Addition of angular momentum

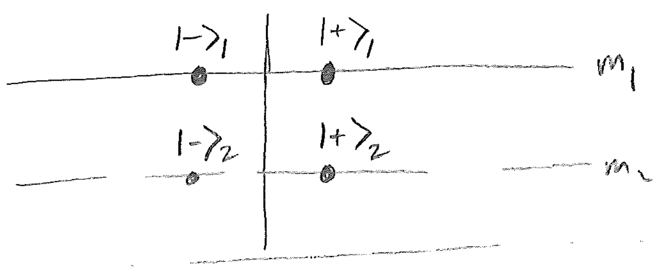
$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

Consider a system of two spin-$\frac{1}{2}$ particles
say p and n (deuteron) or q and $\bar{q}$ (meson)

Each particle is described by a state belonging to the space $\underset{\sim}{2}$

spanned by $|\frac{1}{2}, \frac{1}{2}\rangle = |+\rangle = \uparrow$

and $|\frac{1}{2}, -\frac{1}{2}\rangle = |-\rangle = \downarrow$



$S_{1z} = m_1\hbar$

$S_{2z} = m_2\hbar$

The system is described by a state belonging to
the direct product space $\underset{\sim}{2} \otimes \underset{\sim}{2}$ [Coleman says "direct product"]
spanned by product states

$$|+\rangle \otimes |+\rangle = |+;+\rangle = \uparrow\uparrow$$
$$|+\rangle \otimes |-\rangle = |+;-\rangle = \uparrow\downarrow$$
$$|-\rangle \otimes |+\rangle = |-;+\rangle = \downarrow\uparrow$$
$$|-\rangle \otimes |-\rangle = |-;-\rangle = \downarrow\downarrow$$

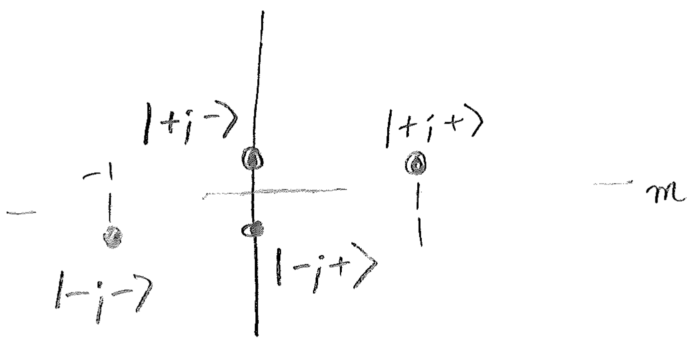
[Note: semicolons]

$\underset{\sim}{2} \otimes \underset{\sim}{2}$ is a 4-dimensional space.

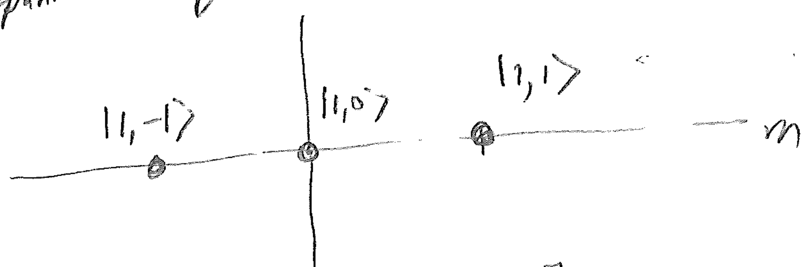
Total spin of the system $S_z = S_{1z} + S_{2z}$

$$= (m_1 + m_2)\hbar$$

$$= m\hbar$$



Since max value of m is 1, some of these states belong to a spin 1 representation, $\underset{\sim}{3}$ spanned by $|j;m\rangle = |1,1\rangle, |1,0\rangle, |1,-1\rangle$



[NB commas vs. semicolons]

[Let's relate these different descriptions.]

triplet
$$\begin{cases} |1,1\rangle = |+;+\rangle \\ |1,-1\rangle = |-;-\rangle \\ |1,0\rangle = \;? \end{cases}$$

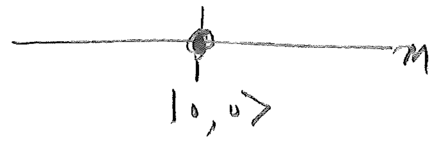
[hard to choose, so split the diff]

$$= \frac{1}{\sqrt{2}}\left[\,|+;-\rangle + |-;+\rangle\,\right]$$

[Proved in 3140. Note $\frac{1}{\sqrt{2}} \Rightarrow \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = 1$.]

$\exists$ one remaining state $\frac{1}{\sqrt{2}}\left[|+;-\rangle - |-;+\rangle\right]$ w/ $m=0$

This state belongs to spin-0 rep, $\underset{\sim}{1}$, spanned by $|0,0\rangle$



singlet $\left\{ |0;0\rangle = \frac{1}{\sqrt{2}}\left[|+;-\rangle - |-;+\rangle\right] \right.$

Thus the full space is a direct sum of spin-1 and spin-0

$\underset{\sim}{3} \oplus \underset{\sim}{1}$, which is 4-dimensional

We write: $\underbrace{\underset{\sim}{2} \otimes \underset{\sim}{2}}_{\text{direct product}} = \underbrace{\underset{\sim}{3} \oplus \underset{\sim}{1}}_{\text{direct sum}}$

$(\text{spin } \tfrac{1}{2}) \otimes (\text{spin } \tfrac{1}{2}) = (\text{spin } 1) \oplus (\text{spin } 0)$

Observe that spin-1 states are symmetric under exchange of the two spins and spin-0 states are antisymmetric

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3}_S \oplus \underset{\sim}{1}_A$$

$|1,0\rangle$ and $|0,0\rangle$ are entangled states.

[z-component of spin of neither ptcl is well-defined but they are correlated.
"I don't know my spin, but whatever it is, it is opposite of yours."
Enemies: if you're for something, I am against it, & vice versa]

Spin-statistics theorem

[G 5.9, p. 175]

(as stated on JI-5)

- Particles w/ half integer spin are fermions (obey Fermi Dirac statistics)

- Particles w/ integer spin are bosons (obey Bose-Einstein statistics)

- The full state (spatial, spin, flavor, color) of two identical fermions / bosons is antisymmetric / symmetric under exchange of particles

$$\psi(\vec{r}_1, m_1, \dots ; r_2, m_2, \dots) = \mp \psi(\vec{r}_2 m_2, \dots ; r_1 m_1, \dots)$$

for fermions / bosons

- The ground state of a bound system of two particles has a symmetric spatial wavefunction

So the rest of the state of two identical fermions / bosons must be antisymmetric / symmetric.

Electrons are spin-½ particles (∴ fermions)

2 electrons in ground state of helium must be in a state antisymmetric under exchange of spins & ∴ must have total spin = 0

$$|0,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

Cannot be

$$|1,1\rangle = \uparrow\uparrow$$
$$|1,-1\rangle = \downarrow\downarrow$$
$$|1,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$

which are symmetric under exchange of spins

First two are ruled out by Pauli principle (no 2 electrons in same state)

$|1,0\rangle$ is more subtle

Naively, say electrons must have opposite spins but more precisely must be $|0,0\rangle$ & not $|1,0\rangle$

If fermions are not identical ⇒ no restriction
eg Bound state of u and $\bar{d}$ in ground state
can have either $S=0$ (π^+ = scalar meson)
or $S=1$ (ρ^+ = vector meson)

General addition of angular momentum

Consider a spin-j_1 particle w/state belonging to the space $\underline{2j_1+1}$, spanned by $|j_1, m_1\rangle$ $m_1 = -j_1, \dots, j_1$

and a spin-j_2 particle w/ state belonging to the space $\underline{2j_2+1}$, spanned by $|j_2, m_2\rangle$ $m_2 = -j_2, \dots, j_2$

The system of two particles is described by a state belonging to the tensor product space $(\underline{2j_1+1}) \otimes (\underline{2j_2+1})$

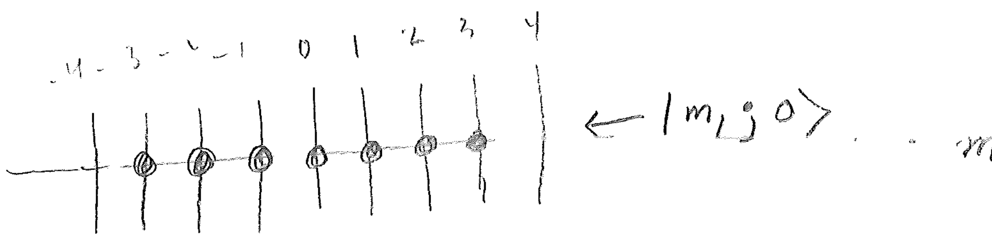
spanned by $(2j_1+1)(2j_2+1)$ product states

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv |m_1 ; m_2\rangle$$

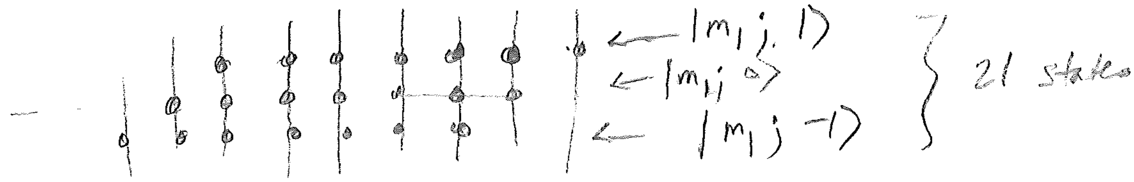
[suppress j_1 & j_2]

As before, the total spin $S_z = S_{1z} + S_{2z} = (\underbrace{m_1 + m_2}_{m})\hbar$

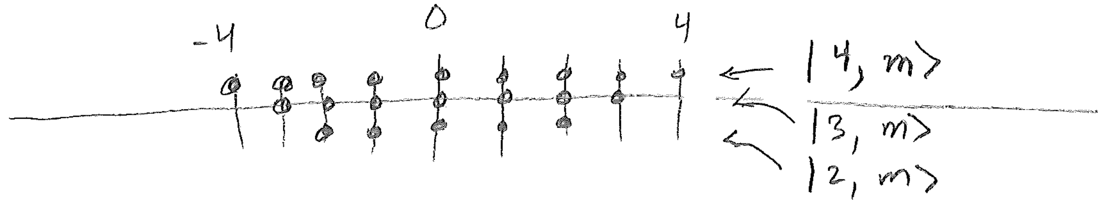
Plot these states on a weight diagram for spin 3 and spin 1: $\underset{\sim}{7} \otimes \underset{\sim}{3}$ $m_1 = -3, -2, \dots, 3$; $m_2 = -1, 0, 1$



Then



Max value of m is 4 so spin-4 rep, $\underset{\sim}{9}$



Also spin-3 rep, $\underset{\sim}{7}$ and spin-2 rep, $\underset{\sim}{5}$

$$\underset{\sim}{7} \otimes \underset{\sim}{3} = \underset{\sim}{9} \oplus \underset{\sim}{7} \oplus \underset{\sim}{5} \quad \Rightarrow 21 \text{ states}$$

$$(\text{spin } 3) \otimes (\text{spin } 1) = (\text{spin } 4) \oplus (\text{spin } 3) \oplus (\text{spin } 2)$$

In general,

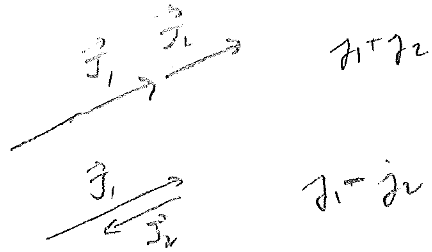
$$(\text{spin } j_1) \otimes (\text{spin } j_2) = \text{spin}(j_1+j_2) \oplus \text{spin}(j_1+j_2-1) \dots \text{spin}|j_1-j_2|$$

(because max value of $m = j_1 + j_2$)

Decomposition of tensor product space into direct sum of spaces

$$\bigoplus_{j=|j_1-j_2|}^{j=j_1+j_2} (\text{spin } j)$$

Picture



$$(2j_1+1) \otimes (2j_2+1) = (\underset{\sim}{2j_1+2j_2+1}) \oplus (\underset{\sim}{2j_1+2j_2-1}) \oplus \dots \oplus (\underset{\sim}{2|j_1-j_2|+1})$$

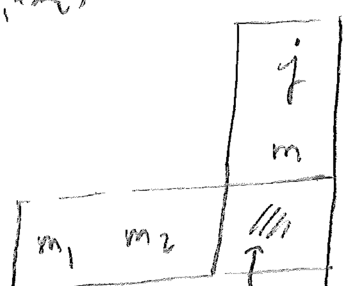
The states $|j, m\rangle$ are linear combinations of the tensor-product states $|m_1; m_2\rangle$, and vice versa.

The coefficients of the linear combinations are called Clebsch-Gordan coefficients $C^{j_1 j_2 j}_{m_1 m_2 m}$

$$|j, m\rangle = \sum_{\substack{m_1, m_2 \\ (m = m_1 + m_2)}} C^{j_1 j_2 j}_{m_1 m_2 m} \, |m_1; m_2\rangle$$

$$|m_1; m_2\rangle = \sum_{\substack{j, m \\ (m = m_1 + m_2)}} C^{j_1 j_2 j}_{m_1 m_2 m} \, |j, m\rangle$$

cf. PPB table



take $\sqrt{\ }$ to get C
(± outside)

1/2×1/2

	1			
	+1	1	0	
+1/2 +1/2	1	0	0	
+1/2 −1/2		1/2	1/2	1
−1/2 +1/2		1/2	−1/2	−1
−1/2 −1/2				1

1×1/2

	3/2					
	+3/2	3/2	1/2			
+1 +1/2	1	+1/2	+1/2			
+1 −1/2		1/3	2/3	3/2	1/2	
0 +1/2		2/3	−1/3	−1/2	−1/2	
0 −1/2				2/3	1/3	3/2
−1 +1/2				1/3	−2/3	−3/2
−1 −1/2						1

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

1/2×1/2

m_1	m_2	1
		+1
+1/2	+1/2	1

m_1	m_2	1	0
		0	0
+1/2	-1/2	1/2	1/2
-1/2	+1/2	1/2	-1/2

m_1	m_2	1
		-1
-1/2	-1/2	1

1×1/2

m_1	m_2	3/2
		+3/2
+1	+1/2	1

m_1	m_2	3/2	1/2
		+1/2	+1/2
+1	-1/2	1/3	2/3
0	+1/2	2/3	-1/3

m_1	m_2	3/2	1/2
		-1/2	-1/2
0	-1/2	2/3	1/3
-1	+1/2	1/3	-2/3

m_1	m_2	3/2
		-3/2
-1	-1/2	1

2×1

m_1	m_2	3
		+3
+2	+1	1

m_1	m_2	3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	-1/3

m_1	m_2	3	2	1
		+1	+1	+1
+2	-1	1/15	1/3	3/5
+1	0	8/15	1/6	-3/10
0	+1	2/5	-1/2	1/10

m_1	m_2	3	2	1
		0	0	0
+1	-1	1/5	1/2	3/10
0	0	3/5	0	-2/5
-1	+1	1/5	-1/2	3/10

m_1	m_2	3	2	1
		-1	-1	-1
0	-1	2/5	1/2	1/10
-1	0	8/15	-1/6	-3/10
-2	+1	1/15	-1/3	3/5

m_1	m_2	3	2
		-2	-2
-1	-1	2/3	1/3
-2	0	1/3	-2/3

m_1	m_2	3
		-3
-2	-1	1

1×1

m_1	m_2	2
		+2
+1	+1	1

m_1	m_2	2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	-1/2

m_1	m_2	2	1	0
		0	0	0
+1	-1	1/6	1/2	1/3
0	0	2/3	0	-1/3
-1	+1	1/6	-1/2	1/3

m_1	m_2	2	1
		-1	-1
0	-1	1/2	1/2
-1	0	1/2	-1/2

m_1	m_2	2
		-2
-1	-1	1

2×1/2

m_1	m_2	5/2
		+5/2
+2	+1/2	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+2	-1/2	1/5	4/5
+1	+1/2	4/5	-1/5

m_1	m_2	5/2	3/2
		+1/2	+1/2
+1	-1/2	2/5	3/5
0	+1/2	3/5	-2/5

m_1	m_2	5/2	3/2
		-1/2	-1/2
0	-1/2	3/5	2/5
-1	+1/2	2/5	-3/5

m_1	m_2	5/2	3/2
		-3/2	-3/2
-1	-1/2	4/5	1/5
-2	+1/2	1/5	-4/5

m_1	m_2	5/2
		-5/2
-2	-1/2	1

3/2×1/2

m_1	m_2	2
		+2
+3/2	+1/2	1

m_1	m_2	2	1
		+1	+1
+3/2	-1/2	1/4	3/4
+1/2	+1/2	3/4	-1/4

m_1	m_2	2	1
		0	0
+1/2	-1/2	1/2	1/2
-1/2	+1/2	1/2	-1/2

m_1	m_2	2	1
		-1	-1
-1/2	-1/2	3/4	1/4
-3/2	+1/2	1/4	-3/4

m_1	m_2	2
		-2
-3/2	-1/2	1

3/2×1

m_1	m_2	5/2
		+5/2
+3/2	+1	1

m_1	m_2	5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	-2/5

m_1	m_2	5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	-1	1/10	2/5	1/2
+1/2	0	3/5	1/15	-1/3
-1/2	+1	3/10	-8/15	1/6

m_1	m_2	5/2	3/2	1/2
		-1/2	-1/2	-1/2
+1/2	-1	3/10	8/15	1/6
-1/2	0	3/5	-1/15	-1/3
-3/2	+1	1/10	-2/5	1/2

m_1	m_2	5/2	3/2
		-3/2	-3/2
-1/2	-1	3/5	2/5
-3/2	0	2/5	-3/5

m_1	m_2	5/2
		-5/2
-3/2	-1	1

3/2×3/2

m_1	m_2	3
		+3
+3/2	+3/2	1

m_1	m_2	3	2
		+2	+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	-1/2

m_1	m_2	3	2	1
		+1	+1	+1
+3/2	-1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	-2/5
-1/2	+3/2	1/5	-1/2	3/10

m_1	m_2	3	2	1	0
		0	0	0	0
+3/2	-3/2	1/20	1/4	9/20	1/4
+1/2	-1/2	9/20	1/4	-1/20	-1/4
-1/2	+1/2	9/20	-1/4	-1/20	1/4
-3/2	+3/2	1/20	-1/4	9/20	-1/4

m_1	m_2	3	2	1
		-1	-1	-1
+1/2	-3/2	1/5	1/2	3/10
-1/2	-1/2	3/5	0	-2/5
-3/2	+1/2	1/5	-1/2	3/10

m_1	m_2	3	2
		-2	-2
-1/2	-3/2	1/2	1/2
-3/2	-1/2	1/2	-1/2

m_1	m_2	3
		-3
-3/2	-3/2	1

2×3/2

m_1	m_2	7/2
		+7/2
+2	+3/2	1

m_1	m_2	7/2	5/2
		+5/2	+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	-3/7

m_1	m_2	7/2	5/2	3/2
		+3/2	+3/2	+3/2
+2	-1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	-2/5
0	+3/2	2/7	-18/35	1/5

m_1	m_2	7/2	5/2	3/2	1/2
		+1/2	+1/2	+1/2	+1/2
+2	-3/2	1/35	6/35	2/5	2/5
+1	-1/2	12/35	5/14	0	-3/10
0	+1/2	18/35	-3/35	-1/5	1/5
-1	+3/2	4/35	-27/70	2/5	-1/10

m_1	m_2	7/2	5/2	3/2	1/2
		-1/2	-1/2	-1/2	-1/2
+1	-3/2	4/35	27/70	2/5	1/10
0	-1/2	18/35	3/35	-1/5	-1/5
-1	+1/2	12/35	-5/14	0	3/10
-2	+3/2	1/35	-6/35	2/5	-2/5

m_1	m_2	7/2	5/2	3/2
		-3/2	-3/2	-3/2
0	-3/2	2/7	18/35	1/5
-1	-1/2	4/7	-1/35	-2/5
-2	+1/2	1/7	-16/35	2/5

m_1	m_2	7/2	5/2
		-5/2	-5/2
-1	-3/2	4/7	3/7
-2	-1/2	3/7	-4/7

m_1	m_2	7/2
		-7/2
-2	-3/2	1

2×2

m_1	m_2	4
		+4
+2	+2	1

m_1	m_2	4	3
		+3	+3
+2	+1	1/2	1/2
+1	+2	1/2	-1/2

m_1	m_2	4	3	2
		+2	+2	+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	-3/7
0	+2	3/14	-1/2	2/7

m_1	m_2	4	3	2	1
		+1	+1	+1	+1
+2	-1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	-1/14	-3/10
0	+1	3/7	-1/5	-1/14	3/10
-1	+2	1/14	-3/10	3/7	-1/5

m_1	m_2	4	3	2	1	0
		0	0	0	0	0
+2	-2	1/70	1/10	2/7	2/5	1/5
+1	-1	8/35	2/5	1/14	-1/10	-1/5
0	0	18/35	0	-2/7	0	1/5
-1	+1	8/35	-2/5	1/14	1/10	-1/5
-2	+2	1/70	-1/10	2/7	-2/5	1/5

m_1	m_2	4	3	2	1
		-1	-1	-1	-1
+1	-2	1/14	3/10	3/7	1/5
0	-1	3/7	1/5	-1/14	-3/10
-1	0	3/7	-1/5	-1/14	3/10
-2	+1	1/14	-3/10	3/7	-1/5

m_1	m_2	4	3	2
		-2	-2	-2
0	-2	3/14	1/2	2/7
-1	-1	4/7	0	-3/7
-2	0	3/14	-1/2	2/7

m_1	m_2	4	3
		-3	-3
-1	-2	1/2	1/2
-2	-1	1/2	-1/2

m_1	m_2	4
		-4
-2	-2	1

d functions

$$d^{1}_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

(spin $\frac{1}{2}$) $\otimes$ (spin $\frac{1}{2}$) = (spin 1) $\oplus$ (spin 0)

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3}_S \oplus \underset{\sim}{1}_A$$

$$|1,1\rangle = |\tfrac{1}{2};\tfrac{1}{2}\rangle = \uparrow\uparrow$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2};-\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|-\tfrac{1}{2};\tfrac{1}{2}\rangle$$
$$= \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$$

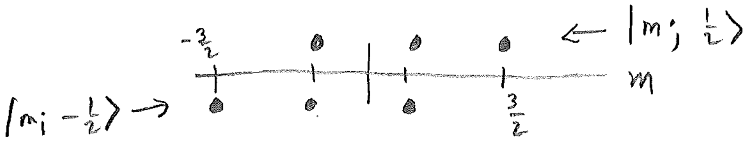
$$|1,-1\rangle = |-\tfrac{1}{2};-\tfrac{1}{2}\rangle = \downarrow\downarrow$$

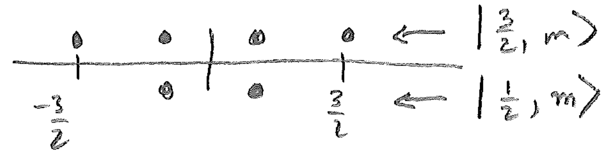
$$|0,0\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2};-\tfrac{1}{2}\rangle - \frac{1}{\sqrt{2}}|-\tfrac{1}{2};\tfrac{1}{2}\rangle$$

$$= \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

$$(\text{spin } 1) \otimes (\text{spin } \tfrac{1}{2}) = (\text{spin } \tfrac{3}{2}) \oplus (\text{spin } \tfrac{1}{2})$$

$$\underset{\sim}{3} \otimes \underset{\sim}{2} = \underset{\sim}{4} \oplus \underset{\sim}{2}$$





[From table]

$$|\tfrac{3}{2}, \tfrac{3}{2}\rangle = |1; \tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, |1; -\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\, |0; \tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, |0; -\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\, |-1; \tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2}, -\tfrac{3}{2}\rangle = |-1; -\tfrac{1}{2}\rangle$$

[NB: $|\alpha|^2 + |\beta|^2 = 1$]
[in 3140, students derive these]

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, |1; -\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\, |0; \tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, |0; -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\, |-1; \tfrac{1}{2}\rangle$$

Spin state of three spin-½ particles (e.g. 3 quarks, 3 nucleons)

$$\underset{\sim}{2} \otimes \underset{\sim}{2} \otimes \underset{\sim}{2} = (\underset{\sim}{3}_S \oplus \underset{\sim}{1}_A) \otimes \underset{\sim}{2}$$

refers to symmetry under exchange of first two particles

$$= \underbrace{(\underset{\sim}{3}_S \otimes \underset{\sim}{2})}_{\underset{\sim}{4}_S \,\oplus\, \underset{\sim}{2}_S} \oplus \underbrace{(\underset{\sim}{1}_A \otimes \underset{\sim}{2})}_{\underset{\sim}{2}_A}$$

$$\underset{\sim}{2} \otimes \underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\text{spin-}\frac{3}{2}}{\underset{\sim}{4}_S} \oplus \underset{\text{spin-}\frac{1}{2}}{\underset{\sim}{2}_S} \oplus \underset{\text{spin-}\frac{1}{2}}{\underset{\sim}{2}_A} \qquad (8 = 8)$$

(can skip to AA 15 here)

A composite particle of 3 spin-½ particle may have total $j = \frac{3}{2}$ or $\frac{1}{2}$

e.g. 3 quarks: Δ baryon has $j = \frac{3}{2}$
N baryon has $j = \frac{1}{2}$

e.g. 3 nucleons: $^3He = ppn$
tritium $^3H = nnp$ } have $j = \frac{1}{2}$

In fact, ^3He + ^3H necessarily have spin $j = \frac{1}{2}$ because contain 2 identical fermions, the spin state of which must be antisymmetric

$\Rightarrow$ state must be $\underset{\sim}{2}_A$

Bound state of nnn not possible (w/ symmetric spatial wavefunction) because can't have a totally antisymmetric spin state

Spin-$\frac{3}{2}$ states ($\underset{\sim}{4}_{TS}$)

$$|\tfrac{3}{2},\tfrac{3}{2}\rangle = |1;\tfrac{1}{2}\rangle = (\uparrow\uparrow)\uparrow = \uparrow\uparrow\uparrow$$

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|1;-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|0;\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{1}{3}}\,(\uparrow\uparrow)\downarrow + \sqrt{\tfrac{2}{3}}\sqrt{\tfrac{1}{2}}(\uparrow\downarrow+\downarrow\uparrow)\uparrow$$

$$= \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$

$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \ldots = \tfrac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow)$$

$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = \downarrow\downarrow\downarrow$$

totally symmetric (+1) under exchange of any two particles

Spin-$\frac{1}{2}$ states ($\underset{\sim}{2}_S$)

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle_S = \sqrt{\tfrac{2}{3}}\,|1;-\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|0;\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{2}{3}}\,(\uparrow\uparrow)\downarrow - \sqrt{\tfrac{1}{3}}\sqrt{\tfrac{1}{2}}(\uparrow\downarrow+\downarrow\uparrow)\uparrow = \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle_S = \sqrt{\tfrac{1}{3}}\,|0;-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|-1;\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{1}{3}}\sqrt{\tfrac{1}{2}}(\uparrow\downarrow+\downarrow\uparrow)\downarrow - \sqrt{\tfrac{2}{3}}(\downarrow\downarrow)\uparrow = \tfrac{1}{\sqrt{6}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - 2\downarrow\downarrow\uparrow)$$

check: sym'c in exchange of first two

Spin-$\frac{1}{2}$ state ($\underset{\sim}{2}_A$)

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle_A = |0;\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)\uparrow = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle_A = |0;-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)\downarrow = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$

check anti symc in exchg of first two

[see: $D \times D \times D = (\overset{3/2}{\square\square\square}, \overset{sym}{\square\square\square}) + (\overset{1/2}{\boxminus}, \overset{dim=2}{\boxplus})$]

Magnetic moments of composite particles

$\vec{\mu}$ = vector sum of moments of constituents

Recall $\mu_p = 2.793\,\mu_N$ where μ_N = nuclear magneton $\equiv \frac{e\hbar}{2m_p}$

$\mu_n = -1.913\,\mu_N$

Deuteron d pn has $J=1 \Rightarrow J_z = 1$ state $\uparrow\uparrow$

only if already discussed isospin → [~~Consider~~ $J_z=1$ state $\frac{1}{\sqrt{2}}(p\uparrow n\uparrow \cancel{- n\uparrow p\uparrow})$]

Spins parallel so $\mu_d = \mu_p + \mu_n = 0.88\,\mu_N$

(expt: $0.857\,\mu_N$)

$^3H = pnn$ has $J = \frac{1}{2}$

Consider $J_z = \frac{1}{2}$ state $\frac{1}{\sqrt{2}}\, p\uparrow(n\uparrow n\downarrow - n\downarrow n\uparrow)$

neutron spins antiparallel $\Rightarrow$ moments cancel

$\mu(^3H) = \mu_p = 2.79\,\mu_N$

(expt. $2.98\,\mu_N$)

$^3He = ppn$ has $J = \frac{1}{2}$

Now proton spins cancel $\Rightarrow \mu(^3He) = \mu_n = -1.913\,\mu_N$

(expt $= -2.13\,\mu_N$)

(Assign as problem) (copy of this in solns)

$2\otimes 2\otimes 2\otimes 2: \quad (4\oplus 2_S\oplus 2_A)\otimes 2$

$$= (4\otimes 2) + (2_S\otimes 2) + (2_A\otimes 2)$$

$$= 5\oplus 3\oplus 3_S\oplus 1_S\oplus 3_A\oplus 1_A$$

5 $\quad |2,0\rangle = \frac{1}{\sqrt{2}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{2}}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$

$$= \frac{1}{\sqrt{6}}(\uparrow\uparrow\downarrow\downarrow + \uparrow\downarrow\uparrow\downarrow + \downarrow\uparrow\uparrow\downarrow + \uparrow\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow\uparrow + \downarrow\downarrow\uparrow\uparrow)$$

3 $\quad |1,0\rangle = \frac{1}{\sqrt{2}}\left|\tfrac{3}{2},\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}\left|\tfrac{3}{2},-\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle$

$$= \frac{1}{\sqrt{6}}(\uparrow\uparrow\downarrow\downarrow + \uparrow\downarrow\uparrow\downarrow + \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow\uparrow - \downarrow\downarrow\uparrow\uparrow)$$

3_S $\quad |1,0\rangle_S = \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_S\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_S\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right)$

$$= \frac{1}{2\sqrt{3}}(2\uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow + \uparrow\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow\uparrow - 2\downarrow\downarrow\uparrow\uparrow)$$

1_S $\quad |0,0\rangle_S = \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_S\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_S\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right)$

$$= \frac{1}{2\sqrt{3}}(2\uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow\uparrow + 2\downarrow\downarrow\uparrow\uparrow)$$

3_A $\quad |1,0\rangle_A = \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_A\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle + \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_A\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right)$

$$= \frac{1}{2}(\uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow + \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow\uparrow)$$

1_A: $\quad |0,0\rangle_A = \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle_A\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle - \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_A\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle\right)$

$$= \frac{1}{2}(\uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow\uparrow)$$

$$= \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\,\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \longleftarrow {}^4He$$

NB: all are normalized and orthonormal

(Previous version of the problem)

4 spin ½ particles

$$\underset{5}{\overset{SU(2)}{(\square\square\square\square}},\ \underset{1}{\overset{S_4}{\square\square\square\square}}) \oplus (\underset{3}{\boxplus\square}, \underset{3}{\boxplus\square}) \oplus (\underset{1}{\boxplus}, \underset{2}{\boxplus})$$

$$2\otimes 2\otimes 2\otimes 2 = (4\oplus 2_S \oplus 2_A)\otimes 2$$

$$= (4\otimes 2) \oplus (2_S\otimes 2) \oplus (2_A \otimes 2)$$

$$= 5 \oplus 3 \oplus 3_S \oplus 1_S \oplus 3_A \oplus 1_A$$

5: $|2,2\rangle = |\tfrac{3}{2},\tfrac{3}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = (\uparrow\uparrow\uparrow)\uparrow = \uparrow\uparrow\uparrow\uparrow$

3: $|1,1\rangle = \frac{\sqrt{3}}{2}|\tfrac{3}{2}\tfrac{3}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle - \frac{1}{2}|\tfrac{3}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle$

$$= \frac{\sqrt{3}}{2}\uparrow\uparrow\uparrow\downarrow - \frac{1}{2\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)\uparrow$$

$$= \frac{1}{2\sqrt{3}}(3\uparrow\uparrow\uparrow\downarrow - \uparrow\uparrow\downarrow\uparrow - \uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow)$$

3_S: $|1,1\rangle_S = |\tfrac{1}{2},\tfrac{1}{2}\rangle_S|\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow\uparrow - \uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow)$

1_S: $|0,0\rangle_S = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle_S|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |\tfrac{1}{2},-\tfrac{1}{2}\rangle_S|\tfrac{1}{2}\tfrac{1}{2}\rangle\right)$

$$= \frac{1}{2\sqrt{3}}(2\uparrow\uparrow\downarrow\downarrow - \uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\downarrow\uparrow + 2\downarrow\downarrow\uparrow\uparrow)$$

$\binom{4}{2} = 6$

3_A: $|1,1\rangle_A = |\tfrac{1}{2}\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow)$

1_A: $|0,0\rangle_A = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle_A|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |\tfrac{1}{2},-\tfrac{1}{2}\rangle_A|\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$

$$= \frac{1}{2}(\uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\uparrow\downarrow - \uparrow\downarrow\downarrow\uparrow + \downarrow\uparrow\downarrow\uparrow)$$

$$= \underset{pp}{\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)}\ \underset{nn}{\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)} \leftarrow \text{Problem 4.}$$