[To explore the structure of atoms]

1910 Rutherford, Geiger, Marsden

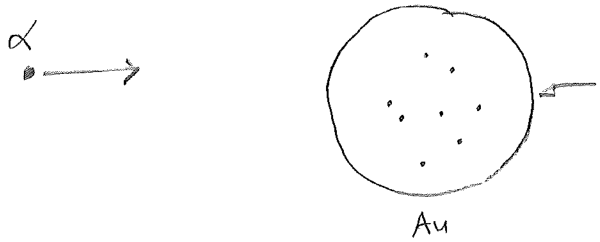
[shot α particles (^4He nuclei) from radioactive decay
at thin gold foil [to reduce likelihood of multiple collisions]

[They expected small deflections.

$m_\alpha \approx 4 m_H$

$m_{Au} \approx 197 m_p$ (distributed)

$m_e \approx \frac{1}{1800} m_H$



Thomson model of atom
(positive jelly w/ pointlike electrons)
overall electrically neutral

[No effect until α goes inside (because neutral)]

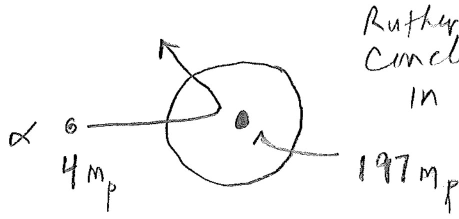
$$\frac{m_\alpha}{m_e} \approx 7000$$

[bowling ball ~ 7 kg, ping pong ball 2.7 g, factor of 2500]

expect little stopping power.

[They measured some large deflections
Rutherford: "It was quite the most incredible event that has ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you" (1936)

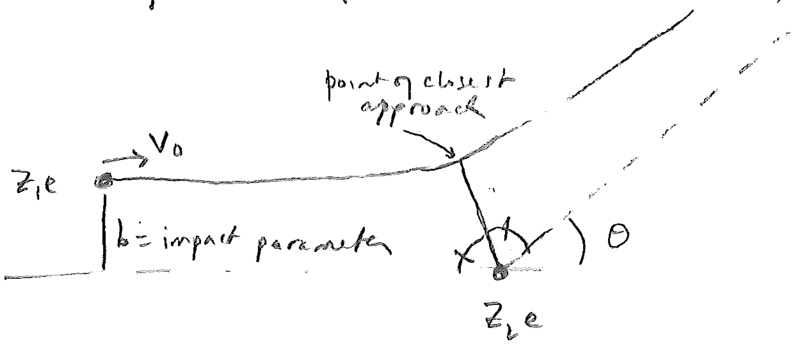
Rutherford
concluded mass of gold atom concentrated
in a small nucleus.



Rutherford scattering (elastic)

[Rutherford calculated]

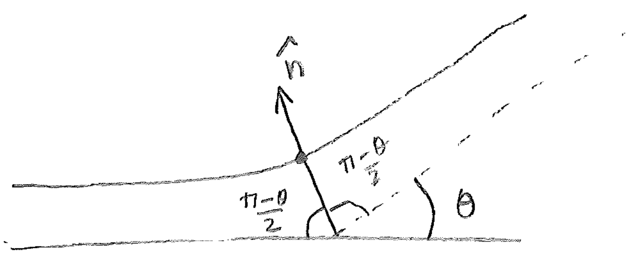
deflection of an α-particle from a fixed, point-like nucleus X



[inverse square ⇒ elliptical, parabolic, hyperbolic]

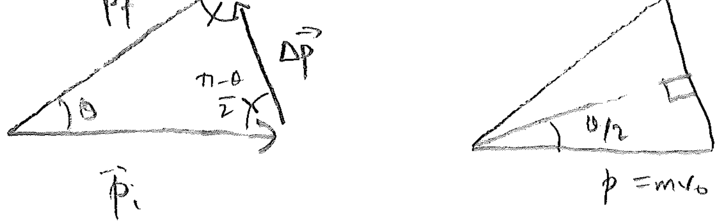
α moves along hyperbolic orbit symmetric w.r.t. point of closest approach

Goal: find θ as a function of v_0 and b

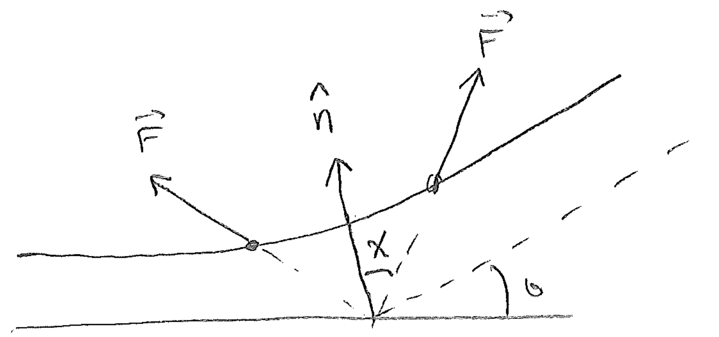


To show: change in momentum $\Delta\vec{p} \parallel \hat{n}$

Energy conservation $\Rightarrow |\vec{p}_f| = |\vec{p}_i|$ [isosceles triangle]



$$\boxed{\Delta p = 2mv_0 \sin\frac{\theta}{2}}$$



$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Delta\vec{p} = \int d\vec{p} = \int \vec{F}\, dt = \text{impulse}$$

Net impulse $\perp$ $\hat{n}$ vanishes

Net impulse $\parallel$ $\hat{n}$

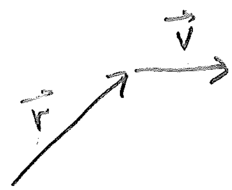
$$|\Delta\vec{p}| = \hat{n}\cdot\Delta\vec{p} = \int \vec{F}\cdot\hat{n}\, dt = \int F\cos\chi\, dt$$

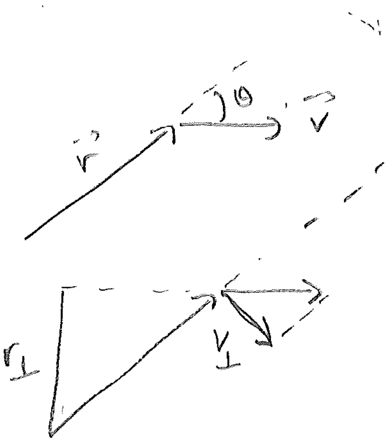
To do this integral, we need to relate dt to $d\chi$.

use cons. of angular moment

Coulomb force is a central force (radial)
$\Rightarrow$ angular momentum is conserved

$$\vec{l} = \vec{r} \times m\vec{v}$$





$$l = rmv\sin\theta$$
$$= r_\perp mv \quad \text{or} \quad r\cdot m\cdot v_\perp$$

Initially
$$r_\perp mv = bmv_0$$

At arbitrary time
$$v_\perp = r\frac{d\chi}{dt}$$
$$rmv_\perp = r^2 m\frac{d\chi}{dt}$$

Set these equal
$$\frac{d\chi}{dt} = \frac{bv_0}{r^2} \Rightarrow dt = \frac{r^2}{bv_0}\,d\chi$$

$$|\vec{F}| = \frac{K(Z_1e)(Z_2e)}{r^2} = \frac{k}{r^2} \quad \text{where } k = Z_1Z_2Ke^2$$

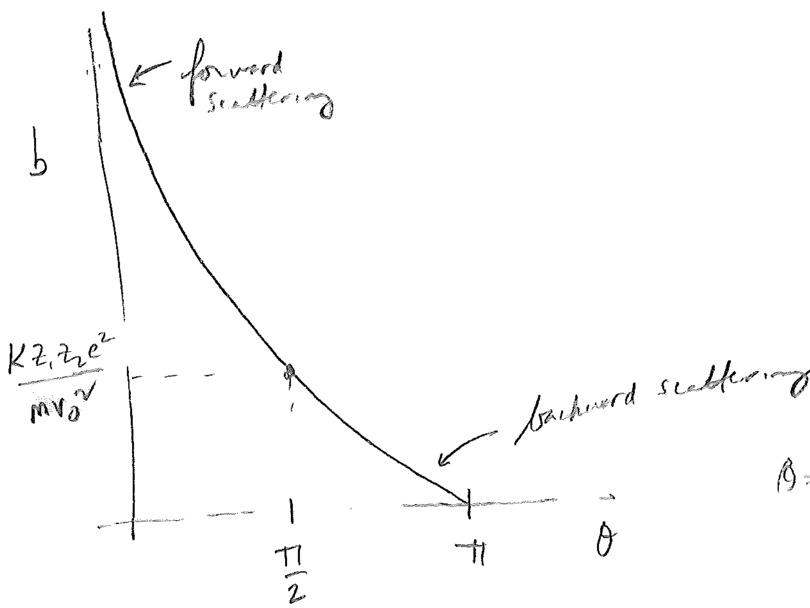
$$|\Delta\vec{p}| = \int F\cos\chi\,dt = \int \frac{k}{r^2}\cos\chi\,\frac{r^2}{bv_0}\,d\chi =$$
$$= \frac{k}{bv_0}\int\cos\chi\,d\chi = \frac{k}{bv_0}\sin\chi\Big|_{-(\frac{\pi-\theta}{2})}^{\frac{\pi-\theta}{2}} = \frac{2k}{bv_0}\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$
$$= \frac{2k}{bv_0}\cos\frac{\theta}{2}$$

Equating two expressions for Δp:

$$2 m v_0 \sin\frac{\theta}{2} = \frac{2k}{v_0 b}\cos\frac{\theta}{2}$$

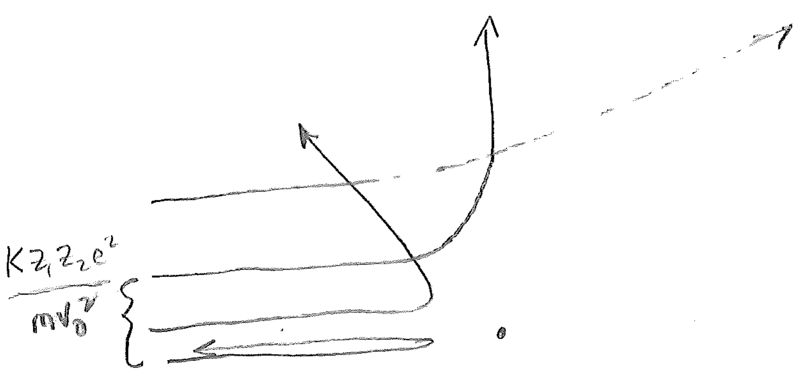
$$\Rightarrow \quad b = \frac{k}{m v_0^2}\cot\frac{\theta}{2}$$

$$\boxed{b = \frac{K Z_1 Z_2 e^2}{m v_0^2}\cot\frac{\theta}{2}}$$



$$\theta = \pi \Rightarrow \cot\frac{\theta}{2} = \frac{\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} = 0$$

$$\theta = \frac{\pi}{2} \Rightarrow \cot\left(\frac{\pi}{4}\right) = \frac{\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})} = 1$$



Scattering cross section

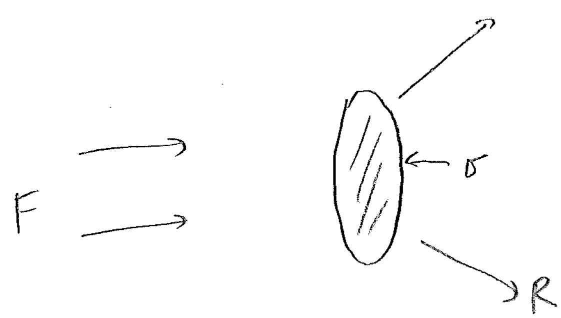
Let F = incident flux of α-particles = $\dfrac{\text{\# incident pcls}}{\text{sec} \cdot \text{area}}$

Let R = scattering rate of α-particles $\dfrac{\text{\# pcls scattered}}{\text{sec}}$

Expect R to be proportional to F

Define $\boxed{\sigma = \dfrac{R}{F}}$

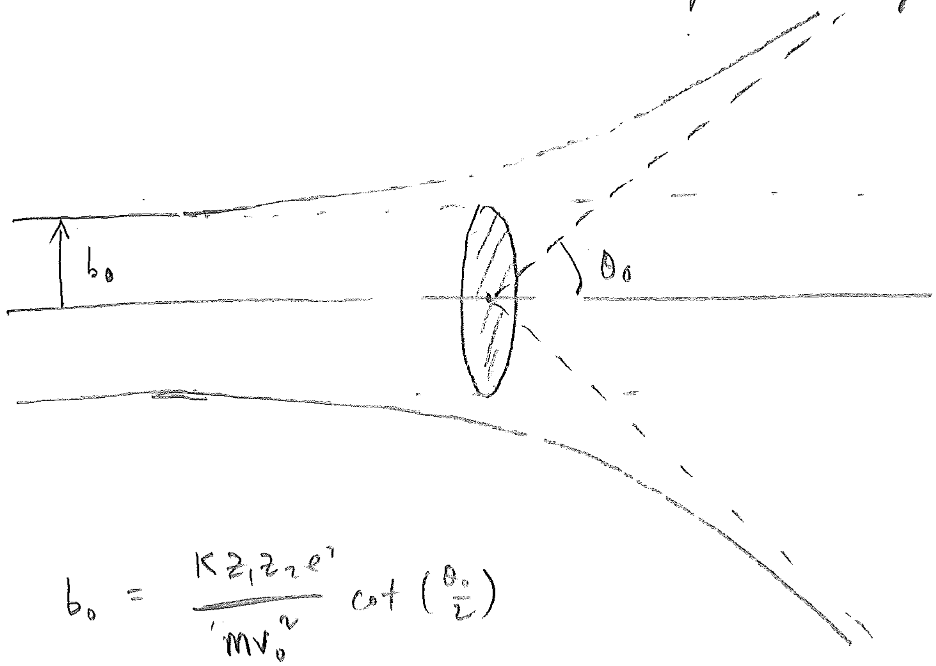
σ has units of area; it is the area of the scatterer incident flux intercepted by the scatterer



$$R = F\sigma$$

σ is called the scattering cross-section

Define $\sigma(\theta > \theta_0)$ = cross-section for scattering thru angle $> \theta_0$



$$b_0 = \frac{KZ_1Z_2e^2}{mv_0^2}\cot\left(\frac{\theta_0}{2}\right)$$

If $b < b_0$ then $\theta > \theta_0$

Any α particle that would have struck a disk of radius b_0 [had it not been deflected,] will be scattered through $\underline{\theta > \theta_0}$

$$\sigma(\theta > \theta_0) = \pi b_0^2$$

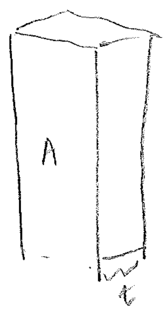
Total cross-section for scattering $\sigma_{tot} = \sigma(\theta > 0)$

is technically infinite because all particles are scattered (deflected) to some extent.

Practically speaking, α particles that pass beyond the electron shells are unscattered (atom is neutral) so effective total cross-sect is no larger than πR_{atom}^2.

What <u>fraction</u> of all incident α particles will be scattered through an angle $> \theta_0$?

Consider a thin foil of thickness t and area A.



Let n = number density of nuclei

Total # of nuclei in foil $N = nAt$

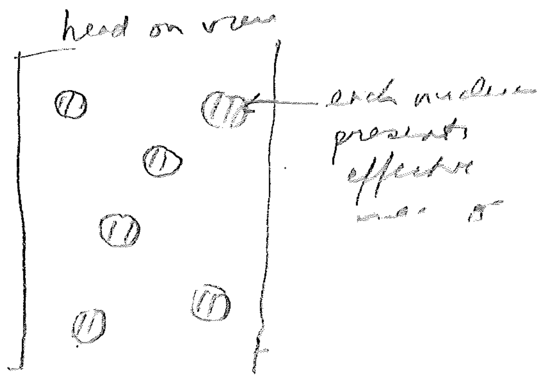
Each nucleus presents a target of size $\sigma(\theta > \theta_0)$.

Total target area $= N\sigma$

Total area $= A$

Fraction occupied by targets

$$f = \frac{N\sigma}{A} = nt\sigma$$



[This is the fraction of α-particles scattered.

$\boxed{\text{fraction scattered } f = nt\sigma}$

[Note: advantage of <u>thin</u> foil is to eliminate possibility of multiple interactions, so each α particle scatters only once

Fraction of α-particles scattered through $\theta > \theta_0$

$$f = nt\,\sigma(\theta>\theta_0) = nt\,\pi b_0^2 \quad \text{where} \quad b_0 = \frac{Z_1 Z_2 K e^2}{m v_0^2} \cot\left(\frac{\theta_0}{2}\right)$$

Let's evaluate this..

- The strength of the electromagnetic force is characterized by the dimensionless fine structure constant, defined by

$$\boxed{\alpha = \frac{Ke^2}{\hbar c} \approx \frac{1}{137}}$$

- A very useful constant to know is

$$\left[\frac{197.327}{137.036} = 1.43996\right]$$

$$\boxed{\hbar c = 197 \text{ MeV}\cdot\text{fm}}$$

- Therefore $Ke^2 = \alpha\hbar c = \frac{197}{137}$ MeV fm $\approx \boxed{1.44 \text{ MeV fm} = Ke^2}$

- Consider α-particle of kinetic energy $T_0 = 5$ MeV incident on gold foil 0.5 microns thick. What fraction is back scattered, i.e. scattered thru $\theta > 90°$?

nonrel. $T_0 = \frac{1}{2}mv_0^2 \Rightarrow mv_0^2 = 10$ MeV

$Z_1 = 2, \quad Z_2 = 79, \quad \theta_0 = \frac{\pi}{2}$

$$b_0 = \frac{2\cdot(79)(1.44 \text{ MeV fm})}{(10 \text{ MeV})} = 22.7 \text{ fm} \qquad [22.75]$$

$t = 5\times10^{-7}$ m, $\quad n = 5.9\times10^{28}$ m^{-3}

$$f = nt\pi b_0^2 \approx 5\times10^{-5} \approx \frac{1}{20{,}000}$$

Probability is small (1 in 20,000) but measurable

[Geiger & Marsden, 1 in 8,000 (Evans, p 2)]

Griffiths, Quantum Mechanics, 2nd ed.

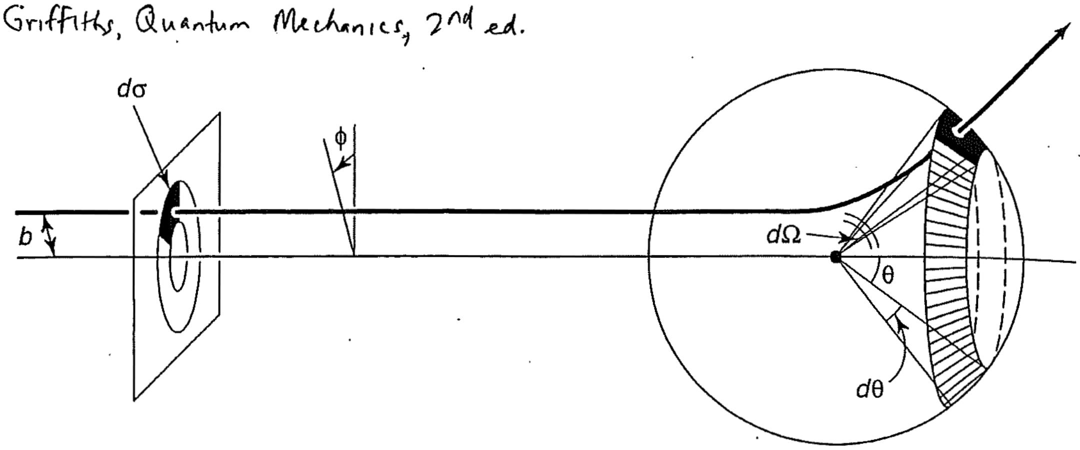
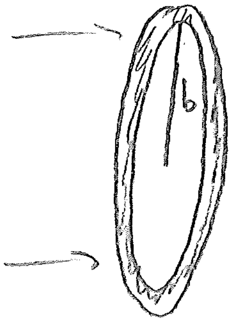


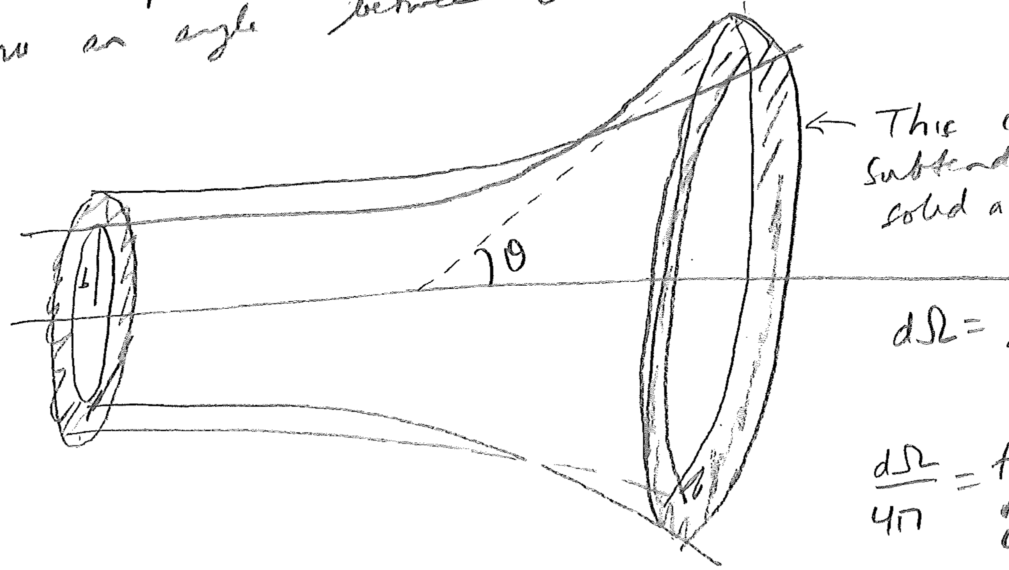
FIGURE 11.3: Particles incident in the area $d\sigma$ scatter into the solid angle $d\Omega$.



Consider incident particles of impact parameter between b and $b + db$

$d\sigma$ = area of annulus = $2\pi b \, db$

All particles passing through this annulus are scattered thru an angle between θ and $\theta + d\theta$



← This collar subtends solid angle

$d\Omega = 2\pi \sin\theta \, d\theta$

$\frac{d\Omega}{4\pi}$ = fraction of area of unit sphere

Define differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b \, db}{2\pi \sin\theta \, d\theta}$$

[absolute value because $\theta \downarrow$ as $b \uparrow$]

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta} \left| \frac{db}{d\theta} \right|}$$

← classical differential cross section

Then $\sigma(\theta > \theta_0) = \int d\sigma = \int \frac{d\sigma}{d\Omega} \, d\Omega$

$$= \int_{\theta=\theta_0}^{\theta=\pi} \frac{d\sigma}{d\Omega} \, 2\pi \sin\theta \, d\theta$$

Total cross section $\sigma = \sigma(\theta > 0) = \int_0^\pi \frac{d\sigma}{d\Omega} \, 2\pi \sin\theta \, d\theta$

For Rutherford scattering, $b = B \cot\left(\frac{\theta}{2}\right)$, $B = \frac{Z_1 Z_2 K e^2}{m v_0^2}$

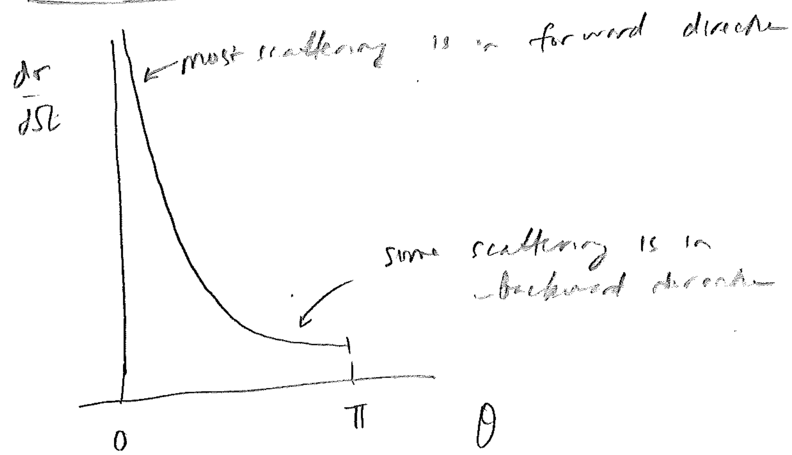
$$\left|\frac{db}{d\theta}\right| = \frac{1}{2} B \csc^2\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} \quad \frac{1}{2} B^2 \frac{\cot\left(\frac{\theta}{2}\right)}{\sin\theta} \csc^2\left(\frac{\theta}{2}\right)$$

$$\sin\theta = 2 \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \quad \cancel{\text{so}}$$

$$\frac{d\sigma}{d\Omega} \quad \frac{1}{4} B^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$

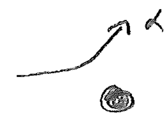
$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{K Z_1 Z_2 e^2}{2 m v_0^2}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)} \quad \text{Rutherford differential cross section}}$$



[HW: show $\sigma(\theta > \theta_0) = \int \frac{d\sigma}{d\Omega} d\Omega = \pi b_0^2$]

Rutherford & colleagues did indeed observe this distribution (w.r.t angular dependence, Z_2, initial v_0)

suggesting that the positive charge of an atom is concentrated in a tiny, point-like nucleus



Suppose nucleus has a radius R.

Then if incident particle comes closer than R, one expects deviations from the Rutherford prediction because:

① nucleus no longer a point charge

② if incident particle is a hadron, the strong force can act.

No deviations for heavy nuclei (w/ $T_\alpha \lesssim 10$ MeV).

Deviations observed for light nuclei.
Also sometimes nuclear transmutation occurs

$$\alpha + {}^{14}_{7}N \longrightarrow {}^{17}_{8}O + p$$

nuclear transmutation

Use deviations from Rutherford scattering to estimate size of nuclei

Various experiments suggested that (e^- scattering)

nuclear radius $R \sim A^{1/3}$

Specifically $R = A^{1/3} r_0$

$r_0 \approx 1.2 \times 10^{-15}$ m $= 1.2$ fm

Thus most nuclei between $1 - 10$ fm

Atomic radii are ~ 1 Å $\approx 10^{-10}$ m, so 4 to 5 orders of magnitude

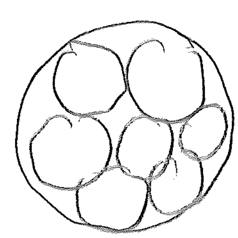
If hydrogen atom were size of this room [~ 10 m]

then size of nucleus is $\sim \underbrace{0.1 \text{ mm}}_{\text{human hair width.}}$ and 1 mm

Volume of nucleus $= \frac{4}{3}\pi R^3 = \left(\frac{4}{3}\pi r_0^3\right) A$

proportional to # nucleons

$\Rightarrow$ nuclei act as a collection of close-packed incompressible nucleons



Bethe + Morrison

size of nuclei determined by

(1) neutron cross-section (for fast but not too fast neutrons)
~20 MeV

treat neutrons geometrically

nucleus becomes transparent

$$\lambda\!\!\!^{-} = \frac{\hbar}{p} = \frac{\hbar c}{\sqrt{2mc^2 T}} = \frac{200}{\sqrt{2\cdot(20)(940)}} \cdot \quad 1 \text{ fm}$$

(2) α-decay lifetime

(3) nuclear σ for chgd ptls (must traverse barrier)

(4) mirror nuclei

^3H, ^3He
^7Li, ^7Be
^{11}C, ^{11}B
^{13}C, ^{13}N
^{15}N, ^{15}O
^{17}O, ^{17}F
^{29}Si, ^{29}P

(5) semi-empirical

(6) e^- scattering

(7) μ-mesic systems