[Background QFT for P2260]

(Jan 2023)

(1-22-23)

[I had intended to present this derivation, but now think I'll just jump straight to Γ and σ, as do Griffiths & Peskin. Too much background is necessary, & even then many things need to be assumed.]

We'll use QFT to compute decay rates & scattering cross sections.

[Srednicki, ch 11; Schwartz, ch 5]

Given initial state i, compute probability that it will transition to another f.

$$(\text{transition probability}) = \sum_{\substack{\text{final}\\\text{states}}} (\text{t.p. } i \to f)$$

= t.p.

$\therefore$ the more final states (channels), the greater the probability

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$$\text{QM} \Rightarrow \quad (\text{t.p } i \to f) = \Big|(\underbrace{\text{transition amplitude}}_{\text{t.a.}}\ i \to f)\Big|^2$$

$$(\text{t.a. } i \to f) \sim \langle f \,|\, T \,|\, i \rangle$$

final state, transition matrix, initial state

[$S = 1 + iT$]

$$(\text{t.p. } i \to f) \sim \langle f|T|i\rangle^* \langle f|T|i\rangle = \langle i|T^\dagger|f\rangle\langle f|T|i\rangle$$

Normalize $$(\text{t.p. } i \to f) = \frac{\langle i|T^\dagger|f\rangle\langle f|T|i\rangle}{\langle i|i\rangle\ \langle f|f\rangle}$$

$$\boxed{\text{t.p.} = \sum_f \frac{|\langle f|T|i\rangle|^2}{\langle i|i\rangle\langle f|f\rangle}}$$

Initial state could be a particle of momentum $\vec{p}_A$

$|i\rangle = |\vec{p}_A\rangle$

In which case t.p. is the probability it will decay

Or it could be a pair of particles with moment $\vec{p}_A$ and $\vec{p}_B$ approaching one another

$|i\rangle = |\vec{p}_A, \vec{p}_B\rangle$

In which case t.p. is probability they will scatter or transform into other particle

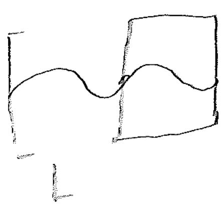
The final state could be could be a set of n_f particle

$|f\rangle = |\vec{p}_1, \vec{p}_2, \dots, \vec{p}_{n_f}\rangle$

free particles of momentum $\vec{p}$ are described by $e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}}$

To be able to normalize these, imagine the universe = box w/ volume $V=L^3$

w/ periodic bound. conds, i.e. (at end, take $V \to \infty$)



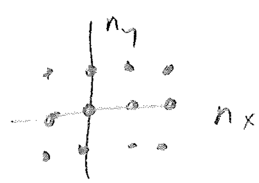
L

$$e^{\frac{ip(x+L)}{\hbar}} = e^{\frac{ipx}{\hbar}}$$

$$e^{\frac{ipL}{\hbar}} = 1 \quad \Rightarrow \quad p = \frac{2\pi\hbar}{L} n$$

3 dimensions: $\vec{p} = \frac{2\pi\hbar}{L}\vec{n} = \frac{2\pi\hbar}{L}(n_x, n_y, n_z)$

Set of possible states characterized by lattice of integers



$$\sum_{n_x, n_y, n_z} \approx \int dn_x\, dn_y\, dn_z$$

$$= \left(\frac{L}{2\pi\hbar}\right)^3 \int dp_x\, dp_y\, dp_z$$

$$= \frac{V}{(2\pi\hbar)^3} \int d^3p$$

Sum over final states

$$\boxed{\sum_f = \prod_{j=1}^{n_f} \frac{V}{\hbar^3} \int \frac{d^3 p_j}{(2\pi)^3}}$$

"integral over phase space"

Free particle state $\underbrace{|\vec{p}\rangle}_{\text{ket}} \sim e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}}$ [we'll pick normalization later]

Complex conjugate (Hermitian) $\underbrace{\langle\vec{p}|}_{\text{bra}} \sim e^{-\frac{i\vec{p}\cdot\vec{x}}{\hbar}}$

Inner product of two states

$$\underbrace{\langle\vec{p}'|\vec{p}\rangle}_{\text{bracket}} \sim \int d^3x \; e^{\frac{i(\vec{p}-\vec{p}')\cdot\vec{x}}{\hbar}}$$

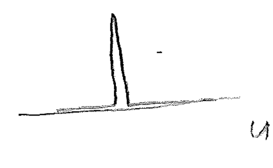
$$\sim \int dx \cdot e^{\frac{i(p_x-p_x')x}{\hbar}} \int dy \, e^{\frac{i(p_y-p_y')y}{\hbar}} \int dz \, e^{\frac{i(p_z-p_z')z}{\hbar}}$$

Consider $I(p-p') = \int dx\; e^{i(p-p')x/\hbar}$

If $p \neq p'$, $I = 0$

If $p = p'$, $I = \int dx \cdot 1 = L$

Dirac delta "function" $\delta(u) \begin{cases} 0 & u \neq 0 \\ \infty & u = 0 \end{cases}$



such that $\int_{-\infty}^{\infty} du\, \delta(u) = 1$ (*)

Claim: $\lim_{L\to\infty} \int dx\; e^{i(p-p')x/\hbar} = \underbrace{(2\pi\hbar)} \delta(p-p')$

necessary to guarantee (*)
(proved in Phys 3140)

$$\lim_{V\to\infty} \int d^3x\; e^{i(\vec{p}-\vec{p}')\cdot\vec{x}/\hbar} = (2\pi\hbar)^3 \underbrace{\delta(p_x-p_x')\,\delta(p_y-p_y')\,\delta(p_z-p_z')}$$

$$\equiv \delta^{(3)}(\vec{p}-\vec{p}')$$

We'll choose to normalize free particle state as

$$|\vec{p}\rangle = \sqrt{\frac{2E}{\hbar^3}}\, e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \quad \text{where} \quad E = \sqrt{\vec{p}^{\,2}+m^2}$$

Then

$$\langle \vec{p}\,'|\vec{p}\rangle = \frac{\sqrt{(2E)(2E')}}{\hbar^3} \int d^3x \; e^{\frac{i(\vec{p}-\vec{p}\,')\vec{x}}{\hbar}}$$

$$\boxed{\langle \vec{p}|\vec{p}\rangle = \frac{V}{\hbar^3}(2E)}$$

$$\langle \vec{p}\,'|\vec{p}\rangle = \frac{\sqrt{(2E)(2E')}}{\hbar^3}(2\pi\hbar)^3\,\delta^{(3)}(\vec{p}-\vec{p}\,')$$

Since $\vec{p}=\vec{p}\,' \Rightarrow E = E'$

$$\boxed{\langle \vec{p}\,'|\vec{p}\rangle = (2E)(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}\,')}$$

This is a Lorentz invariant normalization

What are the dimensions of $|p\rangle$?

$$\langle p|p\rangle \sim \frac{EL^3}{\hbar^3} \sim \frac{EL^3}{(EL)^3} \sim \frac{1}{E^2} \Rightarrow |p\rangle \text{ has units of } E^{-1}$$

What are units of $\delta(p-p')$?

$$\int_{-\infty}^{\infty} dp\, \delta(p-p') = 1 \quad \Rightarrow \delta(p-p') \text{ has units of } E^{-1}$$

transition amplitude from $i \to f$ $\sim \langle f | T | i \rangle$

spacetime translation invariance $\Rightarrow$ conservation of energy + momentum ($\underbrace{\text{energy + momentum}}_{\text{4-momentum}}$)

$$\sum_{j=1}^{n_f} p_j^\mu = \sum_{k=1}^{n_i} p_k^\mu$$

Define $\Delta p^\mu = \sum_{j=1}^{n_f} p_j^\mu - \sum_{k=1}^{n_i} p_k^\mu$

$\langle f | T | i \rangle$ vanishes unless 4-momentum is conserved, i.e. $\Delta p^\mu = 0$
so it is proportional to $\delta^{(4)}(\Delta p^\mu)$

one defines the "invariant amplitude" A as

$$\boxed{\langle f | T | i \rangle = (2\pi)^4 \underbrace{\delta^{(4)}(\Delta p^\mu)}_{\text{guarantees 4-momentum conservation}} A}$$

T is dimensionless (cf. definition of t.p. which has no units)

$\langle f | T | i \rangle$ has units of $E^{-n_i - n_f}$

$\delta^{(4)}(\Delta p^\mu)$ has units of E^{-4}

$\Rightarrow A$ has units of $4 - n_i - n_f$

Recall

$$\text{transition probability} = \sum_f \frac{|\langle f|T|i\rangle|^2}{\langle i|i\rangle\langle f|f\rangle}$$

$$|\langle f|T|i\rangle|^2 \cdot (2\pi)^4\,\delta^{(4)}(\Delta p^\mu)\,(2\pi)^4\,\underbrace{\delta^{(4)}(\Delta p^\mu)}\quad |A|^2$$

$$= \lim_{V\to\infty} \frac{1}{(2\pi\hbar)^4}\underbrace{\int d^4x}_{VT}\; e^{\frac{i\Delta p^\mu x_\mu}{\hbar}}$$

$= 0$ due to other δ-fns

T = time elapsed in experiment

$$= \frac{VT}{\hbar^4}(2\pi)^4\,\delta^{(4)}(\Delta p^\mu)\,|A|^2$$

Let $|f\rangle = |p_1, p_2, \ldots p_f\rangle$

Recall $\sum_f = \int \prod_{j=1}^{n_f} \frac{V}{\hbar^3} \frac{d^3 p_j}{(2\pi)^3}$

$$\langle f|f\rangle = \prod_{j=1}^{n_f} \langle \vec{p}_j | \vec{p}_j \rangle = \prod_{j=1}^{n_f} \frac{V}{\hbar^3}(2E_j)$$

$$\sum_f \frac{1}{\langle f|f\rangle} = \int \prod_{j=1}^{n_f} \frac{d^3 p_j}{(2\pi)^3 (2E_j)}$$ → sometimes designated $\tilde{d}p_j$

$$t.p. = \frac{VT}{\hbar^4} \frac{1}{\langle i|i\rangle} \int \underbrace{\prod_{j=1}^{n_f} \frac{d^3 p_j}{(2\pi)^3 (2E_j)} (2\pi)^4 \delta^{(4)}(\Delta p^\mu)}_{(LIPS)_{n_f}} |A|^2$$

$(LIPS)_{n_f}$ has units $E^{2n_f - 4}$

$|A|^2$ has units $E^{8 - 2n_i - 2n_f}$

$\frac{1}{\langle i|i\rangle}$ has units E^{2n_i}

$\frac{VT}{\hbar^4}$ has units E^{-4}

$\Rightarrow$ t.p. is dimensionless

Decay rate

Consider one particle in initial state $|i\rangle = |\vec{p}_A\rangle$

$$\langle i|i\rangle = \frac{V}{\hbar^3}(2E_A)$$

$$\text{transition probability} = \frac{T}{\hbar}\frac{1}{(2E_A)}\int (LIPS)_f |A|^2$$

$$\text{Rate} = \frac{\text{Probability}}{T}$$

$$= \frac{1}{\hbar}\frac{1}{2E_A}\int (LIPS)_f |A|^2$$

Note: rate independent of volume of box so can now take $V \to \infty$

If initial particle is at rest, $E_A = m_A$
and we call this the decay rate

$$\boxed{\begin{pmatrix}\text{Decay}\\ \text{rate}\end{pmatrix} = \frac{1}{\hbar}\frac{1}{2m_A}\int (LIPS)_f |A|^2}$$

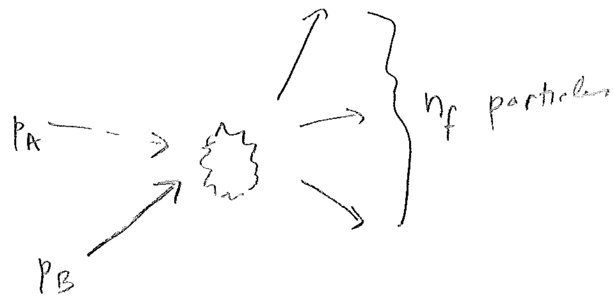
start 7 thru

$$\text{units} = \frac{1}{\hbar}\cdot\frac{1}{E}\cdot E^{2n_f-4}\cdot E^{8-2n_i-2n_f} = \frac{1}{\hbar}E = \frac{1}{\text{time}}$$

$$\text{mean life } \tau = \frac{1}{\text{decay rate}}$$

If particle not at rest, $E_A = \gamma m_A$ so $\text{rate} = \dfrac{\text{decay rate}}{\gamma}$ (slower rate due to time dilat.)

$2 \to n_f$ scattering process



$$|i\rangle = |\vec{p}_A, \vec{p}_B\rangle$$

$$\langle i|i\rangle = \frac{V^2}{\hbar^6}(2E_A)(2E_B)$$

$$t.p. = \frac{VT}{\hbar^4}\frac{1}{\langle i|i\rangle}\int (LIPS)_{n_f}|A|^2$$

$$= \frac{\hbar^2 T}{V(2E_A)(2E_B)}\int (LIPS)_{n_f}|A|^2$$

R = scattering rate $\quad \dfrac{t.p.}{T} = \dfrac{\hbar^2}{V(2E_A)(2E_B)}\displaystyle\int (LIPS)_{n_f}|A|^2$

$$\sigma = \frac{R}{F}$$

$$F = \text{incident flux} = \frac{\#\ \text{incident particle}}{(\text{sec})(\text{area})}$$

First consider fixed target scattering

$\overset{\longrightarrow}{\vec{v}}$ $\qquad\qquad$ $\circ$ target at rest

Particles are described by wave functions in a box of volume L^3 w/ periodic b.c. $\sim e^{i\vec{p}\cdot\vec{x}/\hbar}$

$\#$ incident particles $= 1$, each time it crosses the box it comes back around again

time to cross box $\quad t = \dfrac{L}{v}$

$\vec{v}$ $\quad L$

$$F = \frac{1\ \text{particle}}{\left(\frac{L}{v}\right)L^2} = \frac{v}{L^3} = \frac{v}{V} \qquad [\text{correct units !}]$$

Particles approaching one another.

$\overset{\longrightarrow}{v_A}$ $\qquad$ $\overset{\longleftarrow}{v_B}$

$$t = \frac{L}{|v_A| + |v_B|} = \frac{L}{|v_A - v_B|}$$

$$F = \frac{(v_A - v_B)}{V}$$

start here $\longrightarrow$

$$\boxed{\sigma = \frac{\hbar^2}{(2E_A)(2E_B)(v_A - v_B)} \int (\text{LIPS})_{n_f} |A|^2}$$

independent of size of box so can take $V \to \infty$