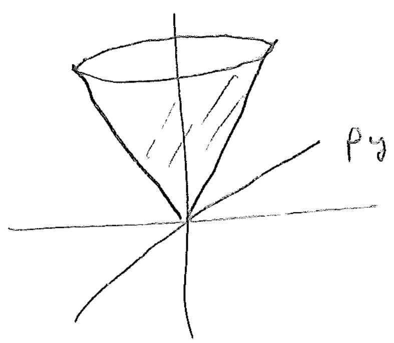
Massless particles

$$p^2 = 0$$

ie $E^2 - |\vec{p}|^2 = 0$

$$E = |\vec{p}| = \sqrt{p_x^2 + p_y^2 + p_z^2}$$



mass-shell : cone

Since $\frac{\vec{p}}{E} = \vec{v}$ massless particle obey $|\vec{v}| = 1$

ie must travel at the speed of light
(photons, gravitons)

For massless particle $T = E$ (all energy is kinetic)

($E = \gamma m c^2$, $\vec{p} = \gamma m \vec{v}$ are useless because $m \to 0$, $\gamma \to \infty$)

Photon energy & momentum depend, not on speed, but on frequency & wavelength (wavenumber)

$$E = hf = \hbar\omega$$
$$|\vec{p}| = \frac{h}{\lambda} = \hbar k$$

$E = |\vec{p}| \iff f = \frac{1}{\lambda}$ and $\omega = k$

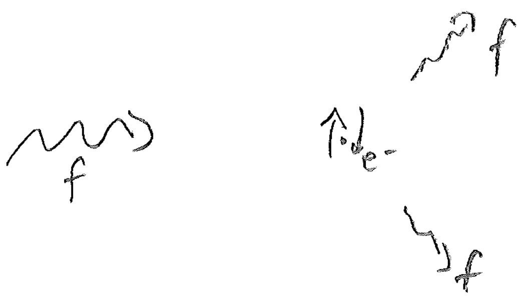
ie $f = \frac{c}{\lambda}$ and $\omega = ck$

[Physicists were skeptical of the photon concept until Arthur Compton did his expt on X-rays (1922)

X-rays were known to be EM waves

Scattering of X-rays by electrons (Thomson or Compton scattering)

Classical picture: electromagnetic waves of frequency f cause electrons to oscillate w/ frequency of f. Those accelerating electrons emit EM waves of frequency f.



Classical cross-section $\sigma = \frac{R}{F}$

R = rate at which energy is radiated by electron

F = flux of incident EM waves

I did this here in 2019 but will probably do it later in QED in future

From 1140, $F = \epsilon_0 c |\vec{E}|^2$

From 3120, Larmor formula $R = \frac{e^2 a^2}{6\pi\epsilon_0 c^3}$

$$\vec{a} = \frac{\vec{F}}{m_e} = \frac{e\vec{E}}{m_e} \Rightarrow R = \frac{e^4 |E|^2}{6\pi\epsilon_0 m_e^2 c^3} \Rightarrow \sigma =$$

$$\Rightarrow \sigma = \frac{e^4}{6\pi\epsilon_0^2 (m_e c^2)^2} = \frac{8\pi}{3}\left(\frac{ke^2}{m_e c^2}\right)^2 = \frac{8\pi}{3} r_0^2$$

where $r_0 \equiv \frac{ke^2}{m_e c^2}$ = classical electron radius $= \frac{1.44 \text{ MeV fm}}{0.511 \text{ MeV}} = 2.8$ fm

$\sigma = 66.5 \text{ fm}^2 = \frac{2}{3}$ barn = Thomson cross-section

[Thomson used it to measure # electrons in atoms]

Nucleus too massive to radiate much.

Compton, however, measured the wavelength of scattered X-rays and found it was slightly longer than incident X-rays

$\lambda' > \lambda$

Compton shift $\Delta\lambda = \lambda' - \lambda$ is proportional to $\frac{h}{m_e c}$,

called the Compton wavelength of the electron

$$\frac{h}{m_e c} = \frac{2\pi\hbar c}{m_e c^2} = \frac{2\pi(197\ \text{MeV fm})}{0.511\ \text{MeV}} = 2400\ \text{fm} \approx 2.4\times10^{-12}\ \text{m}$$

~~[illegible]~~ an unnoticeably small shift for visible light $\lambda \sim 5\times10^{-7}$ m

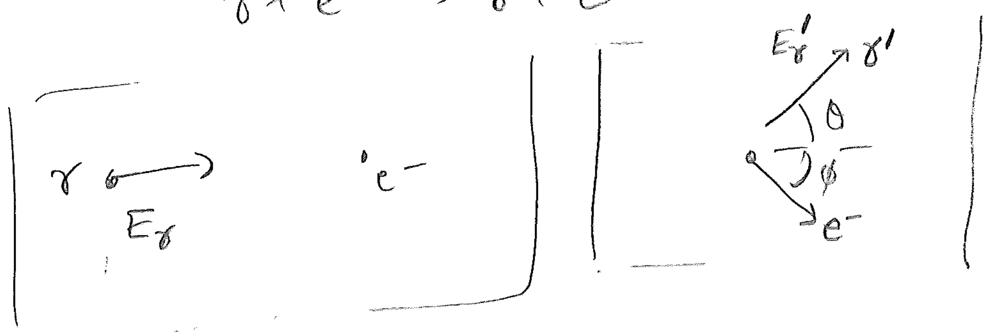
[~~[illegible]~~ is still small,] but measurable, for X-rays $\lambda \sim 10^{-10}$ m

Compton found
The shift depends on the scattering angle

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)$$

Compton scattering can be understood as a collision of a massless photon & a massive electron at rest

$$\gamma + e^- \rightarrow \gamma + e^-$$



Photon gives some of its energy to the electron, so

$E'_\gamma < E_\gamma$

since $E = hf \Rightarrow f' < f$

since $f = \frac{c}{\lambda} \Rightarrow \lambda' > \lambda$

one can compute the Compton shift using energy & momentum conservation

This is most elegantly done using 4-momentum

$$p_\gamma = (E_\gamma, 0, 0, E_\gamma) \qquad p'_\gamma = (E'_\gamma, E'_\gamma \sin\theta, 0, E'_\gamma \cos\theta)$$

$$p_e = (m_e, 0, 0, 0) \qquad p'_e = (E'_e, -|\vec{p}'_e|\sin\phi, 0, |\vec{p}'_e|\cos\phi)$$

$$p_\gamma + p_e = p'_\gamma + p'_e$$

Rewrite this as

$$-p'_e + p_e = p'_\gamma - p_\gamma$$

Consider

$$(-p'_e + p_e)^2 = p_e'^2 - 2p'_e \cdot p_e + p_e^2$$

$$= m_e^2 - 2(E'_e m_e - 0) + m_e^2$$

$$= 2m_e(m_e - E'_e) \underset{\substack{\uparrow \\ \text{energy cons.}}}{=} 2m_e(E'_\gamma - E_\gamma)$$

Now consider

$$(p'_\gamma - p_\gamma)^2 = p_\gamma'^2 - 2p'_\gamma \cdot p_\gamma + p_\gamma^2$$

$$= 0 - 2(E'_\gamma E_\gamma - E'_\gamma E_\gamma \cos\theta)$$

$$= -2E_\gamma E'_\gamma (1 - \cos\theta)$$

Equate these

$$-2m_e(E_\gamma - E'_\gamma) = -2E_\gamma E'_\gamma (1 - \cos\theta)$$

$$\frac{E_\gamma - E'_\gamma}{E_\gamma E'_\gamma} = \frac{1}{m_e}(1 - \cos\theta)$$

$$\frac{1}{E'_\gamma} - \frac{1}{E_\gamma} = \frac{1}{m_e c^2}(1 - \cos\theta)$$

restore

$$E = hf = \frac{hc}{\lambda}$$

$$\Rightarrow \frac{\lambda'}{hc} - \frac{\lambda}{hc} = \frac{1}{m_e c^2}(1 - \cos\theta)$$

$$\lambda' - \lambda = \underbrace{\frac{h}{m_e c}}_{\text{Compton wavelength of an electron}}(1 - \cos\theta)$$

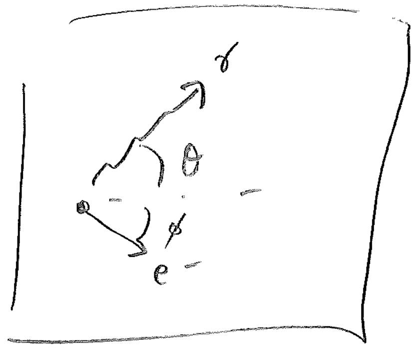
Max shift occurs of back scattered X-rays ($\theta = \pi$)

$$(\Delta\lambda)_{max} = \frac{2h}{m_e c} \approx 4.8 \times 10^{-12}\,m$$

Compton used molybdenum K_α line $\Rightarrow 7 \times 10^{-11}\,m$
(17 keV)

(7% shift)

Afterwards



Can do this more formally using 4-vectors →

$$\begin{cases} E_\gamma + mc^2 = E_\gamma' + E_e' \\ p_\gamma = p_\gamma' \cos\theta + p_e' \cos\phi \\ 0 = p_\gamma' \sin\theta - p_e' \sin\phi \end{cases}$$

Eliminate ϕ:

$$(p_\gamma - p_\gamma' \cos\theta)^2 + (p_\gamma' \sin\theta)^2 = p_e'^2$$

$$\underline{p_\gamma^2} - 2p_\gamma p_\gamma' \cos\theta + \underline{(p_\gamma')^2} = p_e'^2$$

Energy $\Rightarrow$ $(E_\gamma + mc^2 - E_\gamma')^2 = E_e'^2$

$$\underline{E_\gamma^2} + \underline{E_\gamma'^2} + 2E_\gamma mc^2 - 2E_\gamma' mc^2 - 2E_\gamma E_\gamma' = E_e'^2 - (mc^2)^2$$

$$2E_\gamma mc^2 - 2E_\gamma' mc^2 - 2E_\gamma E_\gamma'(1-\cos\theta) = 0$$ Subtract mom from energy

(a) $$\frac{1}{E_\gamma'} - \frac{1}{E_\gamma} = \frac{1-\cos\theta}{mc^2} \Rightarrow \underline{f(\theta) = 1-\cos\theta}$$

(b) $$E = \frac{hc}{\lambda} \Rightarrow \lambda' - \lambda = \underbrace{\frac{h}{mc}}_{2.4\times10^{-12}\,m}(1-\cos\theta)$$

$\theta = \pi \Rightarrow \Delta\lambda = \underline{4.8\ E{-12}\ m}$

$$\lambda = \frac{hc}{17\ KeV} = \frac{1.24E{-6}\ eV\cdot m}{1.7E4\ eV} = 7.3\times10^{-11}\,m$$

$$\frac{\Delta\lambda}{\lambda} \sim 0.066$$

Alternative

Use 4-vectors to compute Compton scattering

$$p_e' = p_e + p_\gamma - p_\gamma'$$

$$(p_e')^2 = p_e^2 + p_\gamma^2 + p_\gamma'^2 + 2p_e \cdot p_\gamma - 2p_e \cdot p_\gamma' - 2p_\gamma \cdot p_\gamma'$$

these cancel   these vanish

so

$$p_e \cdot p_\gamma - p_e \cdot p_\gamma' = p_\gamma \cdot p_\gamma'$$

$$p_e = \begin{pmatrix} mc^2 \\ \vec{0} \end{pmatrix}, \quad p_\gamma = \begin{pmatrix} E \\ E\vec{n} \end{pmatrix}, \quad p_\gamma' = \begin{pmatrix} E' \\ E'\vec{n}' \end{pmatrix}$$

$$mc^2 E - mc^2 E' = EE'(1 - \underbrace{\vec{n}\cdot\vec{n}'}_{\cos\theta})$$

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{mc^2}(1 - \cos\theta)$$

Alternatively: $p_e' - p_e = p_\gamma - p_\gamma'$

$$2m^2 - 2p_e \cdot p_e' = -2p_\gamma \cdot p_\gamma'$$

$$\underbrace{2m^2 - 2m \underbrace{E_e}_{E - E' - m}}_{-2m(E - E')} = -2EE'(1 - \cos\theta)$$

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m}(1 - \cos\theta)$$

done in class