Four-momentum

4-vector $u^\mu = (u^0, u^1, u^2, u^3)$

Transforms as $u^\mu \to \Lambda^\mu{}_\nu u^\nu$ under Lorentz transformation

We'll often omit the index, simply writing u
but it is implied by context

To distinguish 4-vectors from 3-vectors: u vs $\vec{u}$

Primarily we'll be concerned w/

4-momentum $p = \left(\frac{E}{c}, \vec{p}\right)$ or $(E, \vec{p})$ ← set $c=1$ as before (B-8)

Conservation of 4-momentum

$$p^{sys}_{init} = p^{sys}_{final}$$ (each component separately conserved)

$\Rightarrow \begin{cases} \text{conservation of energy} \\ \text{conservation of 3-momentum} \end{cases}$

Given two 4-vectors u and w

define Lorentz-invariant scalar product

$$u \cdot w = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} u^\mu w^\nu$$

where Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ ← particle physics convention "mostly minus metric"

$$u \cdot w = u^0 w^0 - \vec{u} \cdot \vec{w}$$

$$u^2 \equiv u \cdot u = (u^0)^2 - \vec{u}^{\,2}$$

$\vec{u}^{\,2} = \vec{u} \cdot \vec{u} = |\vec{u}|^2$

Use arrow to distinguish square of 3-vector vs 4-vector

[4-momenta & Lorentz scalars are a powerful tool.

For a single particle :

$p = (E, \vec{p})$

$p^2 = E^2 - \vec{p}^{\,2}$

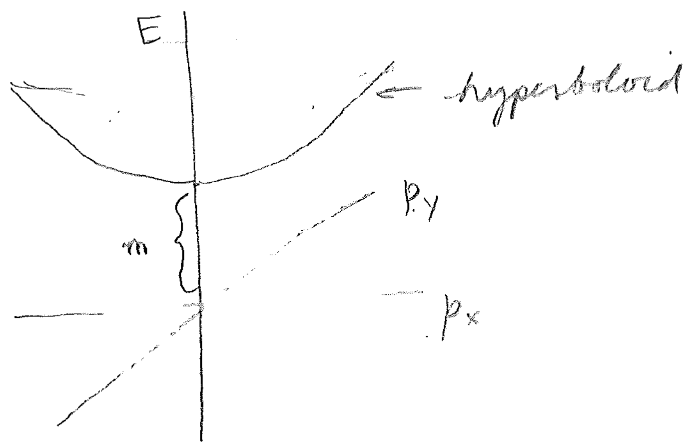
express E, p, m all in terms of energy, e.g. MeV

Recall from P214: $E^2 = (c\vec{p})^2 + (mc^2)^2$ or $E^2 = |\vec{p}|^2 + m^2$

$\Rightarrow \boxed{p^2 = m^2}$ mass is Lorentz scalar

If a particle obeys $p^2 = m^2$ we say it is "on the mass shell"

$E^2 = \vec{p}^{\,2} + m^2$

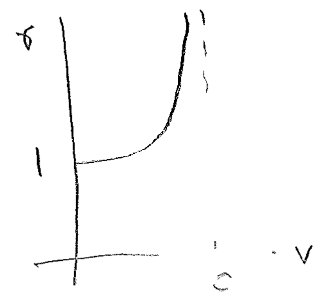


This is true for physical particles, but not virtual ones

If we need to express $E, \vec{p}$ in terms of $\vec{v}$, use

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

gamma factor
time-dilation factor



$\vec{p} = \gamma m \vec{v}$ (relativistic momentum)

$E = \gamma m c^2$

$T = E - mc^2 = (\gamma - 1) mc^2$ (relativistic kinetic energy)

$$\frac{c\vec{p}}{E} = \frac{\vec{v}}{c}$$

omitting c's:

$$\frac{\vec{p}}{E} = \vec{v}$$

where $p, E \sim$ MeV and $v \sim$ dimensionless ($v = 1 \rightarrow$ speed of light)

Often not necessary or helpful to use these

when applying energy + mom conservation

Non relativistic limits ($v \ll c$, $\gamma \simeq 1$)

$$\gamma = 1 + \frac{1}{2}v^2 + \dots$$

$$\vec{p} = \gamma m \vec{v} = m\vec{v} + \dots$$

$$E = \gamma m = m + \frac{1}{2}m v^2 + \dots$$

nonrelativistic kinetic energy

$$E = \sqrt{p^2 + m^2} = m\sqrt{1 + \frac{p^2}{m^2}} = m + \frac{p^2}{2m} + \dots$$

Ultrarelativistic limit ($v \approx c$, $\gamma \gg 1$)

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

$$v = \sqrt{1 - \frac{1}{\gamma^2}} = 1 - \frac{1}{2\gamma^2} + \dots$$

$$E = \sqrt{p^2+m^2} = p\sqrt{1+\frac{m^2}{p^2}} = p + \frac{m^2}{2p} + \dots$$

[proton in LHC has $E = 7$ TeV, $\gamma = 7000$, $v = 1-(10^{-8}) = 0.\underbrace{999\ 999\ 99}_{8\ 9\text{'s}}$]

Two particle decay (e.g. α-decay)

$$X \longrightarrow Y + \alpha$$

Energy + momentum conservation completely determine the kinematics

X at rest $\Rightarrow$ 

momentum: $\vec{p}_Y = -\vec{p}_\alpha$

energy: $E_Y + E_\alpha = m_X$

mass shell conditions $E_Y^2 = m_Y^2 + |\vec{p}_Y|^2$

$E_\alpha^2 = m_\alpha^2 + |\vec{p}_\alpha|^2$

4 eqns 4 unknowns

Elegant solution using 4-mom cpt

$$p_X = p_Y + p_\alpha$$

$$p_X - p_\alpha = p_Y$$

"Square" both sides

$$p_X^2 - 2p_X\cdot p_\alpha + p_\alpha^2 = p_Y^2$$

$$p_X\cdot p_\alpha = E_X E_\alpha - \vec{p}_X\cdot\vec{p}_\alpha = m_X E_\alpha - 0$$

$p_X^2 = m_X^2$ etc.

$$E_\alpha = \frac{m_X^2 + m_\alpha^2 - m_Y^2}{2m_X}$$

Similarly: $$E_Y = \frac{m_X^2 + m_Y^2 - m_\alpha^2}{2m_X}$$

check: $E_\alpha + E_Y = m_X$ ✓

$$T_\alpha = E_\alpha - m_\alpha$$

$$= \frac{(m_X^2 - 2m_X m_\alpha + m_\alpha^2) - m_Y^2}{2m_X}$$

$$= \frac{(m_X - m_\alpha)^2 - m_Y^2}{2m_X}$$

$$= \left(\frac{m_X - m_\alpha + m_Y}{2m_X}\right)\underbrace{(m_X - m_\alpha - m_Y)}_{Q\ =\ \text{total } T \text{ released}}$$

Fraction of total T going to α:

$$\frac{T_\alpha}{Q} = \frac{m_X - m_\alpha + m_Y}{2m_X} \qquad \frac{T_Y}{Q} = \frac{m_X - m_Y + m_\alpha}{2m_X}$$

If $m_Y \approx m_\alpha$ then $\frac{T_\alpha}{Q} \approx \frac{T_Y}{Q} \approx \frac{1}{2}$

If $m_Y \approx m_X$ then $\frac{T_\alpha}{Q} \approx 1$, $\frac{T_Y}{Q} \approx 0$

β-decay was originally thought to be $X \to Y + e^-$ [dm in β-13]

e.g. ${}^3_1H \to {}^3_2He + e^-$

$Q = \Delta({}^3H) - \Delta({}^3He) = 0.019$ MeV $\Rightarrow 18.6$ keV

since $m_e \ll m({}^3He)$, expect $T_e \approx 18.6$ keV

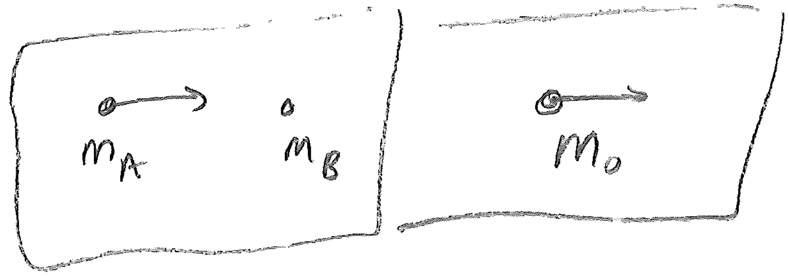
but continuous β-spectrum ! [canvas]

suggests another particle (Pauli predicted)

$X \to Y + e^- + \bar{\nu}_e$

[illegible]

[HW]



Compute $\frac{T_A}{Q}$

$\pi p \to \Delta$ (Fermi)

β-decay

Initially it was thought that

[cf β-13]

$${}^3_1H \longrightarrow {}^3_2He + e^-$$

[obeys charge + baryon # conservation]

kinetic energy released $Q = m_H - m_{He} - m_e$
$= \Delta(3H) - \Delta(3He) = 18.6$ keV

as we calculated earlier.

Initial state tritium at rest $\Rightarrow T_{He} + T_e = Q$

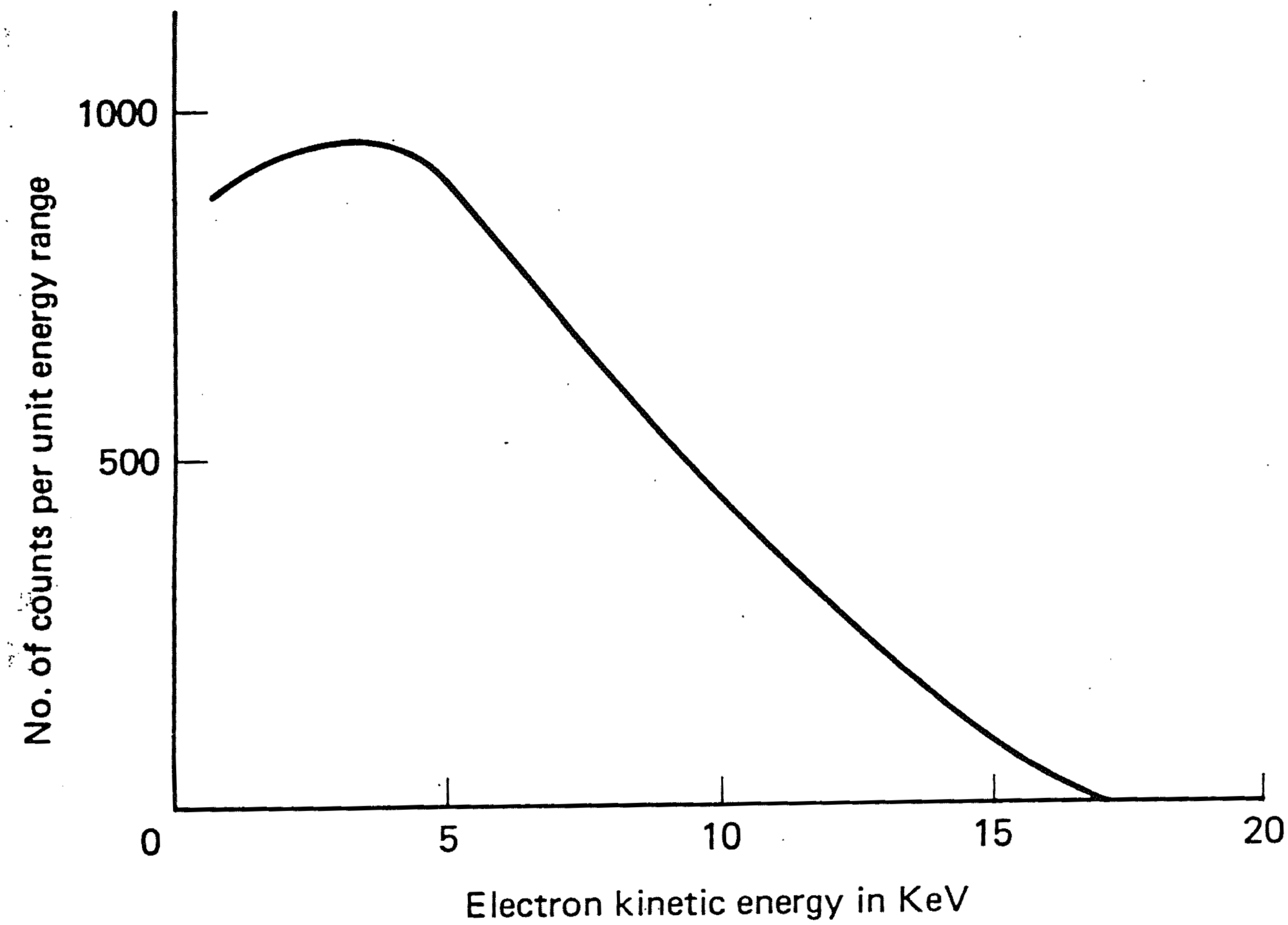
Since $m_e \ll m_H$, $T_{He} \ll T_e$ so expect $T_e \simeq Q$

Experiments, however, revealed a

continuous beta spectrum [Canvas]

ie T_e can take on any value $0 < T_e < Q$

[Chadwick 1914] see graph $\rightarrow$



The beta decay spectrum of tritium ($^3_1H \longrightarrow {}^3_2He$). (*Source:* G. M. Lewis,

Fig. 1.6 Griffiths

[Bohr suggested energy not conserved microscopically but only statistically on macro scale.]

1930 Pauli proposed the existence of a neutral particle ~~that~~ carries off the missing energy (w/ $m < 10$ MeV)
Fermi dubbed it the neutrino (little neutral one)

[quote]

$${}^3_1H \longrightarrow {}^3_2He + e^- + \bar{\nu}_e$$

electron antineutrino

As before, T_{He} is negligible so

$$T_e + E_\nu = Q = 18.6 \text{ keV}$$

[NB not T_ν because didn't include m_ν in definition of Q]

$$(E_\nu)_{min} = m_{\bar{\nu}} c^2 \quad \text{so}$$

$$(T_e)_{max} = Q - m_\nu c^2$$

Since $(T_e)_{max} \gtrsim 18$ keV $\Rightarrow m_\nu \lesssim 500$ eV [Segre, p. 334]

At any rate the earliest reference I know to the new particle is Heisenberg's mention of 'your neutrons' in a letter to Pauli[97] dated 1 December. More details are found in Pauli's letter (its main part follows) of 4 December to a gathering of experts on radioactivity in Tübingen.[60] 1930

Dear radioactive ladies and gentlemen,
I have come upon a desperate way out regarding the 'wrong' statistics of the N- and the Li 6-nuclei, as well as to the continuous β-spectrum, in order to save the 'alternation law' of statistics* and the energy law. To wit, the possibility that there could exist in the nucleus electrically neutral particles, which I shall call neutrons, which have spin 1/2 and satisfy the exclusion principle and which are further distinct from light-quanta in that they do not move with light velocity. The mass of the neutrons should be of the same order of magnitude as the electron mass and in any case not larger than 0.01 times the proton mass.—The continuous β-spectrum would then become understandable from the assumption that in β-decay a neutron is emitted along with the electron, in such a way that the sum of the energies of the neutron and the electron is constant.

There is the further question, which forces act on the neutron? On wave mechanical grounds . . . the most probable model for the neutron seems to me to be that the neutron at rest is a magnetic dipole with a certain moment μ. Experiments seem to demand that the ionizing action of such a neutron cannot be bigger than that of a γ-ray, and so μ may not be larger than $e \times 10^{-13}$ cm.

For the time being I dare not publish anything about this idea and address myself confidentially first to you, dear radioactive ones, with the question how it would be with the experimental proof of such a neutron, if it were to have a penetrating power equal to or about ten times larger than a γ-ray.

I admit that my way out may not seem very probable *a priori* since one would probably have seen the neutrons a long time ago if they exist. But only he who dares wins, and the seriousness of the situation concerning the continuous β-spectrum is illuminated by my honored predecessor, Mr. Debye, who recently said to me in Brussels: 'Oh, it is best not to think about this at all, as with new taxes'. One must therefore discuss seriously every road to salvation.—Thus, dear radioactive ones, examine and judge.—Unfortunately I cannot appear personally in Tübingen since a ball** which takes place in Zürich the night of the sixth to the seventh of December makes my presence here indispensable . . . Your most humble servant, W. Pauli'.