2023 revision

A decay (or other transformation) will generally occur provided it does not violate any conservation laws

["what is not forbidden is mandatory"]

[review of pi-4] Conservation laws

- energy
- momentum
- angular momentum

} a consequence of invariance of laws of physics under translations in space and time, + rotations in space (Noether's thm)

[optional]

- electric charge

} a consequence of gauge invariance of electromagnetism

[believed to be absolute]

- baryon number (quark number)
- lepton number

} "accidental" but valid for standard model

[due to no known symmetry but satisfied by the 4 known forces; # quarks + leptons don't convert into each other;

If a process obeys energy + momentum conservation, it is said to be "kinematically allowed" otherwise "kinematically forbidden"

Energy conservation

[types of energy]

- rest energy mc^2
- kinetic energy T

$$E = \gamma m c^2 \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \dots$$

$$T = E - mc^2 = (\gamma - 1) mc^2$$

non rel. approx. $\cong \frac{1}{2}mv^2 + \frac{3}{8} m \cdot \frac{v^4}{c^2} + \dots$

- potential energy V

associated w/ interactions between particles (EM, strong ..)

Well before and well after the process, the particles are well separated
$\Rightarrow$ ignore any potential energies in initial + final states

$$E_{init} = E_{final}$$

$$\sum_{init} (mc^2 + T) = \sum_{final} (mc^2 + T)$$

Neither kinetic energy nor ~~nor~~ mass is separately conserved
only their sum.

Define the kinetic energy released in a process

$$Q = T_{final} - T_{init}$$

$$= \sum_{init} mc^2 - \sum_{final} mc^2$$

If $Q = 0$, "elastic process" $\Rightarrow$ kinetic energy is conserved
[e.g. scattering of particles which retain their identities]

If $Q < 0$, "inelastic process" $\Rightarrow$ kinetic energy is lost
final state has more mass than initial (collisions)

[lumps of putty: hot object has slightly more mass than a cool one]

[More massive particles can be produced in a collision
e.g. cosmic rays, particle accelerators]

If $Q > 0$, "superelastic process"
$\Rightarrow$ mass "converted" to kinetic energy
(e.g. fission, decay)

If initial state consists of a single particle m_0 at rest ($T_0 = 0$)

then $Q = T_{final} \geq 0$

$$\Rightarrow \quad m_0 c^2 \geq \sum_{final} mc^2$$

$\Rightarrow$ massive particles can only decay into less massive particles (not vice versa)

<u>Generally</u>, decays of larger Q
have larger probability to occur
+ therefore shorter half lives
though for differing reasons

e.g. α-decay, higher Q ∴ small potential barrier
to tunnel through

β-decay, higher Q ⟹ larger phase space

But in some cases, other considerations
cause the relationship to be violated

e.g. parity violation of the weak force
means that $\pi \to \mu$ is more likely
than $\pi \to e$,
despite the smaller phase space available

Conservation laws guarantee stability of some particles

e^- is stable because it is least massive pcl w/ electric charge

[if it decayed into lighter neutral particles, violates charge cons.]

Why is p stable?

[not the lightest positively charged pcl : $p \to e^+ \gamma$]

1938 Stuckelberg proposed conservation of baryon # (A)
(quark #)

p is stable because it is lightest pcl w/ baryon #

All forces of standard model conserve baryon # and lepton #
but extensions of standard model (eg GUTs)
typically violate both

Proton decay expts since 1970's [problem]
(super Kamiokande)
null results so far

$\tau_p > 2 \times 10^{29}$ yrs [need a lot of protons]

(2023)

Abby Chris asked about lifetime limits + confidence levels quoted in PPB

proton decay τ limits

Decay rate $R = \frac{1}{\tau}$

Prob of decay in time of experiment T : $p = RT$

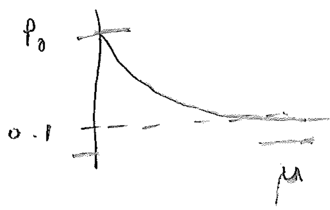
N nuclei

mean # nuclei decays in time T is $\mu = Np = \frac{NT}{\tau}$

Poisson distrib. P_n : prob of n nuclei decaying

$$= e^{-\mu}\frac{\mu^n}{n!}$$

Prob. of no nuclei decays $P_0 = e^{-\mu}$



Suppose no nuclei decay.

This is very likely for small μ. ↳ large τ

Unlikely for large μ ↳ small τ

Set $e^{-\mu} = 0.1$,

$e^{\mu} = 10$

$\mu = \log(10) = 2.3$

W/ 90% confidence we can say that $\mu < 2.3$

↳ are only a 10% chance that no particles would decay in time T

$$\frac{NT}{\tau} < \ln 10$$

$$\tau > \frac{NT}{\ln 10}$$

<u>Mass/energy conversions</u>

Masses of atoms or nuclei are specified in terms of unified atomic mass unit (u).

$$1\,u \equiv \frac{(\text{mass of neutral } ^{12}C \text{ atom in ground state})}{12}$$

[amu was based on ^{16}O; u is based on ^{12}C]

A mole of particles = Avogadro's number of particles

$N_A = 6.022\,140\,76 \times 10^{23}$ (exact)

A mole of particles of mass 1u has mass of 1 g (approx.)

[used to be a <u>definition</u>, & N_A was determined experimentally. Now N_A is defined exactly, so the statement is only approximate. Molar mass of ^{12}C is $11.999\,999\,995\,8(36)$ g.]

$$1\,u = \frac{10^{-3}\,kg}{N_A} = 1.660539\ldots \times 10^{-27}\,kg \quad \text{(approx.)}$$

Using exact $c = 299\,792\,458\ \frac{m}{s}$, a particle of mass 1u has rest energy

$$mc^2 = 1.492418\ldots \times 10^{-10}\,J \quad \text{(approx)}$$

(exact)

$c = 299792458 \frac{m}{s}$

$h = 6.626\,070\,15 \times 10^{-34}$ J/Hz

do not discuss

$e = 1.602\,176\,634 \times 10^{-19}$ C

$k_B = 1.380\,649 \times 10^{-23}$ J/K

1 second defined by
$\nu = 9{,}192{,}631{,}770$ cycles of hyperfine transition of ^{133}Cs.

$E = h\nu = 6.091102297\ldots E{-}24$ J
so $1\,J = 1.64E23$ photons of ^{133}Cs

$E = \frac{h\nu}{e} = 3.80171702616E{-}5$ eV
ie $1\,eV = 26{,}300$ photons

$m = \frac{h\nu}{c^2}$ $1.0136\ldots E{-}42$ kg
= mass equivalent of photon
This defines the kg.

- We'll use eV rather than J

$e = 1.602\,176\,634 \times 10^{-19}$ C (exact)

$1\,eV = 1.602\,176\,634 \times 10^{-19}$ J (exact) [recall J = C·V]

A particle of mass 1 u has rest energy

$mc^2 = 9.31494 \times 10^{8}$ eV (approx).

$= 931.494$ meV (approx)

$$\boxed{1\,u \Leftrightarrow 931.494\ldots\,\text{meV}}$$

[PPB]

- proton 1.0072765u = 938.272 MeV
- neutron 1.0086649u = 939.565 MeV
- electron 0.0005486u = 0.511 MeV

[keep these on the board]

[about $\frac{1}{1800} m_p$]

[Both about 1 GeV]

[Neutron slightly more massive, so decays]

$n \rightarrow p$

[violates charge cons.]

$n \rightarrow p + e^-$

[violates lepton number]

$n \rightarrow p + e^- + \bar{\nu}_e$

[still obey energy conservation?]

- neutrino $\lesssim 1$ eV

$$Q = m_n c^2 - m_p c^2 - m_e c^2 - m_{\bar{\nu}} c^2 = 0.782 \text{ MeV}$$

Neutron decay is kinematically allowed.

A nuclear process involving e^- or e^+ is called β-decay

Most unstable light nuclei decay via β-decay

Mass excess

[It is generally easier to measure the masses of neutral atoms than of bare nuclei]

A neutral atom of Z protons, N neutrons, Z electrons

has an approximate mass $m \approx (Z+N)\,u = A\,u$

ie rest energy

$$mc^2 \approx A\,(931.494\ \mathrm{MeV})$$

It is convenient to characterize the discrepancy as

$$mc^2 \equiv A\,(931.494\ \mathrm{MeV}) + \Delta$$ ← atomic mass (incl. electrons)

Δ = "mass excess" [actually rest energy excess]

$$m = \left(A + \frac{\Delta}{931.494}\right) u$$ [nuclear wallet cards]

Recall: the unified mass unit u is *defined* so that $\Delta \equiv 0$ for $^{12}_{6}C$

[Using our earlier numbers]

- proton $\Delta = 6.778$ MeV
- neutron $\Delta = 8.071$ MeV

[neutron agrees w/ nuclear wallet cards, but not proton]

- hydrogen = proton + electron $\Delta = 7.289$ MeV ✓

$$n \rightarrow p + e^- + \bar{\nu}_e$$

From now on, we can omit the factors of c:

[Drop the c's, w/ this freedom comes responsibility]

(need restore units when doing calculations ⋮)

$$Q = m_n - m_p - m_e - \cancel{m_{\bar{\nu}}}^{\,0}$$

$$= m_n - (m_H - m_e) - m_e$$

$$= m_n - m_H$$

$$= (1u + \Delta(n)) - (1u - \Delta(^1H))$$

$$= \Delta(n) - \Delta(^1H)$$

$$= 0.782 \text{ MeV}$$ ✓

Bound states

Deuteron $d = pn$

[Simplest composite nucleus
p and n bound together by strong force
(residue of color force between quarks]

Composite objects hold together due to their binding energy.

Binding energy = difference between the sum of (rest) energies of the components and the (rest) energy of the composite object

Deuteron binding energy:

$$B_d = m_n + m_p - m_d$$

$$= m_n + [m({}^1H) - \cancel{m_e}] - [m({}^2H) - \cancel{m_e}]$$

$$= [1u + \Delta(n)] + [1u + \Delta({}^1H)] - [2u + \Delta({}^2H)]$$

$$= \Delta(n) + \Delta({}^1H) - \Delta({}^2H)$$

$$= 8.071 + 7.289 - 13.136$$

$$= 2.224 \text{ MeV}$$ (about 0.1% of $m_d \sim 2$ GeV)

$$\Rightarrow m_d = m_n + m_p - B_d$$

Deuteron mass is less than its constituents

[How to understand B ?]

First, let's understand binding energy for hydrogen atom.

$$E_H = (m_p + T_p) + (m_e + T_e) + V_{ep}$$

Coulomb potential energy between p and e

$T_e = \frac{p^2}{2m_e}$, $\quad T_p = \frac{p^2}{2m_p} \ll T_e$ since $m_p \gg m_e$

Estimate T_e. Uncertainty principle $\Rightarrow p \sim \frac{\hbar}{\Delta x}$

where Δx = uncertainty in position of electron $\sim a_0$ = Bohr radius

$$T_e \approx \frac{1}{2m_e}\left(\frac{\hbar}{a_0}\right)^2 = 13.6\,eV$$

more precisely

$$T_e = \frac{1}{2m}\left(\frac{h}{\lambda}\right)^2 = \frac{1}{2m}\left(\frac{\hbar}{a_0}\right)^2$$

$$a_0 = \frac{\hbar^2}{mke^2}$$

$$T_e = \frac{1}{2}mc^2\left(\frac{ke^2}{\hbar c}\right)^2 = \frac{1}{2}mc^2\alpha^2$$

Attractive inverse square force

$\Rightarrow V_{ep} = -2T_e$ (virial theorem)

$\Rightarrow T_e + T_p + V_{ep} \approx T_e + 0 - 2T_e = -T_e = -13.6\,eV$

But $m_H = m_p + m_e - B_{atomic}$ so $B_{atomic} = 13.6\,eV$

$\approx 1\,GeV + \frac{1}{2}MeV - 10\,eV$

B is only 10^{-8} of total mass

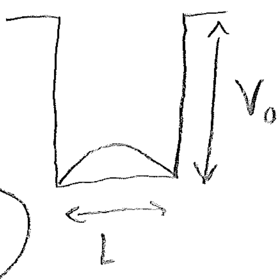
Binding energy of deuteron

strong interaction potential energy

$$m_d = m_p + m_n + \underbrace{T_p + T_n + V_{pn}}_{-B_d} \qquad \text{where } B_d \approx 2.2 \text{ MeV}$$

[Unlike Coulomb force, we have] No simple formula for V_{pn}
[nor do we expect one, since p and n are composites]

Use simplistic potential well model to estimate kinetic energy



$$T = T_p + T_n = 2 \cdot \frac{\vec{p}^2}{2m} = 2 \cdot \frac{3}{2m}\left(\frac{h}{\lambda}\right)^2 \qquad \lambda = \text{de Broglie wavelength}$$

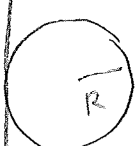
$$\lambda = 2L$$

probably overkill

$$T = \frac{3}{m}\left(\frac{2\pi\hbar}{2L}\right)^2 = \frac{3\pi^2}{mc^2}\left(\frac{\hbar c}{L}\right)^2$$

skip

$$= \frac{3\pi^2}{(1000 \text{ MeV})}\left(\frac{200 \text{ MeV} \cdot \text{fm}}{L}\right)^2 = 1200 \text{ MeV}\left(\frac{\text{fm}}{L}\right)^2$$



$L = 2R$, $R \cdot 2^{1/3} r_0 \approx 1.5$ fm $\Rightarrow L \sim 3$ fm

$T \approx 130$ MeV

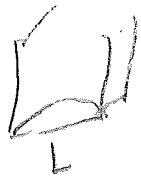
This is a very crude estimate (probably too high by factor of 2 or 4)

but suggests $V_0 \approx -T \approx -130$ MeV

Simpler: $T = T_p + T_n \approx \frac{p^2}{m} \approx \frac{1}{m}\left(\frac{\hbar}{\Delta x}\right)^2 = \frac{1}{mc^2}\left(\frac{\hbar c}{\Delta x}\right)^2 = \frac{1}{1000 \text{ MeV}}\left(\frac{200 \text{ MeV fm}}{1 \text{ fm}}\right)^2 \approx 40$ MeV

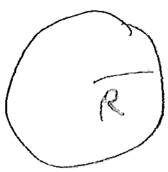
Suggests $V \approx -42$ MeV $\Rightarrow B = -T - V \approx 2$ MeV background

particle in 3d box



$$T = 3 \cdot \frac{1}{2m}\left(\frac{h}{\lambda}\right) = \frac{3}{2m}\left(\frac{2\pi\hbar}{2L}\right)^2 = \frac{3\pi^2\hbar^2}{2mL^2}$$

particle in a sphere



$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}(r\psi) = E\psi$$

$$\frac{d^2}{dr^2}(r\psi) + \frac{2mE}{\hbar^2}(r\psi) = 0$$

$$\psi = \frac{\sin kr}{r} \quad \text{where} \quad \frac{\hbar^2 k^2}{2m} = E$$

$$\psi(R) = 0 \Rightarrow kR = \pi \Rightarrow k = \frac{\pi}{R} \Rightarrow \boxed{E = \frac{\hbar^2\pi^2}{2mR^2}}$$

If $L = \underbrace{2R}_{\text{diameter}}$ then 3d box $\Rightarrow \frac{3\pi^2\hbar^2}{8mR^2}$, slightly less

If volumes equal $L^3 = \frac{4\pi}{3}R^3$

the 3d box $= \frac{3\pi^2\hbar^2}{2m}\left(\frac{3}{4\pi}\right)^{2/3}\frac{1}{R^2} = 0.577\,\frac{\hbar^2\pi^2}{mR^2}$

slightly more (by 15%)

[deuteron is not unstable to $d \to p + n$, but what about β-decay]

$$d \xrightarrow{?} p + p + e^- + \bar{\nu}_e$$

$$Q = m_d - 2m_p - m_e$$

$$= (m(^2H) - m_e) - 2(m(^1H) - m_e) - m_e$$

$$= m(^2H) - 2m(^1H)$$

$$= \Delta(^2H) - 2\Delta(^1H)$$

$$= 13.136 - 2(7.289)$$

$$= -1.442 \text{ MeV}$$ Kinematically forbidden

[Not as much difference as binding energy!
but still kinematically forbidden]

Free neutron is unstable, but neutron
bound to a proton is stable!

[strong force reduces the energy]

Tritium $t = pnn$

$t \xrightarrow{?} d + p + e^- + \bar{\nu}_e$

$Q = \Delta(^3H) - \Delta(^2H) - \Delta(^1H)$

$= 14.95 - 13.136 - 7.289$

$= -5.475$ MeV (forbidden)

[Not possible]

← probably omit this

[How does tritium decay?]

$pnn \longrightarrow ppn + e^- + \bar{\nu}_e$

$Q \cdot [m(^3H) - m_e] - [m(^3He) - 2m_e] - m_e$

$= [3u + \Delta(^3H)] - [3u - \Delta(^3He)$

$= \Delta(^3H) - \Delta(^3He)$

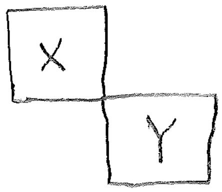
$= 0.019$ MeV (allowed)

[$m(^3H) = 2809.4496$
$m(^3He) = 2809.4310$
0.0186

just barely unstable

β-decay always connects isobars = nuclei of same # nucleons

β^- decay



$$ {}^{A}_{Z}X \xrightarrow{?} {}^{A}_{Z+1}Y + e^- + \bar{\nu}_e $$

one of neutrons converted to a proton

$$Q = m_{nucl}(X) - m_{nucl}(Y) - m_e$$

$$= \left[Au + \Delta(X) - Zm_e\right] - \left[Au + \Delta(Y) - (Z+1)m_e\right] - m_e$$

$$= \Delta(X) - \Delta(Y)$$

If $\Delta(X) > \Delta(Y)$, then X is unstable to β-decay

What if $\Delta(X) < \Delta(Y)$?

Can a proton convert to a neutron?

$$e^- + p \longrightarrow n + \nu_e$$

kinematically forbidden for e^-, p at rest
(but formation of a neutron star)

[conservation laws obeyed]

$$e^- + {}^{A}_{Z+1}Y \xrightarrow{?} {}^{A}_{Z}X + \nu_e \quad \text{(electron capture)}$$

one of the atomic electrons

one of protons converted to a neutron

$$Q = m_e + m_{nucl}(Y) - m_{nucl}(X)$$

$$= m_e + [Au + \Delta(Y) - (Z+1)m_e] - [Au + \Delta(X) - Z m_e]$$

$$= \Delta(Y) - \Delta(X)$$

If $\Delta(Y) > \Delta(X)$, then Y is unstable to EC (electron capture)

EC is followed by X-ray emission, as one of outer shell electrons fills the vacancy created by the s-shell electron

[first observed 1938 Alvarez : cf Krane, p. 272]

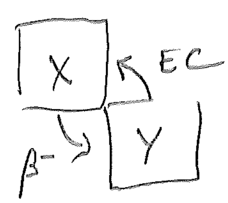
Given two adjacent isobars, one will always be unstable
(but next to adjacent isobars can exist)

[Segre chart]

(see next page for Δ's)

$${}^{113}_{48}Cd \xrightarrow[\beta^-]{} {}^{113}_{49}In \qquad 9.3 \times 10^{15}\ \text{yrs}$$

$${}^{123}_{52}Te \xrightarrow[EC]{} {}^{123}_{51}Sb \qquad 1.3 \times 10^{13}\ \text{yrs}$$



(still shows as stable on Segrè chart).

$^{113}_{48}Cd$, $\Delta = -89.051$, $T_{\frac{1}{2}} = 9.3E15$ yrs β^- decay $J = \frac{1}{2}+$

12.22%

$\downarrow$

$^{113}_{49}In$ $\Delta = -89.367$ 4.3% $J: \frac{9}{2}+$

$^{123}_{51}Sb$ $\Delta = -89.223$ 42.7% $J = \frac{7}{2}+$

$\uparrow$

$^{123}_{52}Te$ $\Delta = -89.172$ 1.3×10^{13} yrs EC $J = \frac{1}{2}+$

0.908%

β^+ decay

$$p \longrightarrow n + e^+ + \nu_e$$ not kinematically possible

$$^{A}_{Z+1}Y \xrightarrow{?} {}^{A}_{Z}X + e^+ + \nu_e$$

proton converted to neutron

$$Q = m_{nucl}(Y) - m_{nucl}(X) - m_e$$

$$= [Au + \Delta(Y) - (Z+1)m_e] - [Au + \Delta(X) - Zm_e] - m_e$$

$$= \Delta(Y) - \Delta(X) - 2m_e$$

If $\Delta(Y) - \Delta(X) \geq 2m_e = 1.022$ MeV

then Y is unstable to β^+ decay (also EC)

β^+ decay is followed by

$$e^+ + e^- \longrightarrow \gamma + \gamma$$ (pair annihilation)

$$Q = m_{e^+} + m_{e^-} = 1.022 \text{ MeV}$$

Each γ carries of 0.511 MeV (tell-tale sign of e^+e^- annihilation)

[first observed 1934 Joliot-Curie; cf Krane p.272]

Binding energy of a nucleus $^A_Z X$

$$B = Z\, m_p + N\, m_n - m_{nucl}(X)$$

$$= Z\left[m(^1H) - m_e\right] + N m_n - \left[m(X) - Z m_e\right]$$

$$= Z\left[1u + \Delta(^1H)\right] + N\left[1u + \Delta(n)\right] - \left[A u + \Delta(X)\right]$$

$$= Z \underbrace{\Delta(^1H)}_{7.289} + N \underbrace{\Delta(n)}_{8.071} - \Delta(X)$$

Rewrite this in terms of $N+Z$ and $N-Z$

$$\boxed{B = (N+Z)(7.680\text{ MeV}) + (N-Z)(0.391\text{ MeV}) - \Delta(X)}$$

For ^{12}C, $N=Z$ and $\Delta = 0$ ~~so~~

[B and Δ not the same]

$$B(^{12}C) = 12\,(7.680\text{ MeV})$$

If we ~~assume~~ ^{12}C is "typical" ie $N \approx Z$ and $\Delta \approx 0$

$$B = (7.7\text{ MeV})\, A$$

ie binding energy per nucleon is approximately constant

[show next page] ⇒ "curve of binding energy"

[need to explain deviations]

$$m_{nucl}(X) = Z m_p + N m_n - B$$

$$\approx (939\text{ MeV})\, A - (8\text{ MeV})\, A$$

$$\approx (931\text{ MeV})\, A$$

≈ 1u, chosen so Δ ≈ 0

Nuclei are ~ 1% less massive than sum of constituents

[recall deuteron about 0.1% → relatively weak binding]

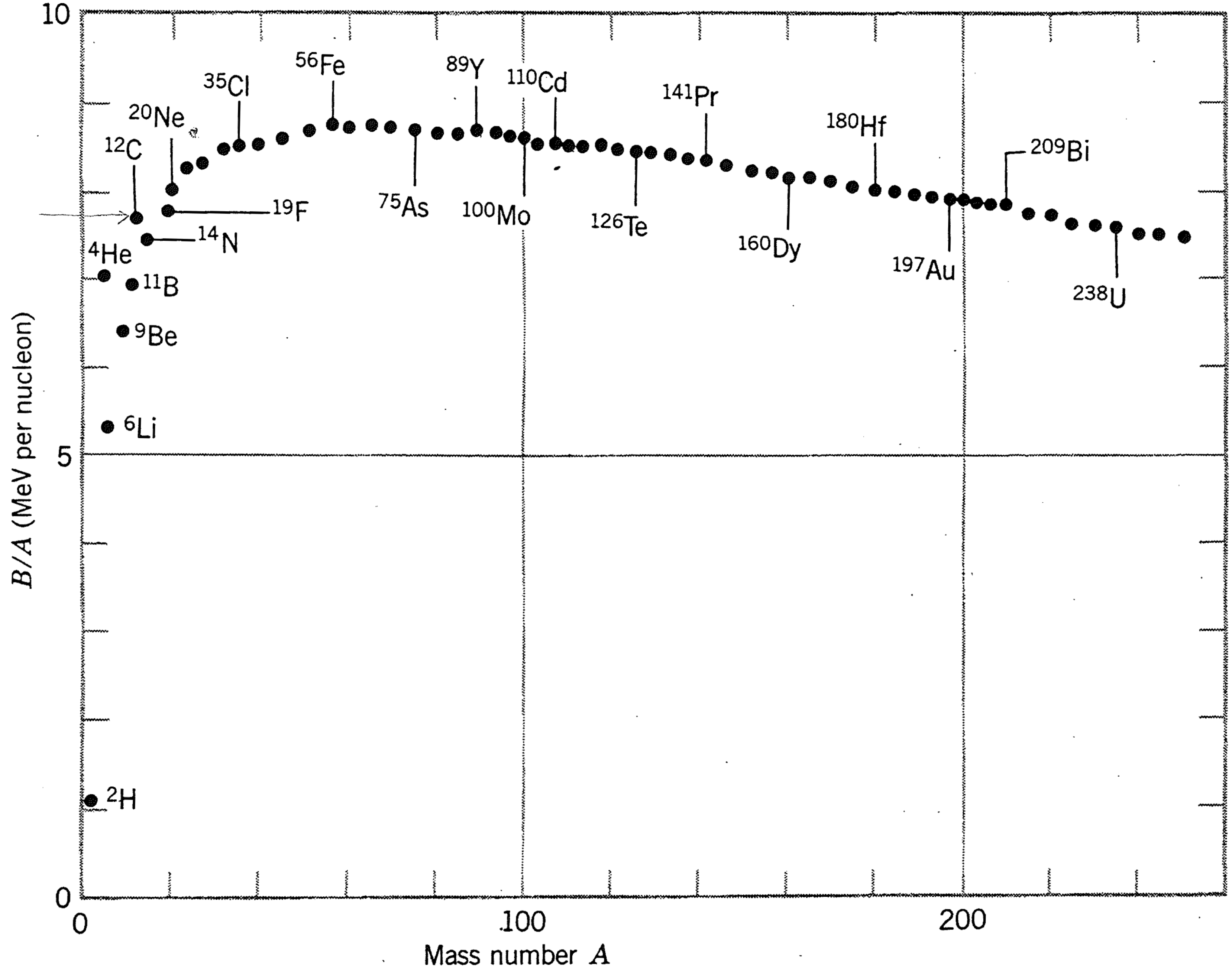
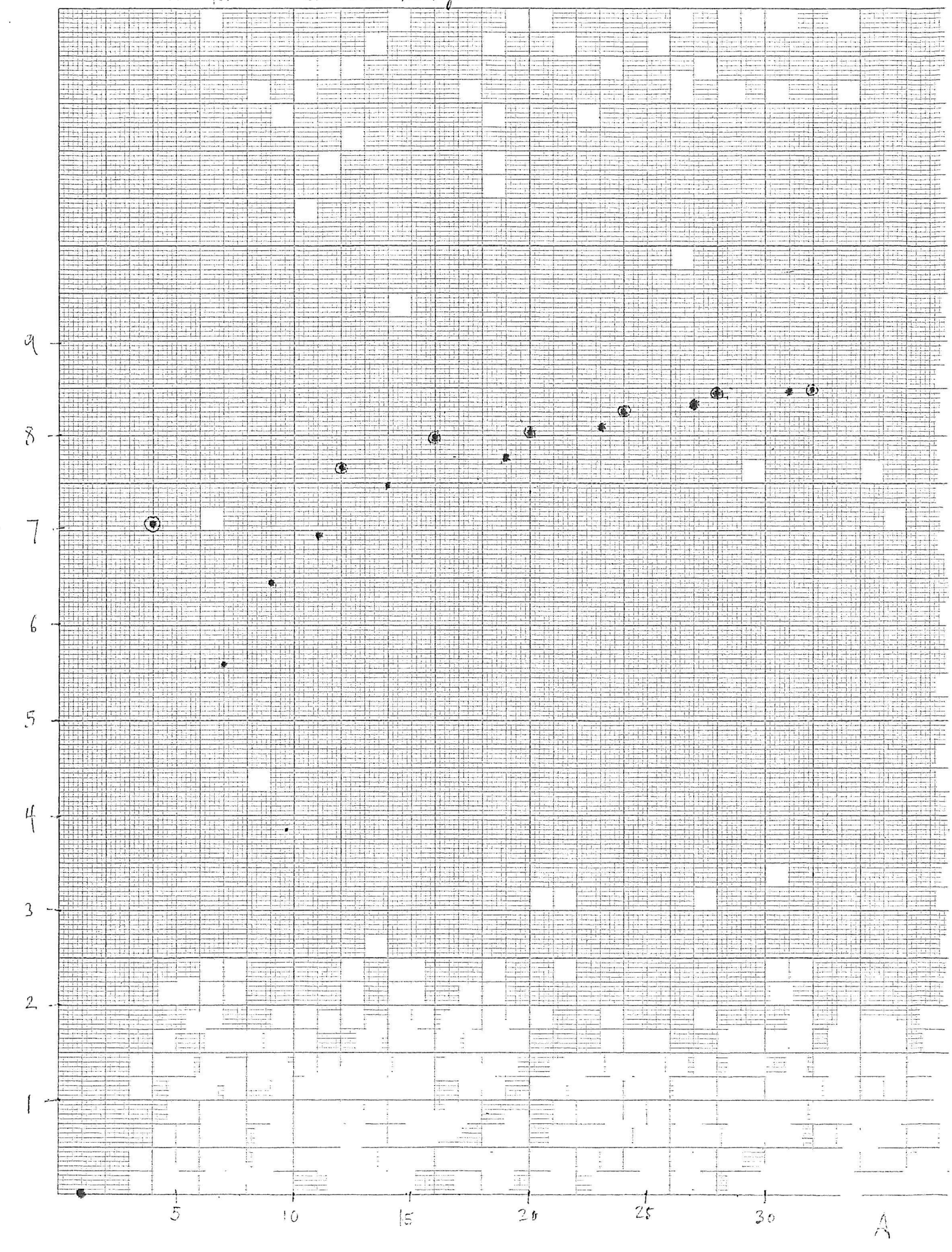


Figure 3.16 The binding energy per nucleon.

B/A for most abundant isotopes

⊙ = even-even nuclides



	Z	A	Δ	A−2Z	$\frac{B}{A} = 7.68 + \frac{(A-2Z)(0.39) - \Delta}{A}$
H	1	1	7.29	−1	0
He	2	4	2.42	0	7.07
Li	3	7	14.91	1	5.61
Be	4	9	11.35	1	6.46
B	5	11	8.67	1	6.93
C	6	12	0	0	7.68
N	7	14	2.86	0	7.48
O	8	16	−4.74	0	7.98
F	9	19	−1.49	1	7.78
Ne	10	20	−7.05	0	8.03
Na	11	23	−9.53	1	8.11
Mg	12	24	−13.93	0	8.26
Al	13	27	−17.20	1	8.33
Si	14	28	−21.49	0	8.45
P	15	31	−24.44	1	8.48
S	16	32	−26.01	0	8.49