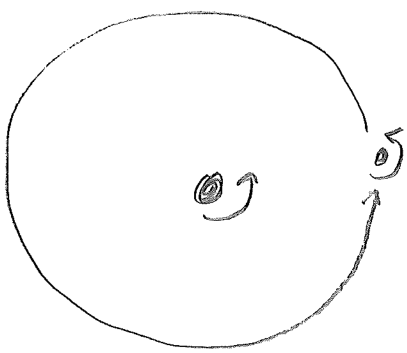
## Spin

Invariance of laws of physics under rotations (isotropy)

$\xrightarrow[\text{Noether}]{}$ conservation of total angular momentum

### Classical analogs

Earth-moon system



$\vec{L}$ = orbital ang. mom.

$\vec{S}$ = rotational ang mom. (spin)

$\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$

[Q: what are directions of $\vec{L}$ & $\vec{S}$?]

$\vec{J}$ conserved but $\vec{L}, \vec{S}_1, \vec{S}_2$ can change

[lengthening of earth's day]

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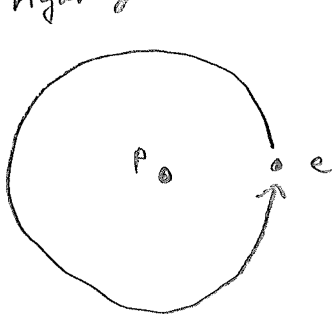
hydrogen atom (Bohr model)



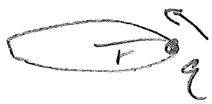
$\vec{L}$: orbital

$\vec{S}$: intrinsic angular momentum (spin)

$\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$

What does spin mean for a point particle?

An orbiting charge acts as a current loop
and therefore has a magnetic moment



$$\mu = IA$$

problem? or do it in class

$$\left[ A = \pi r^2, \quad I = \frac{q}{T}, \quad T = \frac{2\pi r}{v} \Rightarrow \mu = \frac{qvr}{2} \right.$$

$$\left. \text{but } L = Mvr \text{ so } \quad \mu = \frac{q}{2M} L \right]$$

$$\vec{\mu}_L = \underbrace{\left(\frac{q}{2M}\right)}_{\text{gyromagnetic ratio}} \vec{L}$$

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Experimentally, a stationary particle can also have a magnetic moment, which we attribute to the particle's spin

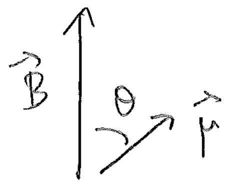
$$\vec{\mu}_S = g \left(\frac{q}{2m}\right) \vec{S}$$

$g$ "Landé factor" (dimensionless #)

The total magnetic moment is $\vec{\mu} = \vec{\mu}_L + \vec{\mu}_S$

Magnetic moment is measured by interaction w/ external $\vec{B}$ field

$$E = -\vec{B}\cdot\vec{\mu}$$




lowest energy when magnet lines up w/ field
highest energy when magnet points opposite

Since $\vec{\mu} \propto \vec{J}$ (where $\vec{J} = \vec{L}$ or $\vec{S}$)..

$$E \propto \vec{B}\cdot\vec{J} = BJ\cos\theta$$

External magnetic field breaks rotational invariance
by picking a preferred direction

w/o field, energy of system is independent of
direction of spin, but w/ field energy depends on $\theta$

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For a particle w/ spin $\quad E = -\vec{B}\cdot\left(\frac{gq}{2M}\vec{S}\right)$

$$= -\frac{gqB}{2M}S_z \quad (\text{if } \vec{B} = B\hat{z})$$

By measuring energy levels in a B-field
one can measure $g$

[actually measure precession: Penning trap] spin

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Rotational invariance is broken by an external $\vec{B}$ field
[because $\vec{B}$ field picks out a preferred direction]

$\Rightarrow$ angular momentum is not conserved
(except component in direction of $\vec{B}$ field)

classical treatment

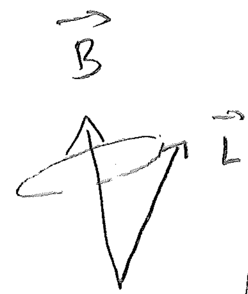
The change in angular momentum is caused by the torque that the $\vec{B}$ field exerts on the moving charge

$$\frac{d\vec{J}}{dt} = \vec{\tau} = \vec{\mu} \times \vec{B}, \qquad \vec{\mu} = \text{magnetic dipole moment}$$

For an orbiting point charge $q$ w/ mass $M$ (and no spin)

$$\vec{\mu} = \underbrace{\left(\frac{q}{2M}\right)}\vec{L}$$

called the gyromagnetic ratio




$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \frac{q}{2M}\,\vec{L} \times \vec{B} \quad \text{(gyroscope eqn)}$$

$$dA = \tfrac{1}{2} r^2 \, d\theta$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \omega = \frac{L}{2m}$$

$$\mu = IA = \frac{q}{T} A$$

$$= q \frac{dA}{dt} = \frac{q}{2m} L$$

$\vec{L}$ precesses around $\vec{B}$

If $\vec{B} = B\hat{z}$, then $L_z$ is still conserved
but $L_x$ + $L_y$ change over time

We can understand that $L_z$ is conserved because $\vec{B}$ field is invariant under rotations about $\hat{z}$

J-4

Dirac eqn predicts that a

<u>fundamental</u> spin-$\frac{1}{2}$ particle (quark, lepton) has $g = 2$.

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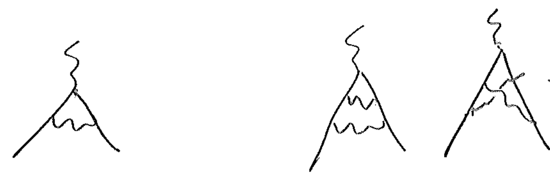
QED predicts small corrections for the electron

$$g_{th} = 2\Big(1 + \underbrace{\frac{\alpha}{2\pi}}_{0.0011614} - \underbrace{0.328\left(\frac{\alpha}{\pi}\right)^2}_{0.000\,0018} + \dots\Big)$$

$\alpha^2$: Sommerfeld 1957 Petermann

$\alpha^3$ Laporta Remiddi 1996

$\alpha^4$: Kinoshita 2014



etc.

$= 2(1.001\ 159\ 652\ 181\ 6\ (7))$ Kinoshita 2014 (talk at Cape Cod)

1948 P. Kusch Zeeman splitting in In, Ga, Na

$g_{exp} = 2(1.001\ 159\ 652\ 180\ 8\ (3))$ ← PPB 2022 "our crown jewel"

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Muon: ← PRD 102 (2020) 033 002 Mercuro

$g_{th} = 2(1.001\ 165\ 918\ 1\ (4))$

$g_{exp} = 2(1.001\ 165\ 920\ 9\ (6))$ ← PPB 2022

$g_{exp} - g_{th}$ $2(0.000\ 000\ 002\ 8\ (8))$ $3.7\sigma$ discrepancy

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proton & neutron are not fundamental particles

proton: $g_{exp} = 2(2.793\dots)$

neutron $g_{exp} = 2(-1.913\dots)$

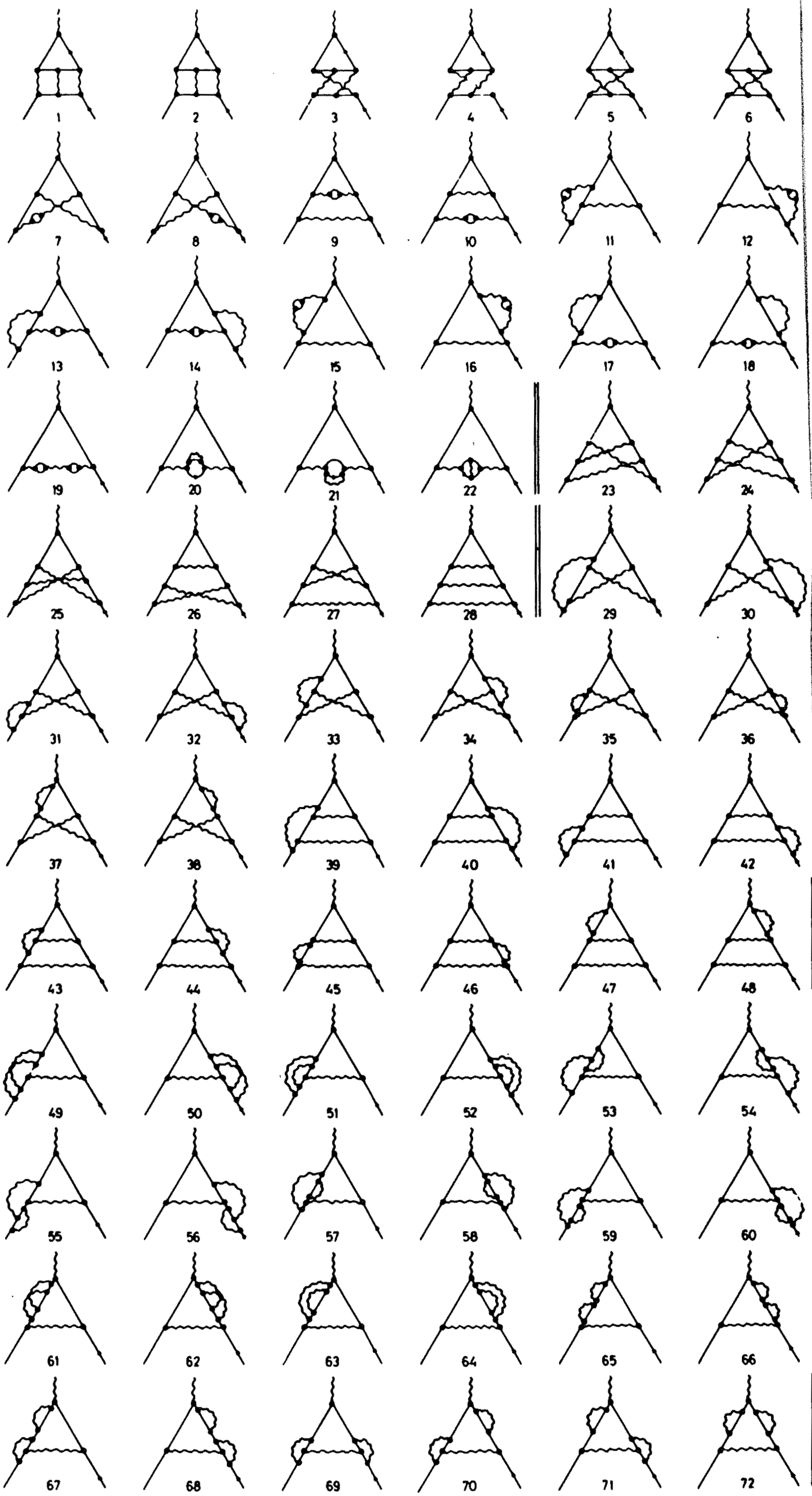


**Fig. 8.2** The Feynman graphs which have to be evaluated in computing the $\alpha^3$ corrections the lepton magnetic moments (after Lautrup *et al.* 1972).

QM: orbital angular momentum is quantized in integer units of $\hbar$ (Bohr)

$$|\vec{L}| = l\hbar \qquad l = 0, 1, 2, \ldots$$

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spin angular moment is quantized in half-integer units of $\hbar$

$$|\vec{S}| = s\hbar \qquad s = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

↑ called the "spin" of the particle

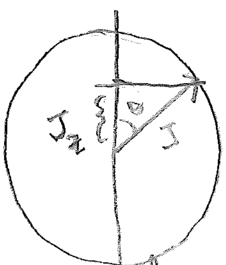
Each type of particle has an intrinsic value of spin.
Particles of integer spin are bosons.
Particles of half-integer spin are fermions.

| spin | type | example: fundamental | example: composite |
|---|---|---|---|
| 0 | scalar | Higgs | scalar meson $\pi$ |
| $\frac{1}{2}$ | spinor | quark, lepton | proton, neutron |
| 1 | vector | $\gamma, W^{\pm}, Z^{0}, g$ | vector meson $\rho$ |
| $\frac{3}{2}$ | spinor-vector | gravitino? | $\Delta$ baryons |
| 2 | tensor | graviton | |

In general, we'll refer to angular momentum $\vec{J}$
[could be $\vec{L}$ or $\vec{S}$ or total $\vec{L}+\vec{S}$]

$$|\vec{J}| = j\hbar$$

Each component of angular momentum is also quantized




spatial quantization

$$J_z = m\hbar \qquad m = -j, -j+1, \ldots, j-1, j$$

$m$ = integer (half-integer) if $j$ = integer (half int)

# of possible values of $J_z$ is called the <u>multiplicity</u> : $2j+1$

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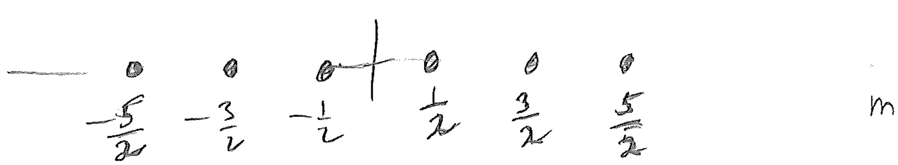
Actually this is true only for particles w/ mass.
Massless particles can only have $m = j$ or $-j$.
ie multiplicity = 2 $\Rightarrow$ photon & graviton polarization

Represent possible values of $m$ on a weight diagram

$j = \frac{5}{2}$



Recall: in a rotationally-invariant environment energy cannot depend on direction of $\vec{J}$ (all directions are equivalent) so the $2j+1$ spin states are <u>degenerate</u>

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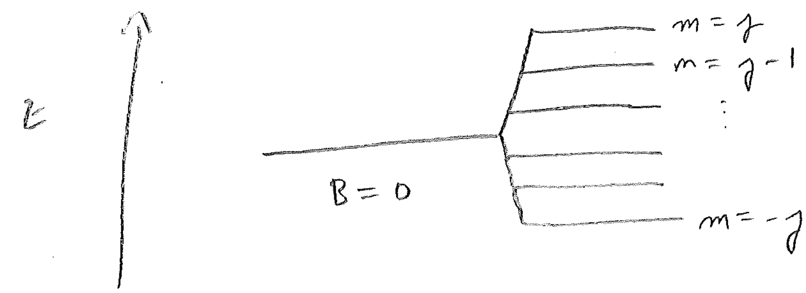
With a $\vec{B}$ field present,

$$E = -\vec{B}\cdot\vec{\mu} \propto \vec{B}\cdot\vec{J} = B J_z \qquad (\text{if } \vec{B} = B\hat{z})$$

const depends on whether $\vec{J} = \vec{L}$ or $\vec{S}$

$$= B\hbar m$$




Zeeman splitting
(if $\vec{J} = \vec{L}$ ⇒ ordinary Zeeman
if $\vec{J}$ includes $\vec{S}$ ⇒ anomalous Zeeman)

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(rotational) symmetry ⇒ (quantum) degeneracy of spin states

broken symmetry ⇒ degeneracy is split

skip

## magnetic moment of Spin-$\frac{1}{2}$ particles

$$\vec{\mu} = \frac{gq}{2M}\vec{S}$$

$$|\vec{\mu}| = \frac{gq}{2M}\frac{\hbar}{2}$$

Electron has $g \approx 2 \Rightarrow \mu_e \approx \frac{e\hbar}{2m_e} \equiv \mu_B$ (Bohr magneton)

(Gaussian $\mu_B = \frac{e\hbar}{2m_e c}$)

Proton has $g = 2(2.793) \Rightarrow \mu_p = 2.793\,\mu_N$

where $\mu_N \equiv \frac{e\hbar}{2m_p}$ (nuclear magneton)

Neutron has $g = 2(-1.913) \Rightarrow \mu_n = -1.913\,\mu_N$

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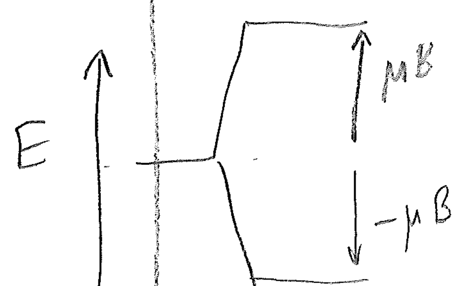



$\mu_B = 5.8 \times 10^{-5}\,\frac{eV}{Tesla}$

20T field $\Rightarrow 0.001\,eV$

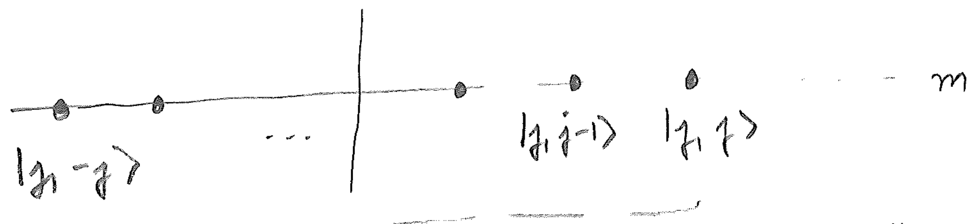
$\mu_N \approx \frac{\mu_B}{1840}$

In QM, describe the <u>state</u> of a system by a wave function $\psi(x,t)$ or a ket: $|\psi\rangle$ which belongs to a vector space (Hilbert space)

Denote the state of $J = j\hbar$ and $J_z = m\hbar$ by

$$|j, m\rangle$$

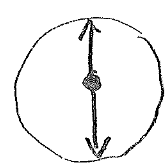
The complete set of states is




Call this set of states "spin-$j$ multiplet"

Denote it by $(\underset{\sim}{2j+1})$ where $2j+1$ = multiplicity of the multiplet

An important example will be the spin-$\frac{1}{2}$ multiplet $\underset{\sim}{2}$ (or doublet)

call this $|-\rangle$ or $\downarrow$ (spin down)

call this $|+\rangle$ or $\uparrow$ (spin up)

Quantum mechanical states live in a vector space,
so arbitrary linear combinations of states are also states

For spin-$\frac{1}{2}$ particle, states are $|+\rangle$ and $|-\rangle$ (spin up & down)

Most general state is

$|\psi\rangle = \alpha |+\rangle + \beta |-\rangle$ where $\alpha, \beta$ are complex constants

$\alpha, \beta$ = probability amplitudes

$$\left.\begin{array}{l}\text{Probability of measuring spin up is } |\alpha|^2 \\ \text{and spin down is } |\beta|^2\end{array}\right\} \quad |\alpha|^2 + |\beta|^2 = 1$$

We say: $|\psi\rangle$ lives in a vector space $\underset{\sim}{2}$
spanned by basis states $|+\rangle$ and $|-\rangle$

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For spin-$j$ particle; states are $|j, m\rangle$, $m = -j, \ldots, j$

Most general state is

$$|\psi\rangle = \sum_{m=-j}^{j} c_m |j, m\rangle$$

Probability of measuring $S_z = m\hbar$ is $|c_m|^2$

$$\sum_{m=-j}^{j} |c_m|^2 = 1$$

We say $|\psi\rangle$ lives in a space $(\underset{\sim}{2j+1})$, spanned by $|j, m\rangle$
$m = -j, \ldots, j$