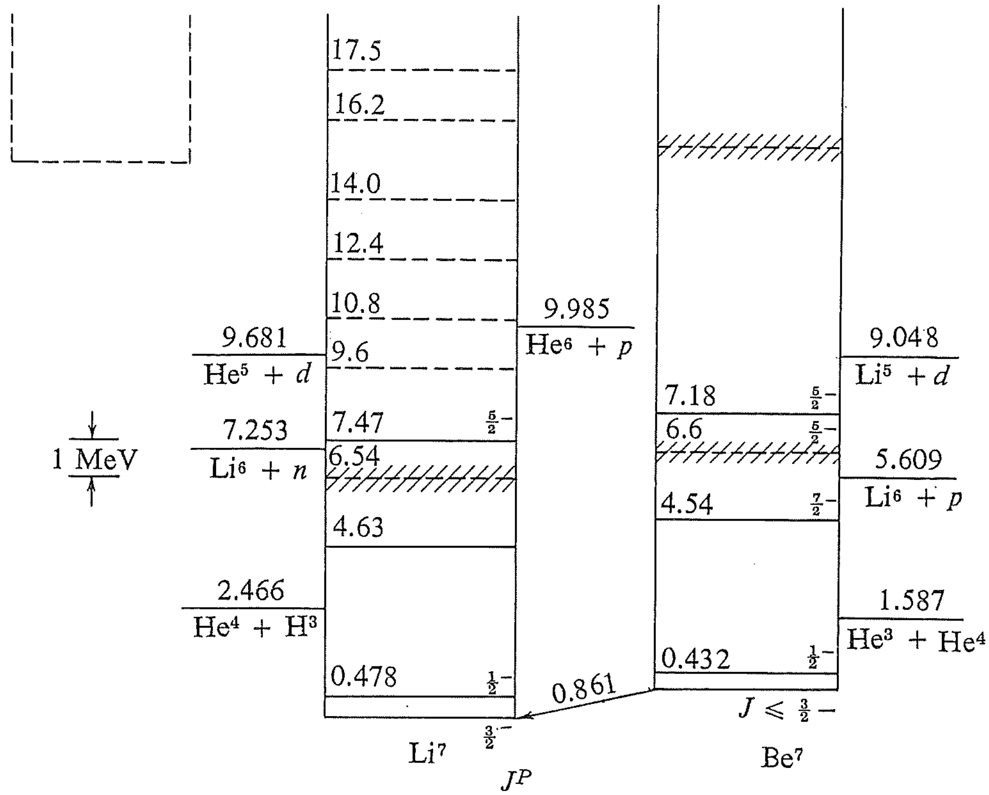
Isospin symmetry

[Consider the proton & neutron.
They have different electric charges,
but nearly the same mass: 938.3 vs 939.6.
Moreover, the strong interaction treats them very
similarly: mirror nuclei have similar energy levels
(see energy level diagram)]

Could this degeneracy be the hint of some symmetry?
[We know that rotational symmetry ⇒ different
spin states $|j, m\rangle$ have degenerate energies.
magnetic field breaks the symmetry & splits the degeneracy]

[Heisenberg suggested the existence of a new symmetry
called isotopic spin, or isospin.
The strong interaction respects isospin symmetry
so that different nuclei have same energies
whereas the electromagnetic interaction
breaks the symmetry, leads to some splitting.]



17.5
16.2
14.0
12.4
10.8
9.6
7.47
6.54
4.63
0.478
9.681
He5 + d
7.253
Li6 + n
2.466
He4 + H3
1 MeV
9.985
He6 + p
0.861
Li7
JP
7.18
6.6
4.54
0.432
9.048
Li5 + d
5.609
Li6 + p
1.587
He3 + He4
J ≤ 3/2 −
Be7

Segre, p. 391

We introduce

Isospin $\vec{I} = (I_1, I_2, I_3)$ analogous to $\vec{S} = (S_x, S_y, S_z)$

Isospin is quantized in integer or half-integer units.

Hadronic particles correspond to states

$|I, I_3\rangle$ analogous to $|j, m\rangle$

$I_3 = -I, \dots, I$ $\qquad m: -j, \dots, j$

Recall; in a rotationally-symmetric (isotropic) environment

the energy of the state $|j, m\rangle$ does not depend on m

$\Rightarrow (2j+1)$ degenerate states

Turning on a magnetic field splits the degeneracy.

Analogously, if only the strong interaction were present (and quark masses were zero)

the mass of the state $|I, I_3\rangle$ would not depend on I_3

$\Rightarrow (2I+1)$ degenerate particles

Strong force is "isotopically isotropic"

(Weak + electromagnetic interactions do depend on I_3, splitting the degeneracy, but the mass splitting is small because these forces are weak.)

Because p and n are a nearly degenerate doublet we regard them as two states of a single isospin-$\frac{1}{2}$ particle N (nucleon).

N has states $|I, I_3\rangle$: $\begin{cases} |\frac{1}{2}, \frac{1}{2}\rangle : p \\ |\frac{1}{2}, -\frac{1}{2}\rangle : n \end{cases}$

We say N belongs to isospin representation $\underset{\sim}{2}$.

A nucleon also has spin-$\frac{1}{2}$ & so belongs to $\underset{\sim}{2}$ of spin

Nucleon belongs to $(\underbrace{\underset{\sim}{2}}_{\text{spin}}, \underbrace{\underset{\sim}{2}}_{\text{isospin}})$

and has 4 states :

$p\uparrow$, $p\downarrow$, $n\uparrow$, $n\downarrow$

[We'll soon see why this way of thinking is useful.]

Recall: p and n are different isospin states of N which belongs to isospin representation $\underset{\sim}{2}$.

A system of two nucleons, NN belongs to tensor-product representation of isospin

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3}_S \oplus \underset{\sim}{1}_A$$

isotriplet $\underset{\sim}{3}_S$

$$|I, I_3\rangle = \begin{cases} |1,1\rangle = pp \\ |1,0\rangle = \frac{1}{\sqrt{2}}(pn + np) \\ |1,-1\rangle = nn \end{cases}$$

isosinglet $\underset{\sim}{1}_A$

$$|I, I_3\rangle = |0,0\rangle = \frac{1}{\sqrt{2}}(pn - np)$$

Recall strong force is insensitive to I_3, but depends on I.

We can see this because $\underbrace{pn}_{^2H}$ forms a bound state, but not $\underbrace{pp}_{^2He}$, nn.

[can blame absence of ^2He on Coulomb, though ^3He exists. Dineutron?]

$I=1$ state is not stable, but $I=0$ is.

Deuteron is an isospin 0 state. [what are the consequences?]

Recall: N belongs to $(\underset{\sim}{2}, \underset{\sim}{2})$ of (spin, isospin)

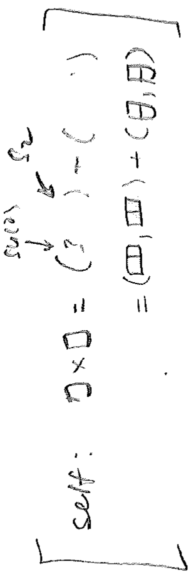
A system of two nucleons belongs to

$$(2,2) \otimes (2,2)$$

$$= (2 \otimes 2,\ 2 \otimes 2)$$

$$= (3_S \oplus 1_A,\ \underbrace{\cancel{3_S}}_{\text{unstable}} \oplus 1_A)$$

$$= \underbrace{(3_S, 1_A)}_{\text{overall antisymmetric}} \oplus \underbrace{(1_A, 1_A)}_{\text{overall symmetric}}$$



$$(1_A, 1_A) = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\tfrac{1}{\sqrt{2}}(pn - np)$$ ← overall symmetric, not allowed for two identical fermions

$$= \tfrac{1}{2}(p\uparrow n\downarrow + n\downarrow p\uparrow - p\downarrow n\uparrow - n\uparrow p\downarrow)$$

sym sym

Therefore a deuteron <u>must</u> belong to

$$(3_S, 1_A) = \begin{pmatrix} \uparrow\uparrow \\ \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ \downarrow\downarrow \end{pmatrix} \tfrac{1}{\sqrt{2}}(pn - np) = \begin{pmatrix} \frac{1}{\sqrt{2}}(p\uparrow n\uparrow - n\uparrow p\uparrow) \\ \frac{1}{2}(p\uparrow n\downarrow - n\downarrow p\uparrow + p\downarrow n\uparrow - n\uparrow p\downarrow) \\ \frac{1}{\sqrt{2}}(p\downarrow n\downarrow - n\downarrow p\downarrow) \end{pmatrix}$$

Isospin predicts: Deuteron must have spin 1 (and it does)

$\hookrightarrow$ $\therefore$ a magnetic moment!

Do this as a problem

System of 3 nucleons NNN ← must be antisymmetric under exchange of any 2 nucleons

$(2,2) \otimes (2,2) \otimes (2,2)$

$\therefore (2 \otimes 2 \otimes 2,\ 2 \otimes 2 \otimes 2)$

$= (\underset{I=\frac{3}{2}}{4} \oplus \underbrace{2_S \oplus 2_A}_{I=\frac{1}{2}},\ \underset{J=\frac{3}{2}}{4} \oplus \underbrace{2_S \oplus 2_A}_{J=\frac{1}{2}})$

isoquartet $4 = \begin{cases} ppp \\ \frac{1}{\sqrt{3}}(ppn + pnp + npp) \\ \frac{1}{\sqrt{3}}(nnp + npn + pnn) \\ nnn \end{cases}$

Not possible ↑ because isospin state is completely symmetric under exchange but cannot have completely antisymmetric spin state
if all nucleons in gnd state

Also trineutron state not observed

Must be an isodoublet $(I = \frac{1}{2})$

$2_S = \begin{cases} \frac{1}{\sqrt{6}}(2ppn - pnp - npp) & \leftarrow {}^3He \\ \frac{1}{\sqrt{6}}(-2nnp + npn + pnn) & \leftarrow {}^3H \text{ (tritium)} \end{cases}$

$2_A = \begin{cases} \frac{1}{\sqrt{2}}(pnp - npp) \\ \frac{1}{\sqrt{2}}(pnn - npn) \end{cases}$

What about spin? Can't be $J = \frac{3}{2}$ because 4 is completely symmetric and can't make isospin state antisymmetric

So 3 nucleon state has $I\ \frac{1}{2}$ and $J\ \frac{1}{2}$

$\Rightarrow$ ${}^3He$, ${}^3H$ have $J = \frac{1}{2}$ ✓

To be antisymmetric under exchange of 1st 2 nucleons must choose either

$(2_S, 2_A)$ or $(2_A, 2_S)$

How to make it completely antisymmetric?

To obtain a state antisymmetric under exchange of 1^{st} 2 nucleons choose

$$(2_S, 2_A) = \frac{1}{\sqrt{12}}(2ppn - pnp - npp)(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

or

$$(2_A, 2_S) = \frac{1}{\sqrt{12}}(pnp - npp)(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

Can we also make it antisymmetric under exchange of 2^{nd} 2 nucleons?

Eliminate $npp\downarrow\uparrow\uparrow$ by taking the difference of these

$$\begin{aligned}(2_S, 2_A) - (2_A, 2_S) = &\frac{1}{\sqrt{12}} 2ppn(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\ &+ \frac{1}{\sqrt{12}} pnp(\cancel{-\uparrow\downarrow\uparrow} + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow + \cancel{\uparrow\downarrow\uparrow} + \downarrow\uparrow\uparrow) \\ &+ \frac{1}{\sqrt{12}} npp(-\uparrow\downarrow\uparrow + \cancel{\downarrow\uparrow\uparrow} + 2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \cancel{\downarrow\uparrow\uparrow})\end{aligned}$$

$$\begin{aligned}= &\frac{1}{\sqrt{3}} ppn(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\ &+ \frac{1}{\sqrt{3}} pnp(\downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow) \\ &+ \frac{1}{\sqrt{3}} npp(\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow\end{aligned}$$

$$\begin{aligned}= \frac{1}{\sqrt{3}}[&p\uparrow p\downarrow n\uparrow - p\downarrow p\uparrow n\uparrow + p\downarrow n\uparrow p\uparrow \\ &- p\uparrow n\uparrow p\downarrow + n\uparrow p\uparrow p\downarrow - n\uparrow p\downarrow p\uparrow]\end{aligned}$$

=

~~SI-4~~

^4He: consists of 4 nucleons

Isospin: $2 \otimes 2 \otimes 2 \otimes 2$: $\underbrace{5}_{I=2} \oplus \underbrace{3 \oplus 3 \oplus 3}_{I=3} \oplus \underbrace{1 \oplus 1}_{I=0}$

But ^4H, ^4Li don't exist so ^4He has $I = 0$

$p\uparrow, p\downarrow, n\uparrow, n\downarrow \Rightarrow {}^4He$ is completely antisym. comb. of these 4 part..
Probably a complicated combination of the $|0,0\rangle, |0,0'\rangle, |0',0\rangle, |0',0'\rangle$
states (where 0 & 0' are the 2 spin 0 combinations) $\quad$ iso $\uparrow$ $\uparrow$ spin

$(5, \square\square\square\square) + (3, \square\square\square/\square)(1, \square\square/\square\square) \leftarrow$ iso
$(5, \square\square\square\square) + (3, \square\square\square/\square)(1, \square\square/\square\square) \leftarrow$ spin

combine to get $\square/\square/\square/\square$ of S_4
$\Rightarrow I : J = 0$

Now consider $A = 6$: $\quad (^4He) + NN$

"inner closed shell"

$NN \to I = 0$ or $I = 1$

Look at nuclide chart $\quad \underbrace{^6He, {}^6Li, {}^6Be}$ $\Rightarrow$

Note $I = 0 \Rightarrow J = 1$
$I = 1 \Rightarrow J = 0$

SC-4

Let $\tilde{P}$ - cyclic perm operator

$\frac{1}{2}\cancel{(2\uparrow\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow)} + \frac{3}{2}\downarrow\uparrow\uparrow\downarrow$

$\frac{1}{2}(-\uparrow\downarrow\uparrow\downarrow - 3\uparrow\downarrow\uparrow\downarrow)$

Neither $|1_A\rangle$ nor $|1_S\rangle$ are eigstts of $\tilde{P}$, but both are eigstts of $\tilde{P}^2$ w eigval 1, so can form eigstts of $\tilde{P}$ w eigval ± 1

Proof related to why spin chain works

Consider $|1\rangle \propto \sqrt{3}\,|1_S\rangle - 3\,|1_A\rangle$

$$= \frac{1}{2}(2\uparrow\uparrow\downarrow\downarrow + 2\downarrow\uparrow\uparrow\downarrow + 2\downarrow\downarrow\uparrow\uparrow + 2\uparrow\downarrow\downarrow\uparrow$$
$$-4\uparrow\downarrow\uparrow\downarrow - 4\downarrow\uparrow\downarrow\uparrow)$$

$$= \uparrow\uparrow\downarrow\downarrow + \downarrow\uparrow\uparrow\downarrow + \downarrow\downarrow\uparrow\uparrow + \uparrow\downarrow\downarrow\uparrow - 2\uparrow\downarrow\uparrow\downarrow - 2\downarrow\uparrow\downarrow\uparrow$$

normalized $|1\rangle = \frac{1}{2}\,|1_S\rangle - \frac{\sqrt{3}}{2}\,|1_A\rangle$

$$= \frac{1}{\sqrt{12}}(\uparrow\uparrow\downarrow\downarrow + \downarrow\uparrow\uparrow\downarrow + \downarrow\downarrow\uparrow\uparrow + \uparrow\downarrow\downarrow\uparrow - 2\uparrow\downarrow\uparrow\downarrow - 2\downarrow\uparrow\downarrow\uparrow)$$

also can explicitly see $S_+|1\rangle = 0$

$$\boxed{\tilde{P}|1\rangle = |1\rangle}$$

orthog. comb

$$|1\rangle' = \frac{\sqrt{3}}{2}|1_S\rangle + \frac{1}{2}|1_A\rangle$$

$$= \frac{1}{4}(2\uparrow\uparrow\downarrow\downarrow - 2\downarrow\uparrow\uparrow\downarrow + 2\downarrow\downarrow\uparrow\uparrow - 2\uparrow\downarrow\downarrow\uparrow)$$
$$\cancel{-\uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\downarrow\uparrow}$$

can also explicitly verify $S_+|1\rangle' = 0$

$$= \frac{1}{2}(\uparrow\uparrow\downarrow\downarrow - \downarrow\uparrow\uparrow\downarrow + \downarrow\downarrow\uparrow\uparrow - \uparrow\downarrow\downarrow\uparrow)$$

$$\boxed{\tilde{P}|1\rangle' = -|1'\rangle}$$