2}\right)}{2 E_{cm} \sin^2 \frac{\theta}{2}} \right)^2$$

$e^-\mu^- \to e^-\mu^-$ scattering

[Griffiths, 2e, p. 245]

$$A = -\frac{e^2}{p_\gamma^2}\left(\bar{u}_1 \gamma^\mu u_A\right)\left(\bar{u}_2 \gamma_\mu u_B\right)$$ [eq. 7.106]

$$\langle |A|^2 \rangle = \frac{2e^4}{(p_\gamma^2)^2}\Big[4\, p_A\cdot p_B\; p_1\cdot p_2 + 4\, p_A\cdot p_2\; p_B\cdot p_1 - 4\, p_A\cdot p_1\, m_\mu^2 - 4\, p_B\cdot p_2\, m_e^2 + 8\, m_e^2 m_\mu^2\Big]$$ [eq. 7.129]

$s = (p_A+p_B)^2 = (p_1+p_2)^2$
$t = (p_A-p_1)^2 = (p_B-p_2)^2$
$u = (p_A-p_2)^2 = (p_B-p_1)^2$

$$= \frac{2e^4}{t^2}\Big[(s - m_e^2 - m_\mu^2)^2 + (u - m_e^2 - m_\mu^2)^2 + 2m_\mu^2(t - 2m_e^2) + 2m_e^2(t - 2m_\mu^2) + 8\, m_e^2 m_\mu^2\Big]$$

$=$

Mott scattering: m_μ [illegible] e^- [illegible] μ

$$\langle |A|^2 \rangle = \frac{8e^4}{(p_\gamma^2)^2} \left[p_A \cdot p_B \, p_1 \cdot p_2 + p_A \cdot p_2 \, p_B \cdot p_1 - p_A \cdot p_1 \, m_\mu^2 - [\text{illegible}] m_e^2 + [\text{illegible}] m_e^2 m_\mu^2 \right]$$

$p_A = (E, 0, 0, p)$ $\qquad p_1 = (E, p\sin\theta, 0, p\cos\theta)$

$p_B = (m_\mu, 0, 0, 0)$ $\qquad p_2 = (m_\mu, -p\sin\theta, 0, -p\cos\theta)$

where $E, p \ll m_\mu$

$p_A \cdot p_B \to m_\mu E$

$p_1 \cdot p_2 \to m_\mu E$

$p_A \cdot p_2 \to m_\mu E$

$p_B \cdot p_1 \to m_\mu E$

$p_A \cdot p_1 = E^2 - p^2\cos^2\theta$

$p_B \cdot p_2 \to m_\mu^2$

$$\langle |A|^2 \rangle = \frac{8e^4}{(p_\gamma^2)^2} \left[(m_\mu E)^2 + (m_\mu E)^2 - m_\mu^2 (E^2 - p^2\cos^2\theta) - m_\mu^2 m_e^2 + 2 m_\mu^2 m_e^2 \right]$$

$$= \frac{8e^4 m_\mu^2}{(4p^2\sin^2\frac{\theta}{2})^2} \left[E^2 + p^2 \underbrace{\cos^2\theta}_{1-2\sin^2\frac{\theta}{2}} + m_e^2 \right]$$

$$2E^2 - 2p^2\sin^2\frac{\theta}{2}$$

equal to correct approx

$$= \left(\frac{e^4 m_\mu^2 E^2}{p^4 \sin^4\frac{\theta}{2}} \right) \left(1 - v^2 \sin^2\frac{\theta}{2} \right)$$

$$\frac{d\sigma}{d\Omega} = \frac{\hbar^2}{(8\pi E_{cm})^2} \langle |A|^2 \rangle = \left(\frac{\hbar \alpha E}{2p^2 \sin^2\frac{\theta}{2}} \right)^2 \left(1 - v^2 \sin^2\frac{\theta}{2} \right)$$

$E_{cm} \to m_\mu$

$$= \left(\frac{\hbar \alpha}{2mv^2 \sin^2\frac{\theta}{2}} \right)^2 (1 - v^2)\left(1 - v^2 \sin^2\frac{\theta}{2} \right)$$

High energy limit of $e^- \mu^- \to e^- \mu^-$

Let $m_e = m_\mu = 0$

$$\langle |A|^2 \rangle = \frac{2e^4}{t^2}\left(s^2 + u^2\right)$$

$$= \frac{8e^4}{\sin^4\frac{\theta}{2}}\left(1 + \cos^4\frac{\theta}{2}\right)$$

$s = 4E^2$

$t = -4E^2 \sin^2\frac{\theta}{2}$

$u = -4E^2 \cos^2\frac{\theta}{2}$

$e^2 = 4\pi\alpha$

$E_{cm} = 2E$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\hbar^2}{(8\pi E_{cm})^2}\langle |A|^2 \rangle$$

$$= \left(\frac{\hbar\alpha}{2E_{cm}\sin^2\frac{\theta}{2}}\right)^2 2\left(1 + \cos^4\frac{\theta}{2}\right)$$

[prob 7.38 (b)] ✓

[Peskin & Schroeder 5.65]

QED

53. **23.07.** *Two-particle final-state momentum.*
Consider a collision of particles (with energy E_{cm} in the center-of-mass frame) that results in a two-particle final state (with masses m_1 and m_2). Compute the magnitude of the momentum p_f of the final-state particles in the CM frame.

54. **23.07.** *Electron-positron annihilation.* Consider the process $e^+e^- \to \mu^+\mu^-$ in the CM frame, where the initial state particles e^+ and e^- are moving along the $+z$ and $-z$ directions, and the final state particle μ^+ and μ^- are moving in the xz plane, with μ^+ making angle θ with respect to e^+.

(a) Show that the energies of each of the particles involved are equal in the CM frame (not their momenta!).

(b) Write the four-momenta of each of the particles involved in terms of the common energy E and the angle θ.

(c) Write down the tree-level Feynman diagram contributing to this process, and compute the four-momentum of the virtual photon.

(d) Calculate the amplitude A, using our rough-and-ready estimate of $\sqrt{(2E_A)(2E_B)(2E_1)(2E_2)}$ for the "spin stuff." Your final answer should be very simple.

(e) Calculate the differential cross section in the CM frame, $(d\sigma/d\Omega)_{\text{cm}}$, expressing everything in terms of the common energy E of the particles.

55. *Retired.* 19.07. ~~*Thomson scattering cross section.*~~
Consider the scattering of a photon from an electron $e^- + \gamma \to e^- + \gamma$ (Compton scattering). Calculate the cross section for this process following the approach we used in class to compute the Rutherford cross section. Carefully state the assumptions you are using at each step. Numerically evaluate your answer in barns.

this was a fluke

To simplify the problem, you may assume that the initial photon energy E_γ is much less than the electron rest energy. Thus, while the photon transfers momentum to the electron, negligible energy is transferred, and so the final state electron energy $\approx m_e c^2$.

The cross section that you obtain will not agree exactly with the classical Thomson cross section (based on scattering of electromagnetic waves), because you have not taken into account the polarization of the photon, but it should be in the same ballpark.

56. *2017-2011.* Griffiths 2.1.
[Calculate the ratio of the gravitational attraction to the electrical repulsion between two stationary electrons. (Do I need to tell you how far apart they are?)]

57. *2005-2011. Delbruck.* Griffiths 2.2.
Sketch the lowest-order Feynman diagram representing Delbruck scattering: $\gamma\gamma \to \gamma\gamma$. (This process, the scattering of light by light, has no analog in classical electrodynamics.)

58. *2005-2011. Delbruck.* Griffiths 2.3.
[Draw all the fourth-order (four vertex) diagrams for Compton scattering. (There are 17 of them; disconnected diagrams don't count.)]

(b. 10^7)

p2760

[problem set 7, 2023]

$e^+e^- \longrightarrow \mu^+\mu^-$

• cm frame: $\Rightarrow$ $E_A = E_B = E_1 = E_2$ by energy cons.

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\hbar^2}{(8\pi E_{cm})^2}\,\frac{p_f}{p_i}\,|A|^2$$

$p_A = (E, 0, 0, p_i)$

$p_B = (E, 0, 0, -p_i)$

$p_1 = (E, p_f \sin\theta, 0, p_f\cos\theta)$

$p_2 = (E, -p_f\sin\theta, 0, -p_f\cos\theta)$

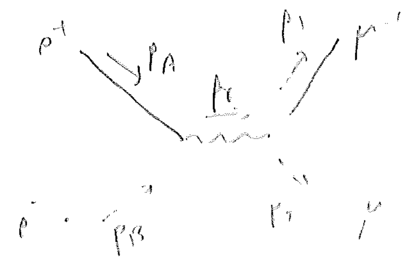
$p_\gamma = (2E, 0, 0, 0)$

$E_{cm} = 2E$

$p_i = \sqrt{E^2 - m_i^2}$

$p_f = \sqrt{E^2 - m_f^2}$

• A =



crude approximation:

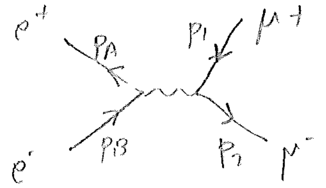
$$A = \frac{e^2}{p_\gamma^2}\sqrt{(2E_A)(2E_B)(2E_1)(2E_2)} = \frac{e^2(2E)^2}{E_{cm}^2} = 4\pi\alpha$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \left(\frac{\hbar\alpha}{2E_{cm}}\right)^2\sqrt{\frac{E^2-m_f^2}{E^2-m_i^2}}$$

$$\sigma = 4\pi\left(\frac{\hbar\alpha}{2E_{cm}}\right)^2\sqrt{\frac{E^2-m_f^2}{E^2-m_i^2}} \xrightarrow{E \gg m_i, m_f} \pi\left(\frac{\hbar\alpha}{E_{cm}}\right)^2$$

(correct answer for spin $\frac{1}{2}$ $\Rightarrow$ $\frac{4\pi}{3}\left(\frac{\hbar\alpha}{E_{cm}}\right)^2$)

$e^+e^- \to \mu^+\mu^-$



$$A = \frac{e^2}{s}\left(\bar v_A \gamma_\mu u_B\right)\left(\bar u_2 \gamma^\mu u_1\right)$$

[Griffiths 2e, 8.1, also 7.109]

$$\langle|A|^2\rangle = \frac{8e^4}{s^2}\left[p_A\cdot p_1\, p_B\cdot p_2 + p_A\cdot p_2\, p_B\cdot p_1 + m_e^2\, p_1\cdot p_2 + m_\mu^2\, p_A\cdot p_B + 2m_e^2 m_\mu^2\right]$$

[Griffiths 8.3]

Use $p_A = (E, 0, 0, p_e)$ $\quad p_1 = (E, p_\mu \sin\theta, 0, p_\mu\cos\theta)$

$p_B = (E, 0, 0, -p_e)$ $\quad p_2 = (E, -p_\mu\sin\theta, 0, -p_\mu\cos\theta)$

$$\langle|A|^2\rangle = e^4\left[1 + \left(\frac{m_e}{E}\right)^2 + \left(\frac{m_\mu}{E}\right)^2 + \left\{1-\left(\frac{m_e}{E}\right)^2\right\}\left\{1-\left(\frac{m_\mu}{E}\right)^2\right\}\cos^2\theta\right]$$

$e^2 = 4\pi\alpha$

[Griffiths, 8.4; agrees w/ Peskin & Schroeder eq 5.11 in $m_e\to 0$ limit]

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar}{8\pi E_{cm}}\right)^2 \frac{p_\mu}{p_e}|A|^2$$

$$= \left(\frac{\hbar\alpha}{2E_{cm}}\right)^2 \sqrt{\frac{1-\frac{m_\mu^2}{E^2}}{1-\frac{m_e^2}{E^2}}}\left[1 + \left(\frac{m_e}{E}\right)^2 + \left(\frac{m_\mu}{E}\right)^2 + \left\{1-\left(\frac{m_e}{E}\right)^2\right\}\left\{1-\left(\frac{m_\mu}{E}\right)^2\right\}\cos^2\theta\right]$$

[P&S, eq 5.12, $m_e\to 0$]

$\langle\cos^2\theta\rangle = \frac{1}{3} \Rightarrow$ $\quad 1 + \left(\frac{m_e}{E}\right)^2 + \left(\frac{m_\mu}{E}\right)^2 + \frac{1}{3}\left(1-\left(\frac{m_e}{E}\right)^2\right)\left(1-\left(\frac{m_\mu}{E}\right)^2\right)$

$$= \frac{4}{3} + \frac{2}{3}\left[\left(\frac{m_e}{E}\right)^2 + \left(\frac{m_\mu}{E}\right)^2\right] + \frac{1}{3}\left(\frac{m_e}{E}\right)^2\left(\frac{m_\mu}{E}\right)^2 = \frac{4}{3}\left(1+\frac{m_e^2}{2E^2}\right)\left(1+\frac{m_\mu^2}{2E^2}\right)$$

$$\sigma = \frac{4\pi}{3}\left(\frac{\hbar\alpha}{E_{cm}}\right)^2 \sqrt{\frac{1-\frac{m_\mu^2}{E^2}}{1-\frac{m_e^2}{E^2}}}\left(1+\frac{m_e^2}{2E^2}\right)\left(1+\frac{m_\mu^2}{2E^2}\right)$$

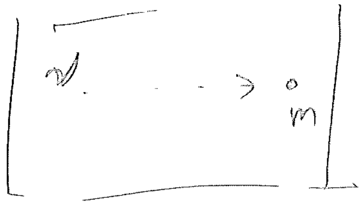
[Griffiths 8.5
P&S 5.13]

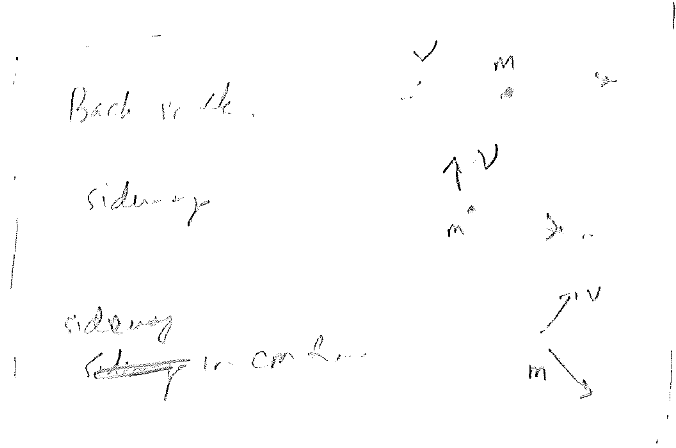
Hi energy limit $\Rightarrow$ $\boxed{\sigma = \frac{4\pi}{3}\left(\frac{\hbar\alpha}{E_{cm}}\right)^2}$

16.10.13

Compton kinematics in lab frame

Consider a massless particle (ν or γ) striking a massive particle at rest (e)





Given $E = E_\nu$
what is E_e in each scenario?

- Backscatter: $E_e = \frac{2E^2 + 2mE + m^2}{2E + m} \to E + \frac{m}{2}$; $E_\nu' \to \frac{m}{2}$ (see next page) (see table of γ)
- sideways (lab) $\Rightarrow$ $E_e = E_\nu$, $E_\nu' = m$
- sideways (cm) $\Rightarrow$ $E_e = \frac{(m+E)^2}{m+2E} \to \frac{E}{2}$, $E_\nu' = \frac{(m+E)E}{m+2E} \to \frac{E}{2}$ (see below)

First determine v_{cm}: $p_\nu' = \gamma(p_\nu - vE_\nu) = \gamma(1-v)E$, $E_\nu' = p_\nu'$
$p_e' = \gamma v m$, $E_e' = \gamma m$

$$\begin{pmatrix} E_e' = \\ = \gamma m \quad v(1-v)E \\ = \gamma \ldots \quad \frac{mE}{E+m} \\ = \gamma m \frac{2E+m}{E+m} \end{pmatrix}$$

cm frame: $\gamma m v = \gamma(1-v)E \Rightarrow v = \frac{E}{m+E}$

sideways in cm frame

$p_e' = (E_e', -p_e', 0, 0)$
$p_\nu' = (E_\nu', +E_\nu', 0, 0)$

Boost back to lab frame
$$\begin{cases} E_e = \gamma(E_e') = \gamma(\gamma m) = \frac{m}{1-v^2} = \frac{(m+E)^2}{m+2E} \\ p_{ex} = \gamma(vE_e') = \gamma^2 vm = \frac{mE}{m+2E} \\ p_{ey} = p_{ey}' = \gamma v m = \ldots \end{cases}$$

check $E_e^2 = p_{ex}^2 + p_{ey}^2$: $m^2(\gamma^4 - \gamma^4 v^2 - \gamma^2 v^2) = 1$ ✓

$$\begin{cases} E_\nu = \gamma(E_\nu') = \gamma^2(1-v)E = \frac{(m+E)E}{m+2E} \\ p_{\nu x} = \gamma(vE_\nu') = \gamma^2 v(1-v)E \\ p_{\nu y} = p_{\nu y}' = \gamma(1-v)E \end{cases}$$

check $E_\nu^2 = p_{\nu x}^2 + p_{\nu y}^2 = (\gamma(1-v)E)^2(\underbrace{\gamma^2 - \gamma^2 v^2 - 1}_{0\ ✓})$

check $E_e + E_\nu = \frac{(m+E)^2}{m+2E} + \frac{E(m+E)}{m+2E} = \ldots = m + E$ ✓

We calculated [illegible] [illegible]

because ([illegible] [illegible] [illegible] in [illegible])

[illegible] may be [illegible]

The [illegible] [illegible] "typical" elect energy is

[illegible]

m [illegible]

$\frac{1}{m}$

[illegible] E_e [illegible]

[illegible] $m \rightarrow E_e$ $\frac{4}{3}$[illegible]

[illegible] $\rightarrow E_e$ [illegible] $\frac{1}{2}$ [illegible]

So we'll say typical [illegible] [illegible] energy
is half the neutrino energy

[illegible] [illegible] [illegible] [illegible] need $E_e \geq \frac{1}{\sqrt{1-v^2}}m > \frac{1}{\sqrt{1-\frac{1}{n^2}}}m$

$\approx 1.56\, m$

Compton ~~scattering~~ kinematics in lab frame

Consider (backward scattering)

$$E + m = E' + E_e$$

$$E = -E' + p_e$$

$$2E + m = E_e + p_e$$

$$(2E + m - E_e)^2 = p_e^2 = E_e^2 - m^2$$

$$(2E+m)^2 - 2E_e(2E+m) = -m^2$$

$$2E_e(2E+m) = 4E^2 + 4mE + 2m^2$$

$$\boxed{E_e = \frac{2E^2 + 2mE + m^2}{2E + m}} \approx E + \frac{m}{2} \quad \text{if } E \gg m$$

if $E_e \approx p_e$ then

$$2E + m = 2E_e$$

$$E_e = E + \frac{m}{2}$$

$$T_e = E - \frac{m}{2}$$

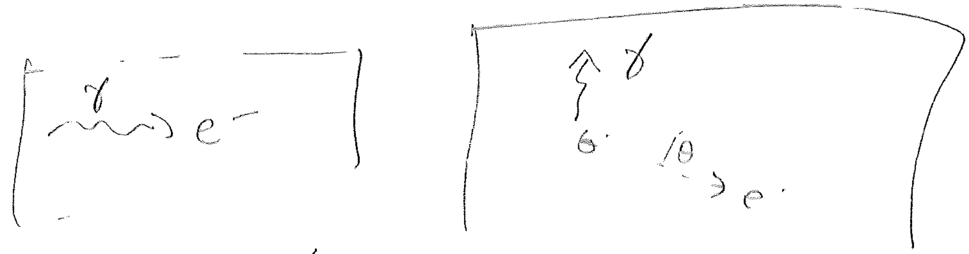
$$E' = \frac{m}{2}$$

At large E_γ and $\theta = \pi$
γ gives almost all its
energy to the electron
only m going to the photon

see next page

(at large E_γ and $\theta = \pi$)
Thus, the γ gives almost all its
energy to the electron
only $\frac{1}{2}m$ going to recoil photon

Compton scatt. γ sideways in lab frame



$$E + m = E' + E_e$$

$$E = p_e \cos\theta$$

$$E' = p_e \sin\theta$$

$$E^2 + E'^2 = p_e^2 = E_e^2 - m^2 = (E + m - E')^2 - m^2$$

$$= E^2 + E'^2 - 2EE' + 2mE - 2mE'$$

$$(E+m)E' = mE$$

$$E' = \frac{mE}{E+m} \quad \therefore \quad \frac{1}{E'} = \frac{1}{m} + \frac{1}{E}$$

exactly right!

$$p_e = \sqrt{E^2 + E'^2}$$

$$E_e^2 = p_e^2 + m^2 = E^2 + E'^2 + m^2$$

$$= E^2 + \left(\frac{mE}{E+m}\right)^2 + m^2$$

$$= \frac{(E^2+m^2)(E+m)^2 + (mE)^2}{(E+m)^2}$$

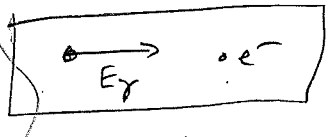
In limit $E \gg m$ we have

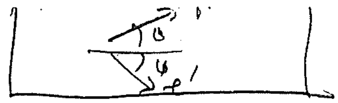
$$E_e^2 \to E^2 \quad \text{i.e.} \quad \left| \begin{array}{l} E_e \to E \\ E' \to m \end{array} \right|$$

$$\sin\theta \to \frac{E'}{p_e} \sim \frac{m}{E}$$

(similar to other $\gamma\gamma$ kinematics

Compton kinematics in lab frame





$$p_\gamma = (E_\gamma, 0, 0, E_\gamma) \qquad p_\gamma' = (E_\gamma', E_\gamma \sin\theta, 0, E_\gamma' \cos\theta)$$

$$p_e = (m_e, 0, 0, 0) \qquad p_e' = (E_e', -|\vec{p}_e'|\sin\phi, 0, |\vec{p}_e'|\cos\phi)$$

$$p_\gamma + p_e = p_\gamma' + p_e'$$

Rewrite this as

$$-p_e' + p_e = p_\gamma' - p_\gamma$$

Consider

$$(-p_e' + p_e)^2 = p_e'^2 - 2p_e' \cdot p_e + p_e^2$$

$$= m_e^2 - 2(E_e' m_e - 0) + m_e^2$$

$$= 2m_e(m_e - E_e') \underset{\uparrow \text{ energy cons.}}{=} 2m_e(E_\gamma' - E_\gamma)$$

Now consider

$$(p_\gamma' - p_\gamma)^2 = p_\gamma'^2 - 2p_\gamma' \cdot p_\gamma + p_\gamma^2$$

$$= 0 - 2(E_\gamma' E_\gamma - E_\gamma' E_\gamma \cos\theta)$$

$$= -2E_\gamma E_\gamma'(1 - \cos\theta)$$

Equate these

$$-2m_e(E_\gamma - E_\gamma') = -2E_\gamma E_\gamma'(1 - \cos\theta)$$

$$\frac{E_\gamma - E_\gamma'}{E_\gamma E_\gamma'} = \frac{1}{m_e}(1 - \cos\theta)$$

$$\frac{1}{E_\gamma'} - \frac{1}{E_\gamma} = \frac{1}{m_e c^2}(1 - \cos\theta)$$

↑ restore

$$E = hf = \frac{hc}{\lambda}$$

$$\Rightarrow \quad \frac{\lambda'}{hc} - \frac{\lambda}{hc} = \frac{1}{m_e c^2}(1 - \cos\theta)$$

$$\lambda' - \lambda = \underbrace{\frac{h}{m_e c}}_{\text{Compton wavelength of an electron}} (1 - \cos\theta)$$

Max shift occurs of back scattered X-rays $(\theta = \pi)$

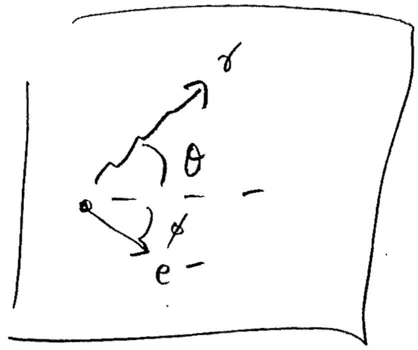
$$(\Delta\lambda)_{max} = \frac{2h}{m_e c} \approx 4.8 \times 10^{-12}\,m$$

Compton used molybdenum K_α line $\Rightarrow 7 \times 10^{-11}\,m$

(17 keV)

(7% shift)

Afterwards



Can do this more formally using 4-vectors →

$$\begin{cases} E_\gamma + mc^2 = E_\gamma' + E_e' \\ p_\gamma = p_\gamma' \cos\theta + p_e' \cos\phi \\ 0 = p_\gamma' \sin\theta - p_e' \sin\phi \end{cases}$$

Eliminate ϕ:

$$(p_\gamma - p_\gamma' \cos\theta)^2 + (p_\gamma' \sin\theta)^2 = p_e'^2$$

$$\underline{p_\gamma^2} - 2 p_\gamma p_\gamma' \cos\theta + \underline{(p_\gamma')^2} = p_e'^2$$

Energy $\Rightarrow$ $(E_\gamma + mc^2 - E_\gamma')^2 = E_e'^2$

$$\underline{E_\gamma^2 + E_\gamma'^2} + 2E_\gamma mc^2 - 2E_\gamma' mc^2 - 2E_\gamma E_\gamma' = E_e'^2 - (mc^2)^2$$

$$2E_\gamma mc^2 - 2E_\gamma' mc^2 - 2E_\gamma E_\gamma' (1 - \cos\theta) = 0$$ Subtract mom from energy

---

ⓐ $$\frac{1}{E_\gamma'} - \frac{1}{E_\gamma} = \frac{1-\cos\theta}{mc^2} \Rightarrow f(\theta) = \underline{1 - \cos\theta}$$

ⓑ $E = \frac{hc}{\lambda} \Rightarrow \lambda' - \lambda = \underbrace{\frac{h}{mc}}_{2.4\times10^{-12}\,m}(1 - \cos\theta)$

$\theta = \pi \Rightarrow \Delta\lambda = \underline{4.8\text{E}{-12}\,m}$

$$\lambda = \frac{hc}{17\,\text{KeV}} = \frac{1.24\text{E}{-6}\ \text{eV}\cdot\text{m}}{1.7\text{E}4\,\text{eV}} = 7.3\times10^{-11}\,m \qquad \frac{\Delta\lambda}{\lambda} \sim 0.066$$

Alternative

Use 4-vectors to compute Compton scattering

$$p_e' = p_e + p_\gamma - p_\gamma'$$

$$(p_e')^2 = \underbrace{p_e^2}_{\text{these cancel}} + \underbrace{p_\gamma^2 + p_\gamma'^2}_{\text{these vanish}} + 2p_e\cdot p_\gamma - 2p_e\cdot p_\gamma' - 2p_\gamma\cdot p_\gamma'$$

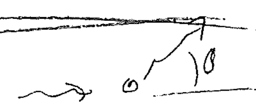
so

$$p_e\cdot p_\gamma - p_e\cdot p_\gamma' = p_\gamma\cdot p_\gamma'$$

$$p_e = \begin{pmatrix} mc^2 \\ \vec{0} \end{pmatrix}, \quad p_\gamma = \begin{pmatrix} E \\ E\vec{n} \end{pmatrix}, \quad p_\gamma' = \begin{pmatrix} E' \\ E'\vec{n}' \end{pmatrix}$$

$$mc^2E - mc^2E' = EE'(1 - \underbrace{\vec{n}\cdot\vec{n}'}_{\cos\theta})$$

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{mc^2}(1-\cos\theta)$$



done in class

Alternatively: $p_e' - p_e = p_\gamma - p_\gamma'$

$$2m^2 - 2p_e\cdot p_e' = -2p_\gamma\cdot p_\gamma'$$

$$\underbrace{2m^2 - 2m\underbrace{E_e}_{E-E'-m}}_{-2m(E-E')} = -2EE'(1-\cos\theta)$$

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m}(1-\cos\theta)$$

I tried to do Compton in 2019 using crude [illegible], but got lucky in getting approx correct answer

~~8-3~~

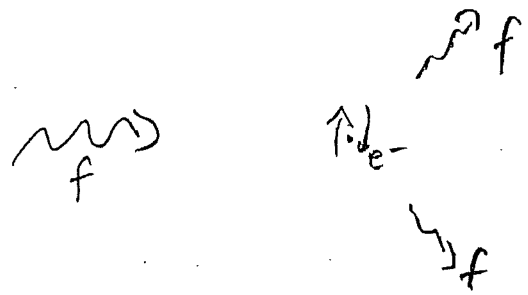
28

~~Physicists were skeptical of the photon concept until Arthur Compton did his expts on X-rays (1922)~~

~~X-rays were known to be EM waves~~

Scattering of X-rays by electrons (Thomson or Compton scattering)

Classical picture: electromagnetic waves of frequency f cause electrons to oscillate of frequency f.
Those accelerating electrons emit EM waves of frequency f.



Classical cross-section $\sigma = \dfrac{R}{F}$

R = rate at which energy is radiated by electron

F = flux of incident EM waves

From 1140, $F = \epsilon_0 c |\vec{E}|^2$

From 3120, Larmor formula $R = \dfrac{e^2 a^2}{6\pi\epsilon_0 c^3}$

$$\vec{a} = \frac{\vec{F}}{m_e} = \frac{e\vec{E}}{m_e} \Rightarrow R = \frac{e^4 |\vec{E}|^2}{6\pi\epsilon_0 m_e^2 c^3} \Rightarrow \sigma =$$

$$\Rightarrow \sigma = \frac{e^4}{6\pi\epsilon_0^2 (m_e c^2)^2} = \frac{8\pi}{3}\left(\frac{ke^2}{m_e c^2}\right)^2 = \frac{8\pi}{3} r_0^2$$

where $r_0 \equiv \dfrac{ke^2}{m_e c^2}$ = classical electron radius $= \dfrac{1.44\,\text{MeV fm}}{0.511\,\text{MeV}} = 2.8\,\text{fm}$

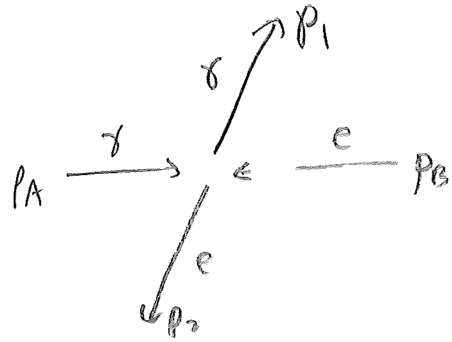
$$\sigma = 66.5\,\text{fm}^2 = \frac{2}{3}\,\text{barn} = \text{Thomson cross-section}$$

[Thomson used it to measure # electrons in atoms]
Nucleus too massive to radiate much.

(1-26-23)

Compton scattering in cm frame

$e^-\gamma \to e^-\gamma$



Kinematics same as Rutherford $\Rightarrow$ all momenta equal

Also $E_\gamma = p$, E_e $\sqrt{p^2+m_e^2} = E$

$p_A = (p, 0, 0, p)$ $\quad p_1 = (p, p\sin\theta, 0, p\cos\theta)$

$p_B = (E, 0, 0, -p)$ $\quad p_2 = (E_e, -p\sin\theta, 0, -p\cos\theta)$

$p_A + p_B : (E+p, 0, 0, 0)$

$s = (p_A+p_B)^2 = (E+p)^2 = m^2 + 2p(E+p)$

$p_B - p_1 = (E-p, -p\sin\theta, 0, -p(1+\cos\theta))$

$u = (p_B - p_1)^2 : (E-p)^2 - p^2s^2 - p^2(1+2c+c^2) =$

$= E^2 - 2Ep + p^2 - p^2 - p^2 - 2p^2c$

$= E^2 - 2Ep - p^2 - 2p^2\cos\theta = m^2 - 2p(E + p\cos\theta)$

$p_1 - p_A : (0, p\sin\theta, 0, p(\cos\theta - 1))$

$t : (p_1 - p_A)^2 : -p^2s^2 - p^2(1 - 2c + c^2) = -2p^2(1-\cos\theta)$

$s+t+u = E^2 + \cancel{2Ep} + \cancel{p^2} + E^2 - \cancel{2Ep} - \cancel{p^2} - 2\cancel{p^2c} - 2p^2 + \cancel{2p^2c} = 2(E^2-p^2) = 2m^2$ ✓

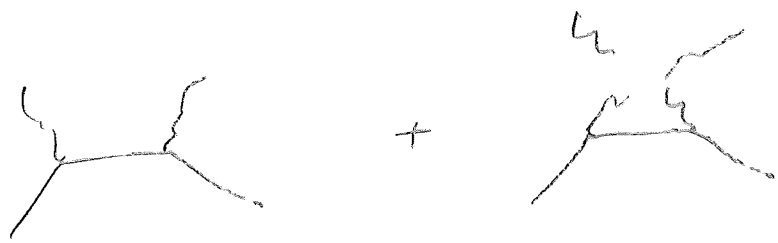
$s - m^2 = 2p(E+p)$

$u - m^2 = -2p(E + p\cos\theta)$

$t = -2p^2(1-\cos\theta)$

Compton scattering discussed by most authors.

- I'll use results from Peskin + Schroeder (p. 161-2)
- Mandl + Shaw is also good (p. 144-5)
- Srednicki also has results for pair annihilation (p. 354-60) that can be converted to Compton using prob 59.1



$$A = e^2 \left[\frac{N_s}{s-m^2} + \frac{N_u}{u-m^2} \right]$$

where N_s, N_u depend on spins + polarizations

$$|A|^2 = e^4 \left[\frac{|N_s|^2}{(s-m^2)^2} + \frac{(N_s^* N_u + N_u^* N_s)}{(s-m^2)(u-m^2)} + \frac{|N_u|^2}{(u-m^2)^2} \right]$$

Now imagine summing over the spins + polarizations

eq (5.81) $$= \frac{e^4}{4} \left[\frac{\mathrm{I}}{(s-m^2)^2} - \frac{(\mathrm{II}+\mathrm{III})}{(s-m^2)(u-m^2)} + \frac{(\mathrm{IV})}{(u-m^2)^2} \right]$$

eq (5.84) see errata

$$\mathrm{I} = 16\left[2m^4 + m^2(s-m^2) - \tfrac{1}{2}(s-m^2)(u-m^2)\right]$$

$$\mathrm{IV} = 16\left[2m^4 + m^2(u-m^2) - \tfrac{1}{2}(s-m^2)(u-m^2)\right]$$

$$\mathrm{II}+\mathrm{III} = -16\left[4m^4 + m^2(s-m^2) + m^2(u-m^2)\right]$$

Thus

$$\begin{array}{cccc} & \mathrm{I} & \mathrm{II}+\mathrm{III} & \mathrm{IV} \end{array}$$

$$\begin{aligned}
\langle |A|^2 \rangle = 4e^4 \Bigg[& \frac{2m^4}{(s-m^2)^2} + \frac{4m^4}{(s-m^2)(u-m^2)} + \frac{2m^4}{(u-m^2)^2} \\
& + \frac{m^2}{(s-m^2)} + \frac{m^2}{(s-m^2)} + \frac{m^2}{(u-m^2)} + \frac{m^2}{(u-m^2)} \\
& - \frac{\frac{1}{2}(u-m^2)}{(s-m^2)} \qquad - \frac{\frac{1}{2}(s-m^2)}{(u-m^2)} \Bigg]
\end{aligned}$$

$$\begin{aligned}
= 4e^4 \Bigg[& 2m^4 \left(\frac{1}{s-m^2} + \frac{1}{u-m^2} \right)^2 \\
& + 2m^2 \left(\frac{1}{s-m^2} + \frac{1}{u-m^2} \right) \\
& - \frac{1}{2} \left(\frac{u-m^2}{s-m^2} \right) - \frac{1}{2} \left(\frac{s-m^2}{u-m^2} \right) \Bigg]
\end{aligned}$$

which agrees w/ eq (5.87), using eqn (5.83), of Peskin & Schroeder!

Let's take the $p \ll m$ limit (Thomson) of $\langle |A|^2 \rangle$

$\Rightarrow E \to m$

since
$$s - m^2 = 2p(E+p)$$
$$u - m^2 = -2p(E + p\cos\theta)$$

we have
$$\frac{u-m^2}{s-m^2} \to -1$$

and
$$\frac{1}{s-m^2} + \frac{1}{u-m^2} = \frac{1}{2p(E+p)} - \frac{1}{2p(E+p\cos\theta)}$$
$$= \frac{(\cos\theta - 1)}{2(E+p)(E+p\cos\theta)}$$
$$\to \frac{\cos\theta - 1}{2m^2}$$

Then

$$\langle |A|^2 \rangle = e^4 \Big[\underbrace{8m^4 \left(\frac{\cos\theta - 1}{2m^2} \right)^2}_{2\cos^2\theta - 4\cos\theta + 2} + \underbrace{8m^2 \left(\frac{\cos\theta - 1}{2m^2} \right)}_{4\cos\theta - 4} + 4 \Big]$$

$$= 2e^4 \left[1 + \cos^2\theta \right]$$

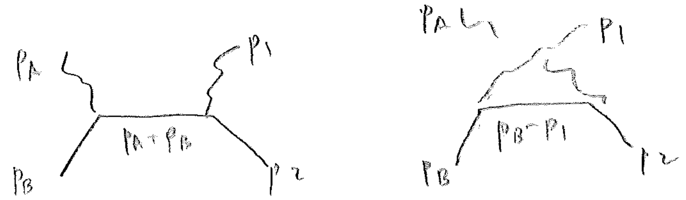
$$\left(\frac{d\sigma}{d\Omega} \right)_{cm} = \frac{\hbar^2}{(8\pi E_{cm})^2} \left(\frac{p_f}{p_i} \right) |A|^2 \xrightarrow[p \ll m]{} \frac{\hbar^2 (4\pi\alpha)^2}{(8\pi m)^2} 2(1+\cos^2\theta)$$

$$= \frac{\hbar^2 \alpha^2}{m^2} \frac{(1+\cos^2\theta)}{2} = \frac{\hbar^2 \alpha^2}{m^2} \left(1 - \frac{1}{2}\sin^2\theta \right)$$

$$\sigma = \frac{\hbar^2 \alpha^2}{m^2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_{-1}^{1} d(\cos\theta) \left(\frac{1+\cos^2\theta}{2} \right)}_{\frac{4}{3}} = \frac{8\pi}{3} \cdot \frac{\hbar^2 \alpha^2}{m^2}$$

Thomson cross section

Compton scattering is too complicated for our crude approach, whereby to make the amplitude dimensionless, we multiply by $\sqrt{(2E_A)(2E_B)(2E_1)(2E_2)} = (2E)(2p)$



$$A = e^2\left[\frac{(2E)(2p)}{s-m^2} + \frac{(2E)(2p)}{u-m^2}\right]$$

$$= e^2\left[\frac{(2E)(2p)}{(2p)(E+p)} - \frac{(2E)(2p)}{(2p)(E+p\cos\theta)}\right]$$

$$= e^2\left[\frac{2E}{E+p} - \frac{2E}{E+p\cos\theta}\right] \quad p \ll E \quad \Rightarrow O(p) \text{ not const!}$$

The actual computation shows that the numerators go (in the limit $p \ll E \sim m$) as m^2, not mp and that there is cancellation between the terms so that the leading contribution to the amplitude ~ const.

In 2019, I used the crude approx above and also neglected the 2nd Feynman diag, to get (although I was using different conventions)

$$A = e^2 \frac{2E}{E+p} \to 2e^2 \quad (\text{isotropic!})$$

& then $\dfrac{d\sigma}{d\Omega} = \dfrac{1}{(8\pi m)^2}|A|^2 = \dfrac{4(4\pi\alpha)^2}{(8\pi m)^2} = \dfrac{\alpha^2}{m^2}$

or $4\pi \dfrac{\alpha^2}{m^2}$ but this is not really kosher old calc based on Sakurai notes ...

see 313 notes

QM Compton scattering : non rel limit

$(\hbar\omega \ll mc^2)$ $\rightarrow$ Thomson $\rightarrow$ $\omega' = \omega$

Time dep. pert theory $\Rightarrow$ Fermi golden rule

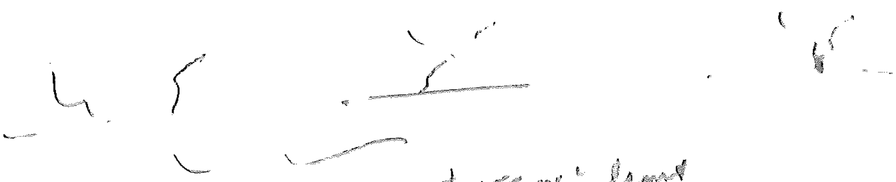
(set $\hbar = c = 1$)

$$dR = \text{transition rate} = 2\pi |\mathcal{M}|^2 \, \delta(E_f - E_i)$$

$$\mathcal{M} = \langle p', k'\alpha' | H_{int} | p, k\alpha \rangle$$

$$H_{int} = -\frac{e}{m} \vec{A}\cdot\vec{p} + \frac{e^2}{2m} \vec{A}^2$$

$$\vec{A} = \frac{1}{\sqrt{2V}} \sum_{k\alpha} \frac{1}{\sqrt{\omega}} \left[a_{k\alpha} \vec{\epsilon}^{\,\alpha} e^{i\vec{k}\cdot\vec{x}} + a^\dagger_{k\alpha} \vec{\epsilon}^{\,\alpha *} e^{-i\vec{k}\cdot\vec{x}} \right]$$



suppressed in $\hbar\omega \ll mc^2$ limit

$$A^2 \rightarrow \langle k' | (a + a^\dagger)(a + a^\dagger) | k \rangle$$

$$= \langle k' | a_k a^\dagger_{k'} + a^\dagger_{k'} a_k | k \rangle$$

$$= 2$$

$$\mathcal{M} = \langle p' | \frac{e^2}{2m} \frac{\vec{\epsilon}\cdot\vec{\epsilon}'}{2V\omega} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} | p \rangle$$

Now we'll set $\hbar = 1$

$$|\mathcal{M}|^2 = \left(\frac{e^2}{2mV^2}\right)^2 \frac{(\vec{\epsilon}\cdot\vec{\epsilon}')^2}{\omega\omega'} \int d^3x\, e^{i(k+p-k'-p')\cdot x} \int d^3x\, e^{i(k+p-k'-p')x}$$

In a box, the momenta are discrete, so the integral gives $V\,\delta_{k+p-k'-p'}$

but we can use the continuum approximation in which case

$\int d^3x\, e^{i(k+p-k'-p')x} = (2\pi)^3\,\delta(k+p-k'-p')$.

Then the exponent in 2nd integral vanishes & $\int d^3x = V$

$$|\mathcal{M}|^2 = \left(\frac{e^2\hbar\,\epsilon\cdot\epsilon'}{2m}\right)^2 \frac{1}{\omega\omega' V^4}\cdot(2\pi)^3\,\delta(k+p-k'-p')\;V$$

$$\text{Then } R = \sum_{\text{final state}} dR = \underbrace{\frac{V}{(2\pi)^3}\int d^3p'}_{\text{this kills } (2\pi)^3\delta(\ )\,V} \;\frac{V}{(2\pi)^3}\int d^3k'$$

$$= \left(\frac{e^2\hbar^2\,\vec{\epsilon}\cdot\vec{\epsilon}'}{2m}\right)^2 \frac{1}{V^4\omega\omega'}\frac{V}{(2\pi)^3}\int d^3k'$$

We could have gotten to here by ignoring the phase space factor of the final electron &

At this pt we may proceed by ignoring the spatial dependence of the electrons (ie treat them as infinitely massive) + summing over the phase space of the final photon only (ie using momentum conservation to eliminate the final electron phase space)

Then

$$R = \int dR \quad \frac{V}{(2\pi\hbar)^3}\int d^3k' \quad \frac{2\pi}{\hbar}\left(\frac{e^2\hbar}{m\, 2V}\right)^2 \frac{(\vec{\epsilon}\cdot\vec{\epsilon}\,')^2}{\omega\omega'}$$

My 315 note

Fermi golden rule

$$d\Gamma = \frac{2\pi}{\hbar}|\mathcal{M}|^2\,\delta(E_f - E_i + \hbar\omega' - \hbar\omega)$$

$$\text{Rate} = \sum d\Gamma = \frac{2\pi}{\hbar}\int \frac{V\,d^3k'}{(2\pi)^3}|\mathcal{M}|^2\,\delta(E_f - E_i + \hbar\omega' - \hbar\omega)$$

sum over final photon states

$$= \frac{V}{4\pi^2}\frac{(\omega')^2}{\hbar^2 c^3}\int d\Omega' \sum_{\alpha'}|\mathcal{M}|^2$$

$$\text{Flux} = \frac{c}{V}$$

$$\frac{d\sigma}{d\Omega} = \frac{\text{Rate}}{\text{Flux}} = \frac{V^2(\omega')^2}{4\pi^2\hbar^2c^4}\sum_{\alpha'}|\mathcal{M}|^2$$

$$= \frac{(\omega')^2}{4\pi^2\cancel{\hbar^2}c^4}\,\frac{e^4\cancel{\hbar^2}}{4m^2\omega\omega'}\sum_{\alpha}\Big|[\quad]\Big|^2$$

$$= \left(\frac{\omega'}{\omega}\right)\underbrace{\left(\frac{e^2}{4\pi mc^2}\right)^2}_{r_0^2}\sum_{\alpha}\Big|[\quad]\Big|^2 \qquad \text{Kramers Heisenberg}$$

$$= \left(\frac{m r_0^2}{\hbar}\right)^2 \omega\omega'^3 \sum_{\alpha}\left|\sum_I \frac{(\epsilon'\cdot x)_{BI}(\epsilon\cdot x)_{IA}}{\omega_{AI}+\omega} + \frac{(\epsilon\cdot x)_{B}(\epsilon'\cdot x)_{IA}}{\omega_{AI}-\omega'}\right|^2$$

If $\omega \ll \omega_{AI}$ (Rayleigh) then $\sum|\;|^2 \sim$ indep of ω

$$\Rightarrow \left(\frac{m r_0^2}{\hbar}\right)^2 \omega^4 \quad \# \qquad \text{(Rayleigh)}$$

$$|M|^2 = \left(\frac{e^2\hbar}{2mV^2}\right)^2 \frac{(\vec{\epsilon}\cdot\vec{\epsilon}')^2}{\omega\omega'} \int d^3x\, e^{i(k+p-k'-p')\cdot x} \int d^3x\, e^{i(k+p-k'-p')x}$$

in a box, all momenta are descrete

$= 0$

$$= \delta_{k+p-k'-p'} \cdot V \quad \cdot V$$

$$R = \sum_{\substack{\text{final}\\\text{states}}} dR = \left(\frac{V}{(2\pi\hbar)^3}\int d^3p'\right)\left(\frac{V}{(2\pi\hbar)^3}\int d^3k'\right)$$

$\langle B| \quad |A\rangle$

$(2\hbar'$

Gr

$$\left[(2\pi)^3\,\delta(\quad)\right]\cdot V$$

$\delta \qquad \int \delta(k-k)\, dk \quad 2\pi$

$\sum \quad \int$

$p \cdot \frac{n}{L}$

$$\int_0^L dx\; e^{i(p-p')\cdot x}$$

$=$

$$\sigma = \frac{R}{F}$$

$T \cdot$ Flux of photons $= \frac{c}{V} \sim \frac{1}{V}$

$$R = \sum dR = \frac{V}{(2\pi)^3} \int d^3k' \, (dR)$$

$$\sigma = \frac{V^2}{(2\pi)^3} \int d^3k' \, (2\pi) \left(\frac{e^2}{2m} \frac{\vec{\epsilon} \cdot \vec{\epsilon}\,'}{2\omega} \right)^2 \frac{1}{V^2} \left| \langle p' | e^{i(\vec{k} - \vec{k}') \cdot \vec{x}} | p \rangle \right|^2$$

mm / misc / compton.nb

Compton scattering

```
In[33]:= s = m^2 + 2 p (e + p);
In[34]:= u = m^2 - 2 p (e + p cos);
In[35]:= t = -2 p^2 (1 - cos);
In[36]:= e = Sqrt[p^2 + m^2];
```

Srednicki, p. 359

```
In[37]:= num1 = - 2 (s u - m^2 (3 s + u) - m^4);
In[38]:= num2 = -2 (s u - m^2 (3 u + s) - m^4);
In[39]:= num3 = - 4 m^2 (t - 4 m^2);
In[40]:= T1 = Series[num1 / (s - m^2) ^2, {p, 0, 2}];
In[41]:= T2 = Series[num2 / (u - m^2) ^2, {p, 0, 2}];
In[42]:= T3 = Series[num3 / (s - m^2) / (u - m^2), {p, 0, 2}];
In[43]:= sred = Simplify[T1 + T2 + T3]
```

Out[43]= $$2\left(1+\cos^2\right)-\frac{4\left(\cos\left(-1+\cos^2\right)\right)p}{\sqrt{m^2}}+\frac{\left(4-4\cos-6\cos^2+6\cos^4\right)p^2}{m^2}+O[p]^3$$

Peskin and Schroeder

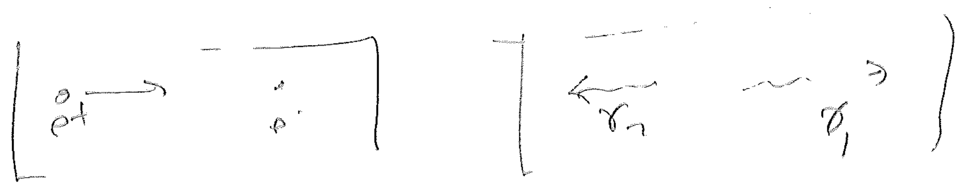
```
In[44]:= X1 = Series[ 8 m^4 ( 1 / (s - m^2) + 1 / (u - m^2) ) ^2, {p, 0, 2}];
In[45]:= X2 = Series[ 8 m^2 ( 1 / (s - m^2) + 1 / (u - m^2) ), {p, 0, 2}];
In[46]:= X3 = Series[-2 ( (u - m^2) / (s - m^2) + (s - m^2) / (u - m^2)) , {p, 0, 2}];
In[47]:= ps = Simplify[X1 + X2 + X3]
```

Out[47]= $$2\left(1+\cos^2\right)-\frac{4\left(\cos\left(-1+\cos^2\right)\right)p}{\sqrt{m^2}}+\frac{\left(4-4\cos-6\cos^2+6\cos^4\right)p^2}{m^2}+O[p]^3$$

```
In[48]:= sred - ps
```

Out[48]= $O[p]^3$

$e^+e^- \to \gamma\gamma$ in lab frame



$E+m = E_1+E_2$

$\sqrt{E^2-m^2} = E_1-E_2$

$E_1 = \frac{1}{2}\left(E+m+\sqrt{E^2-m^2}\right)$

$E_2 = \frac{1}{2}\left(E+m-\sqrt{E^2-m^2}\right)$

(See 2260 prob set 6 in 2023)

$E+m = E_1+E_2$

$\sqrt{E^2-m^2} = E_1\cos\theta$

$E_2 = E_1\sin\theta$

[illegible]

$$\begin{cases} E_1 = m \\ E_2 = E \\ \sin\theta = \dfrac{m}{E} \end{cases}$$

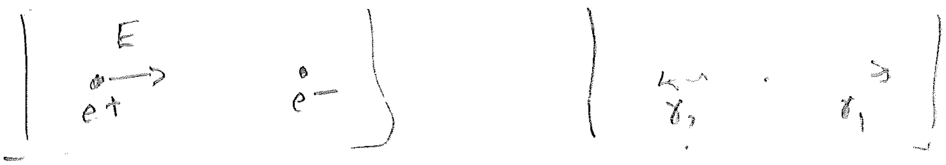
p2260

2017 problem set 3

PAIR ANNIHILATION KINEMATICS IN LAB FRAME

$e^- e^+ \to \gamma\gamma$, electron at rest

(2023 midterm problem)



Energy cons: $E + m = E_1 + E_2$

mom cons: $\sqrt{E^2 - m^2} = E_1 - E_2$

$$\Rightarrow \begin{cases} E_1 = \frac{1}{2}\left(E + m + \sqrt{E^2 - m^2}\right) \longrightarrow \begin{cases} E_1 = m & E \to m \\ E + \frac{m}{2} & E \gg m \end{cases} \\ E_2 = \frac{1}{2}\left(E + m - \sqrt{E^2 - m^2}\right) \longrightarrow \begin{cases} E_2 = m & E \to m \\ E_2 \to \frac{m}{2} & E \gg m \end{cases} \end{cases}$$

(a) (b) (c)

(-1) if say $E_2 \to 0$ when $E \gg m$

Using 4-vectors (much less useful) → see next pg

$$p_+^\mu = p_1^\mu + p_2^\mu - p_-^\mu \quad \left(\text{or } p_-^\mu = p_1^\mu + p_2^\mu - p_+^\mu\right)$$

$$\Rightarrow m^2 = \underbrace{2p_1 \cdot p_2}_{\underbrace{2(E_1E_2 - \vec{p}_1\cdot\vec{p}_2)}_{4E_1E_2}} - 2mE_1 - 2mE_2 + m^2$$

$$\Rightarrow 2E_1E_2 - m(E_1 + E_2) = 0$$

Also $E_1 - E_2 = p_+$ so

$$2(E_2 + p_+)E_2 - m(2E_2 + p_+) = 0$$

$$2E_2^2 + (2p_+ - 2m)E_2 - mp_+ = 0 \xrightarrow{p_+ \to -p_+} 2E_1^2 + (-2p_+ - 2m)E_1 - mp_+ = 0$$

$$E_2 = \frac{1}{2}(m - p_+) + \frac{1}{4}\sqrt{\underbrace{4(p_+ - m)^2 + 8p_+m}_{4(p_+^2 + m^2) = E_+^2}} \quad (\text{since } E_2 > 0)$$

$$= \frac{1}{2}\left(m - p_+ + E_+\right)$$

$$= \frac{1}{2}\left(E_+ + m - \sqrt{E_+^2 - m^2}\right) \text{ as above}$$

Another approach using 4-vectors (Mark [illegible])

$$p_+ + p_- - p_1 = p_2 \leftarrow \text{to eliminate } \underline{\underline{p_2}}$$

$$m^2 + m^2 + 0 + \underbrace{2p_+\cdot p_-}_{2Em} - \underbrace{2p_+\cdot p_1}_{2EE_1 - 2pE_1} - \underbrace{2p_-\cdot p_1}_{2mE_1} = 0$$

$$2m^2 + 2mE = (2E - 2p + 2m)\,E_1$$

$$E_1 = \frac{m(m+E)}{E - \sqrt{E^2 - m^2} + m}$$

$$E_2 = \frac{m(m+E)}{E + \sqrt{E^2 - m^2} + m} \xrightarrow{E \gg m} \frac{m}{2}$$

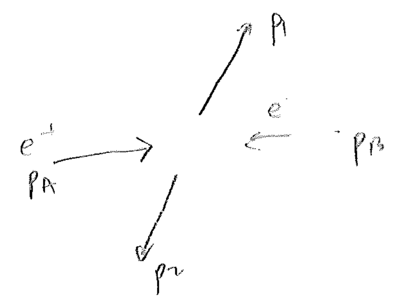
(1-27-23)

Pair production $e^+e^- \to \gamma\gamma$ (CM frame)

In cm frame the kinematics are the same as $e^+e^- \to \mu^+\mu^-$
w/ all energies equal, and w/ $p_\gamma = E$, $p = p_e = \sqrt{E^2 - m^2}$

$p_A = (E, 0, 0, p)$ $p_1 = (E, E\sin\theta, 0, E\cos\theta)$

$p_B = (E, 0, 0, -p)$ $p_2 = (E, -E\sin\theta, 0, -E\cos\theta)$



$p_A + p_B = (2E, 0, 0, 0)$

$s = (p_A + p_B)^2 = 4E^2$

$p_1 - p_A = (0, E\sin\theta, 0, E\cos\theta - p)$

$t = (p_1 - p_A)^2 = -(E\sin\theta)^2 - (E\cos\theta - p)^2 = -E^2 + 2Ep\cos\theta - p^2$

$t - m^2 = -2E^2 + 2Ep\cos\theta = -2E(E - p\cos\theta)$

$p_2 - p_A = (0, -E\sin\theta, 0, -E\cos\theta - p)$

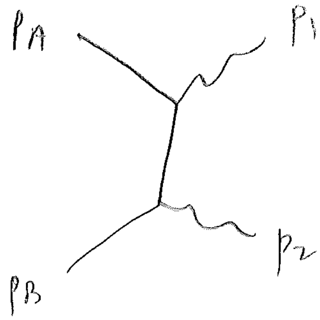
$u = (p_2 - p_A)^2 = -(E\sin\theta)^2 - (E\cos\theta + p)^2 = -E^2 - 2Ep\cos\theta - p^2$

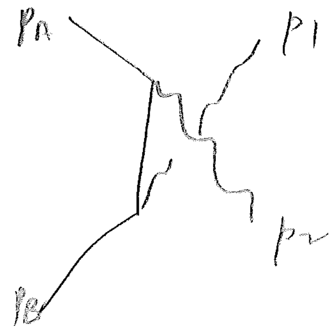
$u - m^2 = -2E(E + p\cos\theta)$

$s + t + u = 2E^2 - 2p^2 = 2m^2$ ✓

$t = -m^2$, $u = -m^2$

In limit of e^+e^- initially at rest. $s = 4m^2$, $t - m^2 = -2m^2$, $u - m^2 = -2m^2$ cont





$$A = e^2 \left[\frac{N_t}{t-m^2} + \frac{N_u}{u-m^2} \right]$$

Our crude approximation says: $N_t = N_u = (2E)^2 = s$

$$A = e^2 \left[\frac{(2E)^2}{-2E(E-p\cos\theta)} + \frac{(2E)^2}{-2E(E+p\cos\theta)} \right]$$

$$= e^2 \left[\frac{-(2E)^2}{(E-p\cos\theta)(E+p\cos\theta)} \right]$$

That is:

$$e^2 \left[\frac{s}{t-m^2} + \frac{s}{u-m^2} \right] = e^2 \frac{-s^2}{(t-m^2)(u-m^2)}$$

This is far from the mark: see Srednicki p. 359-60 (next page)

However assume $p \ll m$ i.e. $\boxed{e^+e^- \text{ initially at rest}}$

then our crude assumption gives $A = e^2 \left[\frac{-(2m)^2}{m \cdot m} \right] = -4e^2$

which lo & behold! agrees w/ Griffiths's (2e) ch 7, eq. 7.163.

(But this is just a lucky fluke. E.g. in Srednicki the cross term vanishes in ... ...)

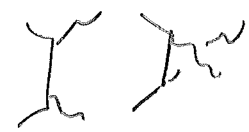
Then $\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi E_{cm})^2} \frac{p_f}{p_i} |A|^2 = \frac{1}{(8\pi(2m))^2} \frac{m}{(mv)} [4(4\pi\alpha)]^2 = \frac{\alpha^2}{m^2 v}$

$$\sigma = \frac{4\pi\alpha^2}{m^2 v} \qquad \text{(Griffiths 7.168)}$$

Srednicki
p. 359-60

$$\boxed{e^+e^- \to \gamma\gamma}$$

invt under $t \leftrightarrow u$ due to



$$\langle |A|^2 \rangle = e^4 \Bigg[\frac{2(tu - m^2[3t+u] - m^4)}{(t-m^2)^2}$$

eq (59.18)
eqs (59.22-25)

$$+ \frac{2(tu - m^2[3u+t] - m^4)}{(u-m^2)^2}$$

cross term

$$+ \frac{4m^2(s-4m^2)}{(t-m^2)(u-m^2)} \Bigg]$$

e^+e^- initially at rest
$\Downarrow$
$s = 4m^2$
$t = -m^2$
$u = -m^2$

note cross term vanishes!

$$\Rightarrow e^4 \left[\frac{2(1+4-1)}{4} + \frac{2(1+4-1)}{4} + 0 \right] = 4e^4$$

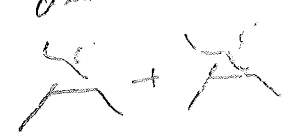
seems to be $\frac{1}{4}$ of Griffiths result

$$e^-\gamma \to e^-\gamma \quad \text{(Compton)}$$

Srednicki
prob. 59.1

Same as above, but w/ $s \leftrightarrow t$ and overall minus.

invt under $s \leftrightarrow u$ due to



$$|A|^2 = -e^4 \Bigg[\frac{2(su - m^2[3s+u] - m^4)}{(s-m^2)^2}$$

$$+ \frac{2(su - m^2[3u+s] - m^4)}{(u-m^2)^2}$$

$$+ \frac{4m^2(t-4m^2)}{(s-m^2)(u-m^2)} \Bigg]$$

$s \sim m^2$
$u \sim m^2$
$t \sim 0$

$$\Rightarrow \; = +e^4 \left[\frac{8m^4}{(s-m^2)^2} + \frac{8m^4}{(u-m^2)^2} + \frac{16m^4}{(s-m^2)(u-m^2)} + \dots \right]$$