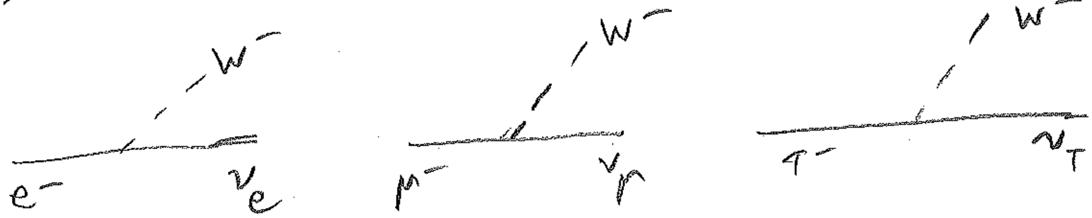
In the standard model L_e, L_μ, L_τ are separately conserved, because the weak interaction vertices conserve them.



Solar neutrino problem suggests neutrino "mixing" which is only possible if neutrinos have mass.

let ν_1, ν_2, ν_3 be mass eigenstates w/ masses m_1, m_2, m_3

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

3x3 mixing matrix, analogous to CKM matrix.
w/ angles $\theta_{12}, \theta_{13}, \theta_{23}, e^{i\delta}$

Experiments indicate that, unlike CKM angles, these angles are large

$\theta_{12} \sim 34°$ [Griffiths]

$\theta_{23} \sim 45°$

$\theta_{13} < 10°$

[To simplify]

Assume mixing between 2 generations only

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

If ν_1 + ν_2 have different masses, and if $\theta \neq 0$, then neutrino oscillations between ν_e + ν_μ can occur.

Let ν_e be emitted from $x = 0$ at $t = 0$

$$\psi(0,0) = \nu_e = \nu_1 \cos\theta + \nu_2 \sin\theta$$

Let it travel to $x = L$ at $t = T$.
Each mass component picks up a phase (to be calculated later)

$$\psi(L,T) = \nu_1 \cos\theta \, e^{i\phi_1} + \nu_2 \sin\theta \, e^{i\phi_2}$$

What is the probability that the neutrino will be detected as ν_μ?

$$\nu_\mu = -\nu_1 \sin\theta + \nu_2 \cos\theta$$

Amplitude

$$\langle \nu_\mu | \psi(L,T) \rangle = -\sin\theta \underbrace{\langle \nu_1 | \psi(L,T) \rangle}_{\cos\theta \, e^{i\phi_1}} + \cos\theta \underbrace{\langle \nu_2 | \psi(L,T) \rangle}_{\sin\theta \, e^{i\phi_2}}$$

$$= \sin\theta\cos\theta \left[e^{i\phi_2} - e^{i\phi_1} \right]$$

$$\langle \nu_\mu | \Psi(L,T)\rangle = \sin\theta\cos\theta\, e^{i\left(\frac{\phi_1+\phi_2}{2}\right)} \underbrace{\left[e^{i\frac{(\phi_2-\phi_1)}{2}} - e^{-i\frac{(\phi_2-\phi_1)}{2}} \right]}_{2i\sin\left(\frac{\Delta\phi}{2}\right)}$$

$$= i e^{i\frac{(\phi_1+\phi_2)}{2}} \sin(2\theta)\sin\left(\frac{\Delta\phi}{2}\right)$$

$$\Delta\phi = \phi_2 - \phi_1$$

Probability of detecting a ν_μ:

$$P_{\nu_\mu} = |\langle \nu_\mu | \Psi(L,T)\rangle|^2 = \sin^2(2\theta)\sin^2\left(\frac{\Delta\phi}{2}\right)$$

A free particle has wavefunction $e^{i\left(\frac{px}{\hbar} - \frac{Et}{\hbar}\right)}$ so the phase is $\phi = \frac{pL - ET}{\hbar}$

A wavepacket is composed of superposition of momenta.

The wavefunction at $x = L$ and $t = T$ is dominated by momenta such that $v = \frac{L}{T}$

Relativistically $v = \frac{c^2 p}{E} = \frac{c^2 p}{\sqrt{(cp)^2 + (mc^2)^2}} = \frac{L}{T}$

Solve for $p = \frac{mc\,L}{\sqrt{(cT)^2 - L^2}} \Rightarrow E = \frac{mc^3 T}{\sqrt{(cT)^2 - L^2}}$

$$\Rightarrow \phi = \frac{pL - ET}{\hbar} = \frac{mc\,L^2 - mc^3T^2}{\hbar\sqrt{(cT)^2 - L^2}} = -\frac{mc}{\hbar}\sqrt{(cT)^2 - L^2}$$

[if $m = 0$, $\phi = 0$ because group velocity = phase velocity]

$$\Delta\phi = -\frac{(\Delta m)\,c}{\hbar}\sqrt{(cT)^2 - L^2} \qquad \Delta m = m_2 - m_1$$

If $\Delta m = 0$, then $\Delta\phi = 0 \Rightarrow$ no oscillation.

Let's now eliminate T from this

$$(cT)^2 - L^2 = \left(\frac{cL}{v}\right)^2 - L^2 = L^2\,\frac{c^2 - v^2}{v^2}$$

$$= L^2\,\frac{\left(1 - \left(\frac{cp}{E}\right)^2\right)}{\left(\frac{cp}{E}\right)^2} = L^2\,\frac{E^2 - (cp)^2}{(cp)^2}$$

The p + E for each mass component are slightly different but let's take an average $\Rightarrow$ $E^2 - (cp)^2 = (mc^2)^2$

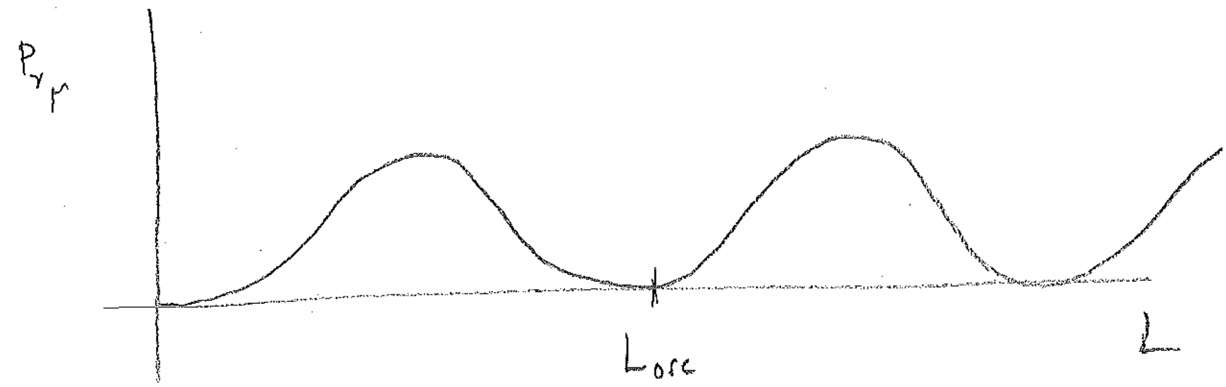
$$\Rightarrow \sqrt{(cT)^2 - L^2} = \frac{mc}{p}L$$

$$\Delta\phi = -\frac{(\Delta m)\,m\,c^2\,L}{\hbar\,p}$$

$$m\,\Delta m = \frac{1}{2}(m_1 + m_2)(m_2 - m_1) = \frac{1}{2}(m_2^2 - m_1^2) = \frac{1}{2}\Delta m^2$$

$$\Delta\phi = -\frac{1}{2}\,\frac{\Delta m^2\,c^2\,L}{\hbar p}$$

$$\boxed{P_{\nu_\mu} = \sin^2(2\theta)\ \sin^2\left(\frac{(\Delta m^2)\, c^2 L}{4\hbar p}\right)}$$



$$\frac{(\Delta m^2)\, c^2 L_{osc}}{4\hbar p} = \pi \quad \Rightarrow \quad \boxed{L_{osc} = \frac{4\pi\hbar p}{(\Delta m^2)\, c^2}}$$

Prob. of obtaining ν_μ is maximised at $\frac{1}{2} L_{osc}$

Solve for mass difference:

$$(\Delta m^2)\, c^4 = \frac{4\pi\hbar c\ cp}{L_{osc}} = \frac{(2.5 \times 10^{-6}\ \text{m} \cdot \text{eV})\,(cp)}{L_{osc}}$$

Suppose there is significant oscillation between Sun & earth

$L_{osc} < 10^{11}\,\text{m}$, $cp = 1\ \text{MeV}$ $\Rightarrow$ $\Delta(mc^2)^2 > 2.5 \times 10^{-11}\ (\text{eV})^2$

If $m_1 = 0$ then $m_2 c^2 > 5 \times 10^{-6}\ \text{eV}$

Background on massive neutrinos

Wave-packet spread

$$\psi(x,t) = \int dk \;\alpha(k)\; e^{i\Theta(k,x,t)}$$

cf Eckart 1948 on stationary phase

$$\Theta(k,x,t) = kx - \omega(k)t$$

Stationary phase occurs when

$$\frac{d\Theta}{dk} = 0 = x - \frac{d\omega}{dk}t$$

Given x, t, solve this for k: $\left.\frac{d\omega}{dk}\right|_{k=K} = \frac{x}{t}$

Then $\psi(x,t)$ will be significant if $\alpha(K) \neq 0$, that is if wavepacket contains components K such that the group velocity $V_g(K) = \frac{d\omega}{dk}(K)$ is such as to travel to x at time t.

This is essentially the point of my 3140 problem to find the "classical" spread of the wavepacket

If $\alpha(k) \sim e^{-\left[\frac{k-k_0}{\Delta k}\right]^2}$ then the spread of speeds is

$$\frac{d\omega}{dk}(k_0 \pm \Delta k) = \frac{d\omega}{dk}(k_0) \pm \frac{d^2\omega}{dk^2}\Delta k$$ so the wavepacket

Spread is $\Delta x \sim \frac{d^2\omega}{dk^2}\Delta k \cdot t$

or just use $v = \frac{c^2 p}{E}$

Let $\omega(k) = \sqrt{k^2+m^2}$ (note $\frac{\omega(k)}{k} > 1$!)

$$\frac{d\omega}{dk} = \frac{k}{\sqrt{k^2+m^2}} = \left(1+\frac{m^2}{k^2}\right)^{-\frac{1}{2}} \simeq 1 - \frac{m^2}{2k^2}$$

$$\frac{d^2\omega}{dk^2} = -\frac{1}{2}\left(1+\frac{m^2}{k^2}\right)^{-3/2}\left(-\frac{2m^2}{k^3}\right) \simeq \frac{m^2}{k^3}\left(1+\frac{m^2}{k^2}\right)^{-3/2}$$

$$\frac{d\omega}{dm} = \frac{m}{\sqrt{k^2+m^2}} \approx \frac{m}{k}$$

$\therefore$ a wavepacket w/ central momentum k_0 & spread Δk will travel to $x = \left(1-\frac{m^2}{2k_0^2}\right)t$ w/ spread $\Delta x \sim \frac{m^2}{k_0^3}\Delta k\, t$

If wave has different mass components m_1 + m_2 then the components will be separated by $\frac{m_1^2-m_2^2}{2k_0^2}t$

If $\frac{\Delta k}{k_0} \gg \frac{m_1^2-m_2^2}{2m^2}$ then wavepackets will still overlap

otherwise they will separate

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we'll assume this, so that their phases can interfere

Consider a fixed point x.

At time t, we expect waves of momentum k to be passing through, where $\frac{d\omega}{dk} = \left(1+\frac{m^2}{k^2}\right)^{-1/2} = \frac{x}{t}$

$$\Rightarrow k = \frac{mx}{\sqrt{t^2-x^2}} \qquad \Rightarrow \omega = \frac{mt}{\sqrt{t^2-x^2}}$$

The (stationary!) phase of these waves is

$$kx - \omega(k)t = \frac{mx^2}{\sqrt{t^2-x^2}} - \frac{mt^2}{\sqrt{t^2-x^2}} = -m\sqrt{t^2-x^2}$$

If two mass components are present, the difference in phases is $+\Delta m\sqrt{t^2-x^2}$

Finally, $t = \left(1+\frac{m^2}{k^2}\right)^{1/2} x$

$$t^2 - x^2 = x^2\left[1+\frac{m^2}{k^2}\right] - x^2 = \frac{x^2m^2}{k^2}$$

$$\sqrt{t^2-x^2} = \frac{mx}{k}$$

$$\text{so} \quad \Delta\phi = \frac{(\Delta m)\, m x}{k} \approx \frac{1}{2}\frac{\Delta m^2}{k}x$$

Or exactly $\frac{x}{t} = v = \frac{c^2 p}{E} \Rightarrow (ct)^2 - x^2 = x^2\frac{(E^2-(cp)^2)}{(cp)^2} = x^2\frac{(mc^2)^2}{(cp)^2}$

$$\sqrt{(ct)^2-x^2} = x\,\frac{cm}{p}$$

old notes

Assume mixing between only 1st + 2nd generations (for simplicity)

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$$

mass eigenstates w/ masses m_1 + m_2

weak interaction eigenstates, i.e. ν_e couples to e^- + ν_μ " to μ^- (lepton # eigenstates)

If ν_e is emitted at the origin, what fraction will have been converted to ν_μ after a distance x?

$$\psi(0,0) = \nu_e = (\cos\theta)\,\nu_1 + (\sin\theta)\,\nu_2$$

For an energy (mass) eigenstate $\psi(x,t) = \psi(0,0)\, e^{\frac{ipx}{\hbar} - \frac{iEt}{\hbar}}$

(Nearly) massless particles travel x in time $t \approx \frac{x}{c}$

$$\psi(x,t) = \psi(0,0)\, e^{\frac{i}{\hbar}\left(p - \frac{E}{c}\right)x}$$

$$E = \sqrt{(cp)^2 + (mc^2)^2} = cp\sqrt{1 + \left(\frac{mc}{p}\right)^2}$$

$$\approx cp\left(1 + \frac{1}{2}\left(\frac{mc}{p}\right)^2\right) = c\left(p + \frac{m^2c^2}{2p}\right)$$

$$\psi(x,t) \approx \psi(0,0)\, e^{-\frac{i\, m^2 c^2}{2\hbar p}x}$$

$$\Psi(x,t) = (\cos\theta)\,\nu_1\, e^{-\frac{i m_1^2 c^2}{2\hbar p}x} + (\sin\theta)\,\nu_2\, e^{-\frac{i m_2^2 c^2}{2\hbar p}x}$$

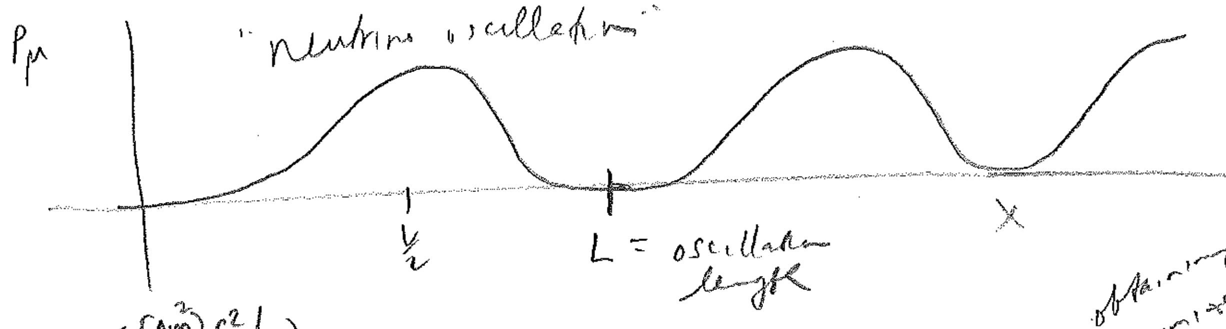
Now $\nu_\mu = -\sin\theta\,\nu_1 + \cos\theta\,\nu_2$ As amplitude that the ν is a μ-neutrino is

$$\langle \nu_\mu | \Psi\rangle = -\sin\theta\,\langle \nu_1|\Psi\rangle + \cos\theta\,\langle\nu_2|\Psi\rangle$$

$$= -\sin\theta\cos\theta\,\Big[\underbrace{e^{-\frac{i m_1^2 c^2}{2\hbar p}x} - e^{-\frac{i m_2^2 c^2}{2\hbar p}x}}_{e^{-\frac{i(m_1^2+m_2^2)c^2 x}{4\hbar p}}\cdot 2i\sin\left(\frac{(m_2^2-m_1^2)c^2 x}{4\hbar p}\right)}\Big]$$

$$P_\mu = |\langle\nu_\mu|\Psi\rangle|^2 = \underbrace{4\sin^2\theta\cos^2\theta}_{(\sin 2\theta)^2}\,\sin^2\left(\frac{(\Delta m^2)c^2 x}{4\hbar p}\right)$$

↑ oscillation depends on differences of m^2



$$\sin^2\left(\frac{(\Delta m^2)c^2 L}{4\hbar p}\right) = 0$$

$$\Rightarrow \frac{(\Delta m^2)c^2 L}{4\hbar p} = \pi \quad \Rightarrow \quad L = \frac{4\pi\hbar p}{(\Delta m^2)c^2}$$

Prob of obtaining ν_μ maximized at $\frac{L}{2}$.