Liquid drop model of nucleus

Semi-empirical binding energy formula
(Bethe-Weizsäcker)

(Fits data well for $A \gtrsim 15$)

$$B = c_1 A - c_2 A^{2/3} - c_3 \frac{Z^2}{A^{1/3}} - c_4 \frac{(A-2Z)^2}{A} + \delta_{pair}$$

A fit to the empirical data gives

[Krane]

$$\begin{cases} c_1 = 15.5 \text{ MeV} \\ c_2 = 16.8 \text{ MeV} \\ c_3 = 0.72 \text{ MeV} \\ c_4 = 23 \text{ MeV} \\ c_5 = 34 \text{ MeV} \end{cases}$$

$$\delta_{pair} = \begin{cases} c_5 A^{-3/4} & Z, N \text{ even} \\ 0 & Z+N \text{ odd} \\ -c_5 A^{-3/4} & Z, N \text{ odd} \end{cases}$$

c_1 due to strong interaction (volume term) and kinetic energy

c_2 due to strong interaction (surface term)

c_3 due to EM int. (Coulomb repulsion) $\Rightarrow$ acts to reduce Z

c_4 due to kinetic energy & strong force $\Rightarrow$ acts to keep $Z \approx N$

c_5 due to spin pairing $\Rightarrow$ favors even-even nuclei

Recall $B = -(\sum T + \sum V)$

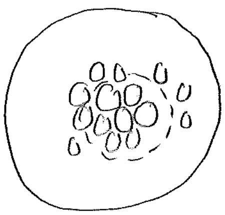
[First think about potential energies due to forces

(Force primarily responsible for existence of nucleus is

strong nuclear force

- acts between pairs of nucleons (pp, pn, nn) ↗ think about "bonds" between each pair
  # bonds: $\frac{A(A-1)}{2}$
  ↓
  Binding energy $\sim \frac{A^2}{2}$ (energy per bond)

<ins>but</ins> - short range) range << size of nucleus R



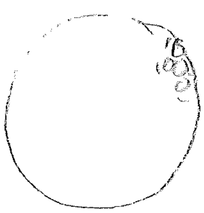
Suppose each nucleon acts on n others for some $n \ll A$

total # of pairs ("bonds") is $\frac{1}{2} n A$

Potential energy = − (# pairs)(energy of "bond")

$$\Rightarrow B \approx c_1 A \qquad \text{(volume term)}$$

Then $\frac{B}{A} \approx c_1$ explains why binding energy per nucleon is ≈ const (due to short range nature of force, else $B \sim A^2$)



Nucleons on surface form fewer "bonds"

"missing bonds" ~ # of nucleons on surface ~ $4\pi R^2 \sim A^{2/3}$

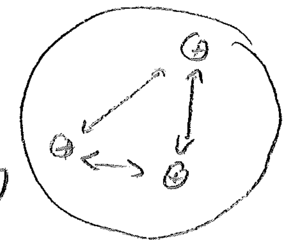
$$B \approx c_1 A - c_2 A^{2/3} \quad \text{(surface term)}$$

Surface term causes nucleus to minimize surface area → sphere

Liquid drop model of nucleus [Bohr]

Electromagnetic potential energy

- acts between pairs of protons: $\frac{Z(Z-1)}{2}$ pairs
- long range: affects all protons in nucleus
- repulsive $\Rightarrow$ $V > 0$ $\Rightarrow$ reduces binding energy



Treat protons as uniformly distributed charge $q = Ze$

Potential energy of sphere of charge $V = \frac{3}{5}\frac{Kq^2}{R} = \underbrace{\left(\frac{3Ke^2}{5r_0}\right)}_{c_3}\frac{Z^2}{A^{1/3}}$

Estimate $c_3 = \frac{3Ke^2}{5r_0} = \frac{(0.6)(1.44\ \text{MeV}\cdot\text{fm})}{(1.2\ \text{fm})} = \underline{\underline{0.72\ \text{MeV}}}$

[Krane, p. 67] $$B \approx c_1 A - c_2 A^{2/3} - c_3 \frac{Z^2}{A^{1/3}} + \dots$$

$$\frac{B}{A} \approx c_1 - \frac{c_2}{A^{1/3}} - c_3 \frac{Z^2}{A^{4/3}}$$

causes curve to bend down for small A

[hydrogen has $B = 0$]

causes curve to bend down for large Z & A & eventually nucleus becomes unstable

[EM acts on all protons, strong force only on nearby nucleons]

$$B = -(\Sigma T) - (\Sigma V)$$

Binding energy depends, not only on potential energy but also on the kinetic energy of the nucleons (not the nucleus itself, which is at rest)

Nucleons must have kinetic energy because they are confined to a small space, so by Heisenberg uncertainty principle cannot have zero momentum,

Let's estimate the kinetic energy of A nucleons confined to a nucleus.

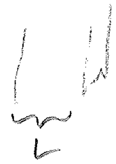
We will treat the nucleons as quantum particles confined to a box, but within the box they are noninteracting (not very realistic).

Box is really a sphere of radius $R = A^{1/3} r_0$ but we'll treat it as a cube of the same volume

$$L^3 = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi A r_0^3$$

Boundary conditions don't make much difference.

Particle in a 3d cube of volume L^3. LD-5 ③

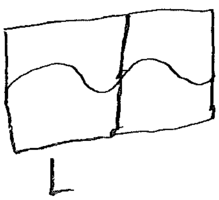


Impose periodic boundary conditions
so we can use travelling waves (not standing waves)

free particle of momentum $\vec{p}$ are described by $e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}}$

~~To be able to normalize these, imagine the universe = box of volume $V=L^3$~~

w/ periodic boundary conditions ~~(at end, take $V\to\infty$)~~

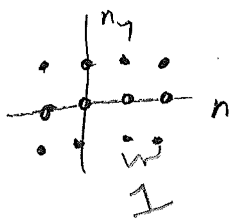


$$e^{\frac{ip(x+L)}{\hbar}} = e^{\frac{ipx}{\hbar}}$$

$$e^{\frac{ipL}{\hbar}} = 1 \quad \Rightarrow \quad p = \frac{2\pi\hbar}{L}n = \frac{h}{L}n$$

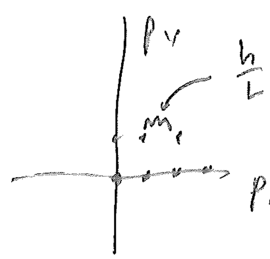
3 dimensions $\quad \vec{p} = \frac{h}{L}\vec{n} = \frac{h}{L}(n_x, n_y, n_z)$

Set of possible states characterized by lattice of integers



$$\sum_{n_x,n_y,n_z} \approx \int dn_x\, dn_y\, dn_z$$

$$= \left(\frac{L}{h}\right)^3 \int dp_x\, dp_y\, dp_z$$



$$= \frac{V}{h^3}\int d^3p$$

Sum over final states

$$\boxed{\sum_f = \prod_{j=1}^{n_f} \frac{V}{\hbar^3}\int\frac{d^3p_j}{(2\pi)^3}}$$

"integral over phase space"

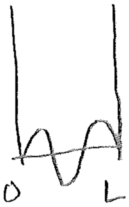
Skip

First question: what are the allowed momenta?

~~LD~~/5

periodic vs Dirichlet

1. Consider a 1-dimensional box of length L.



If we impose Dirichlet b.c. on the particle (i.e. $\psi(x)$ vanishes at edges of box)

then de Broglie wavelength $\lambda = \frac{2L}{n}$

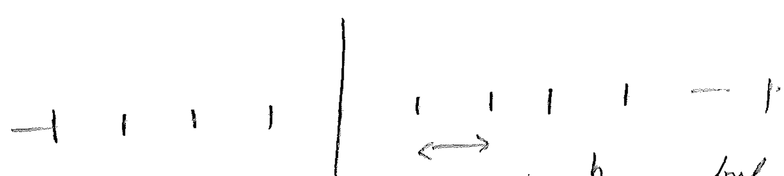
then $|\vec{p}| = \frac{h}{\lambda} = \frac{hn}{2L}$, $n = 1, 2, 3, \ldots$

momentum spacing is $\frac{h}{2L}$

Alternatively one can impose periodic b.c. $\psi(x+L) = \psi(x)$

Then $\psi = e^{\frac{ipx}{\hbar}} \Rightarrow e^{\frac{ipL}{\hbar}} = 1 \Rightarrow \frac{pL}{\hbar} = 2\pi n$

so $p = \frac{hn}{L}$, $n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$



spacing is $\frac{h}{L}$ but negative p are allowed

These give equivalent results because both give same # of states for $|\vec{p}| \leq p_{max}$,

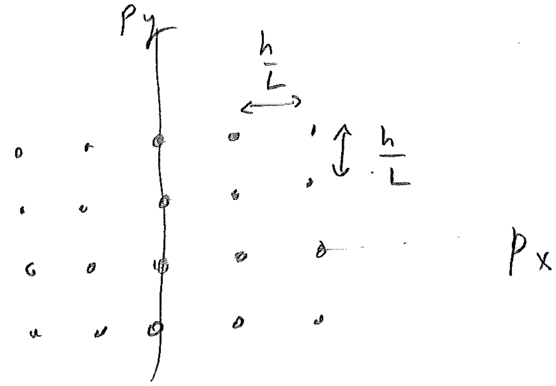
namely $N_{states} = \frac{p_{max}}{\left(\frac{h}{2L}\right)} = \frac{2\, p_{max}}{\left(\frac{h}{L}\right)}$

skip

For a particle in a cube w/ periodic boundary cond.

$$\vec{p} = (p_x, p_y, p_z) = \frac{h}{L}(n_x, n_y, n_z) = \frac{h}{L}\vec{n}$$

$\Rightarrow$ Momenta live on a 3d lattice w/ spacing $\frac{h}{L}$ in "momentum space"



[If we used Dirichlet b.c., lattice points would be more closely spaced – 8 times more dense – but we would only consider points in the first octant.]

If we want to sum over points on the lattice

$$\sum_{\vec{n}} \approx \int d^3\vec{n} = \int dn_x\, dn_y\, dn_z$$

$$= \int \left(\frac{L}{h}dp_x\right)\left(\frac{L}{h}dp_y\right)\left(\frac{L}{h}dp_z\right)$$

$\vec{n} = \frac{L}{h}\vec{p}$

$$= \left(\frac{L}{h}\right)^3 \int d^3\vec{p}$$

$\Rightarrow$ just integrate over $\vec{p}$-space w/ measure $\left(\frac{L}{h}\right)^3$

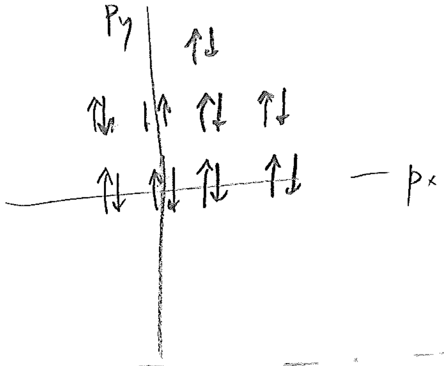
[to account for density of states]

Protons and neutrons are both spin-$\frac{1}{2}$ fermions.

Spin-$\frac{1}{2}$ particles have 2 spin states, $\uparrow$ and $\downarrow$.

The quantum state of a nucleon is determined by its momentum and its spin.

$\Rightarrow \exists$ 2 distinct quantum states for each lattice point in momentum space



explain exclusion principle first

Let's count the number of <u>distinct</u> quantum states w/ $|\vec{p}|$ less than a certain value, call it p_F.

$$N_{states} = \sum_{\vec{n}} \underset{\text{spin multiplicity}}{2} = 2\left(\frac{L}{h}\right)^3 \int d^3\vec{p}.$$

sum over lattice points

$$= 2\left(\frac{L}{h}\right)^3 \underbrace{\int_0^{p_F} 4\pi p^2\, dp}_{\frac{4\pi}{3}p_F^3}.$$

volume of sphere

$$= \frac{8\pi}{3}\left(\frac{Lp_F}{h}\right)^3$$

$$\Rightarrow p_F = \frac{h}{L}\left(\frac{3}{8\pi}N_{states}\right)^{1/3}$$

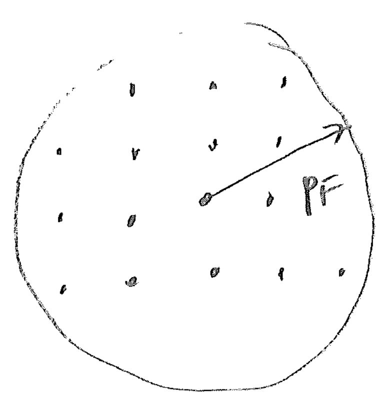
No two identical fermions can occupy the same quantum state, so a collection of identical fermions must occupy lattice points of larger & larger momentum.

At zero temperature, N neutrons will occupy the N quantum states of <u>lowest energy</u> (& ∴ smallest momentum $|\vec{p}|$) up to some maximum momentum $p_F \left[= \frac{h}{L}\left(\frac{3N}{8\pi}\right)^{\frac{1}{3}} \right]$

This is called a <u>degenerate fermi gas</u> and p_F is the <u>fermi momentum</u>.

Let's compute p_F (go back to previous page)

Z protons occupy their own degenerate fermi gas w/ their own fermi momentum $p_F = \frac{h}{L}\left(\frac{3Z}{8\pi}\right)^{\frac{1}{3}}$



What is the kinetic energy of this degenerate fermi gas?

$$T^{tot} = 2 \sum_{\vec{n}} T_{\vec{n}} = 2\left(\frac{L}{h}\right)^3 \int (4\pi p^2\,dp)\,T(p)$$

spin multiplicity

For nonrelativistic particles ($v \ll c$), $T \approx \frac{p^2}{2m}$

For ultrarelativistic particles ($v \approx c$), $T \approx cp$

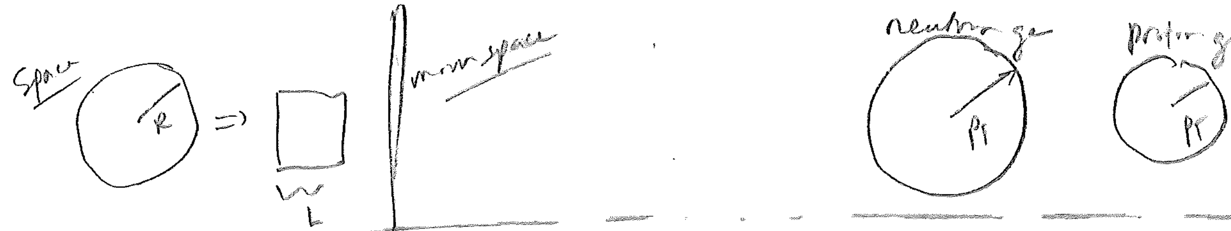
[HW] Show that average kinetic energy $\bar{T} = \frac{3}{5}\epsilon_F$

where $\epsilon_F = \frac{p_F^2}{2m}$ is the nonrelativistic fermi energy

The total kinetic energy of a degenerate gas of N nonrelativistic neutrons is then

$$T^{tot} = N\bar{T} = \frac{3}{10}\frac{p_F^2}{m}N$$

[HW] Calculate the average kinetic energy of a degenerate gas of ultrarelativistic particles



Recall fermi momentum of a degenerate gas of N neutrons

$$p_F = \frac{h}{L}\left(\frac{3N}{8\pi}\right)^{\frac{1}{3}} = \hbar\left(\frac{3\pi^2 N}{L^3}\right)^{\frac{1}{3}}$$

Replace cube volume L^3 by sphere volume $\frac{4\pi}{3} A r_0^3$

$$p_F = \frac{\hbar}{r_0}\left(\frac{9\pi}{4}\frac{N}{A}\right)^{\frac{1}{3}}$$

Degenerate gas is <u>nonrelativistic</u> if $\frac{v}{c} \ll 1 \Rightarrow \frac{p}{mc} \ll 1$

The max momentum of the gas is p_F

$$\frac{p_F}{mc} = \frac{\hbar c}{mc^2 r_0}\underbrace{\left(\frac{9\pi}{4}\frac{N}{A}\right)^{\frac{1}{3}}}_{\text{of order } 1} \sim \frac{(200\text{ MeV}\cdot\text{fm})}{(940\text{ MeV})(1.2\text{ fm})} \sim \frac{1}{5} \quad \checkmark$$

we can use nonrelativistic kinetic energy

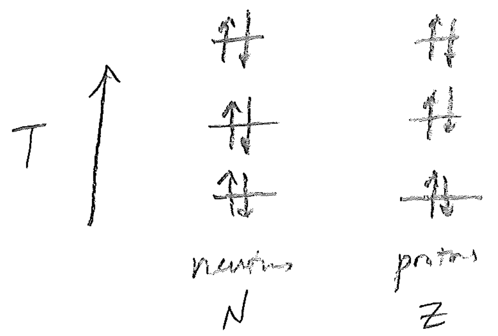
$$\Rightarrow T^{tot} = \frac{3}{10}\frac{p_F^2}{m}\frac{N}{A}\cdot A = \underbrace{\frac{3}{10}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}}_{\sim 1.105}\frac{\hbar^2}{m r_0^2}\left(\frac{N}{A}\right)^{\frac{5}{3}}\cdot A$$

Total kinetic energy of neutron and proton gas is

$$T = 1.105\,\frac{\hbar^2}{m r_0^2}\left[\left(\frac{N}{A}\right)^{\frac{5}{3}} + \left(\frac{Z}{A}\right)^{\frac{5}{3}}\right] A$$

(masses about the same)

First, consider nuclei w/ $Z \simeq N \simeq \frac{1}{2}A$



← ^{12}C

$$\frac{(\hbar c)^2}{(mc^2)\, r_0^2} = \frac{(197)^2}{(940)(1.2)^2} = 29 \text{ MeV}$$

$$T^{tot} = \underbrace{1.105 \left[\left(\tfrac{1}{2}\right)^{5/3} + \left(\tfrac{1}{2}\right)^{5/3} \right]}_{\simeq\, 0.696} \underbrace{\frac{\hbar^2}{m r_0^2}}_{29 \text{ MeV}} A$$

$\approx (20 \text{ MeV})\, A$ ← reduces binding energy

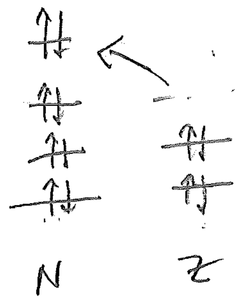
[For deuteron, we found $T = 130$ MeV]

[This is a contribution to the binding energy volume term

Suggests that potential energy per nucleon is ~ -35 MeV

$B = -T^{tot} - V^{tot} \approx (15 \text{ MeV})\, A$]

If $N \neq Z$, kinetic energy increases. Why?

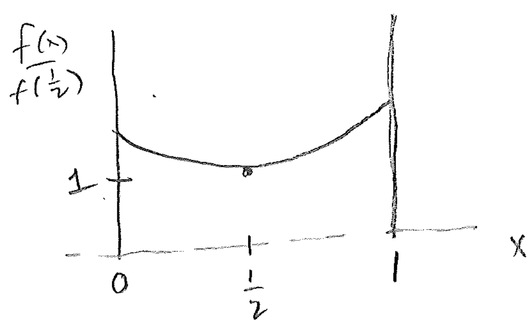


Let's estimate change in T^{tot}.

Let $x = \frac{Z}{A}$

Then $T^{tot} = (20\,\mathrm{MeV})\, A \cdot \frac{f(x)}{f(\frac{1}{2})}$

where $f(x) = x^{5/3} + (1-x)^{5/3}$ $\qquad f(\frac{1}{2}) \cong 0.63$



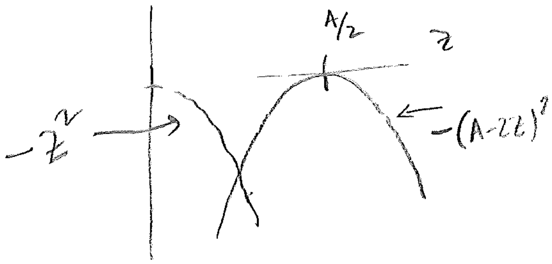
Near $x = \frac{1}{2}$, curve is $\approx$ parabolic

$$\frac{f(x)}{f(\frac{1}{2})} = 1 + \#\,(x - \tfrac{1}{2})^2 + \ldots \qquad [\# = \tfrac{20}{9}]$$

HW: calculate # & use it estimate c_4 in semi-empirical
Hint: it doesn't agree too well, but is right order of mag

[$c_4 = 11$ MeV, whereas actual $c_4 = 23$ MeV]

$$B = c_1 A - c_2 A^{2/3} - \underbrace{c_3 \frac{Z^2}{A^{1/3}}} - \underbrace{c_4 \frac{(A-2Z)^2}{A}} + \delta_{pair}$$



favors $Z \ll N$

more effective at large A

⇓

heavy nuclei have fewer protons than neutrons

favors $Z = \frac{A}{2}$

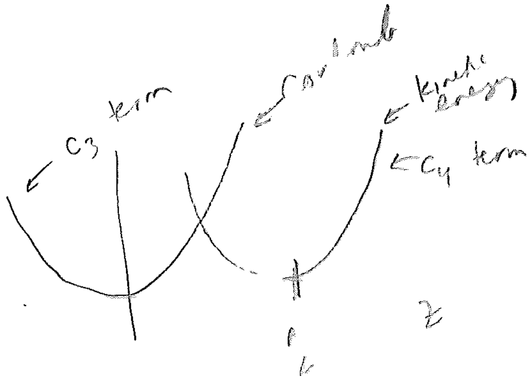
or $Z = N$

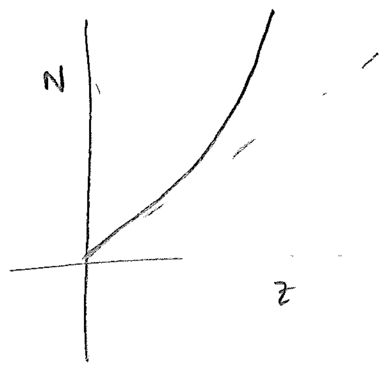
more effective at small A

⇓

light nuclei have $Z \simeq N$

^4He, ^{12}C, ^{16}O





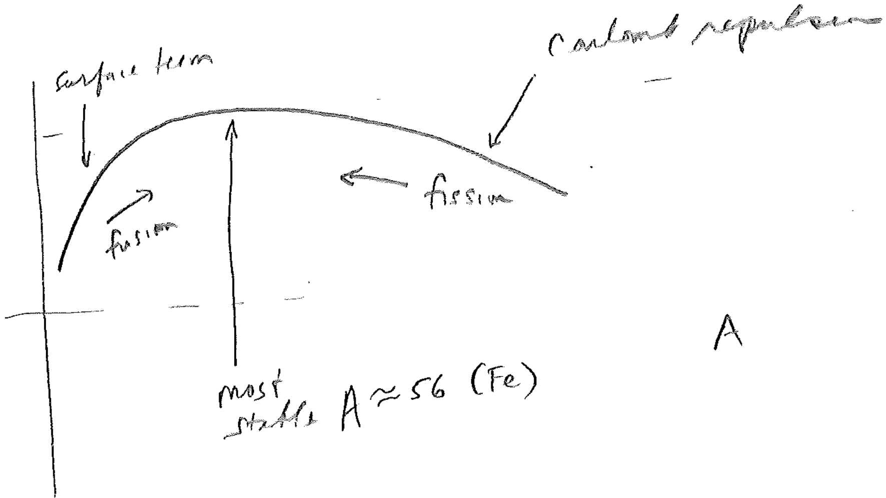
[problem: solve for Z as a fcn of A that minimizes Δ (not B)]

Binding energy per nucleon

$$\frac{B}{A} = c_1 - c_2 A^{-1/3} - c_3 \frac{Z^2}{A^{4/3}} - c_4 \frac{(A-2Z)^2}{A^2} + \frac{\delta_{pair}}{A}$$

assume $Z \approx N \approx \frac{A}{2}$

$$\frac{B}{A} = c_1 - \frac{c_2}{A^{1/3}} - c_3 A^{2/3}$$



already discussed

$$X \to Y + Z \quad \text{iff} \quad m_X > m_Y + m_Z$$

A nucleus is stable only if its mass is less than the sum of masses of nuclei into which it could decay

9

Calc of asymmetry term in semi-empirical formula

Kinetic energy of nucleus

$$\vec{k} = \frac{2\pi}{L}\vec{n}$$

$$\frac{4\pi}{3} n_{max}^3 = \frac{N}{2}$$

$$\epsilon_F = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 \left(\frac{3N}{8\pi}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$U = \frac{3}{5} N \epsilon_F$$

$r_0 = 1.2$ fm (Serway)
$= 1.5$ fm (Fermi)

For a nucleus. $V = A V_0$, $V_0 = \frac{4}{3}\pi r_0^3$

$$\epsilon_F = \underbrace{\frac{\hbar^2 c^2}{2 m c^2 r_0^2}\left(\frac{9\pi}{4}\right)^{2/3}}_{C_0} \left(\frac{N}{A}\right)^{2/3}$$

53 MeV (Serway)
34 MeV (Fermi)

$\rightarrow N_p = N_n = \frac{A}{2} \Rightarrow \epsilon_F \simeq 21$ MeV (Fe)

if only neutrons, $\epsilon_F = \begin{Bmatrix} 53 \\ 34 \end{Bmatrix}$

(27 MeV, say K+K, $r_0 = 1.3$ f)

$$U_{Nuc} = \frac{3}{5} N_p C_0 \left(\frac{N_p}{A}\right)^{2/3} + \frac{3}{5} N_n C_0 \left(\frac{N_n}{A}\right)^{2/3}$$

$$= \frac{3}{5}\frac{C_0}{A^{2/3}}\left[\underbrace{Z^{5/3} + (A-Z)^{5/3}}_{f(Z,A)}\right]$$

$$\rho = Z - \frac{A}{2}$$

$$f(Z,A) = \left(\frac{A}{2}+\rho\right)^{5/3} + \left(\frac{A}{2}-\rho\right)^{5/3}$$

$$= \frac{A^{5/3}}{2^{3/2}}\left[1 + \frac{20}{9}\frac{\rho^2}{A^2}\right] = (.63) A^{5/3} + (.35) A^{-1/3}(A-2Z)^2$$

$$U_{Nuc} = \underbrace{(.38)\, C_0}_{\substack{20 \text{ MeV (Serway)} \\ 13 \text{ MeV (Fermi)}}} A + \underbrace{(.21)\, C_0}_{\substack{11 \text{ MeV (Serway)} \\ 7 \text{ MeV (Fermi)}}} \frac{(A-2Z)^2}{A}$$

If all neutrons, $(A = N_n, Z = 0)$ $U_{Nuc} = \frac{3}{5} C_0 A = \begin{Bmatrix} 32 \text{ MeV} \\ 20 \text{ MeV} \end{Bmatrix} A$,