(5-29-19) Review of t.d.p.t. approach to potential scattering

~~(1-3-19)~~

Background: time-dep. pert. th. + Fermi golden rule

$$i\hbar \frac{\partial U}{\partial t} = \left[H_0 + V(t)\right] U \qquad \text{where } |\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle$$

$$\Rightarrow \quad U(t,t_0) = U_0(t,t_0) - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, U_0(t,t')\, V(t')\, U(t',t_0)$$

where $U_0(t,t_0) = e^{-\frac{iH_0}{\hbar}(t-t')}$

Define $U_I(t,t_0) = U_0(0,t)\, U(t,t_0)\, U_0(t_0,0)$

$$\Rightarrow U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, U_0(0,t')\, V(t')\, U_0(t',0)\, U_I(t',t_0)$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, U_0(0,t')\, V(t')\, U_0(t',0) + \dots$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, e^{\frac{iH_0 t'}{\hbar}}\, V(t')\, e^{-\frac{iH_0 t'}{\hbar}} + \dots$$

Energy basis

$$\langle E_f|\, U_I(t,t_0)\,|E_i\rangle = \langle E_f|E_i\rangle - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, e^{\frac{i(E_f-E_i)t'}{\hbar}} \langle E_f|V|E_i\rangle + \dots$$

$$\langle E_f|\, S\,|E_i\rangle = \delta_{fi} - \frac{i}{\hbar}\int_{-\infty}^{\infty} dt'\, e^{\frac{i(E_f-E_i)t'}{\hbar}} \langle E_f|V|E_i\rangle + \dots$$

$S \equiv U_I(\infty,-\infty)$

$$= \delta_{fi} - 2\pi i\, \delta(E_f - E_i)\langle E_f|V|E_i\rangle + \dots$$

$$|\langle E_f|S|E_i\rangle|^2 \underset{(f\neq i)}{=} 2\pi\delta(E_f-E_i)\,\underbrace{2\pi\delta(E_f-E_i)}_{\underbrace{\int \frac{dt'}{\hbar} e^{\frac{i(E_f-E_i)t'}{\hbar}}}_{\frac{t}{\hbar}}}\,|\langle E_f|V|E_i\rangle|^2$$

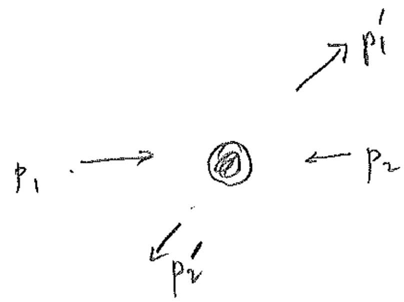
$$\text{Rate} = \frac{d}{dt}|\langle E_f|S|E_i\rangle|^2 = \frac{2\pi}{\hbar}\delta(E_f-E_i)\,|\langle E_f|V|E_i\rangle|^2$$

(5-29-19)

$2 \to 2$ potential scattering

[cf Holstein, p. 73ff]

This can also be used for single particle scattering by treating the other particle as infinitely heavy



The transition element is

$$\langle E_f | V | E_i \rangle = \langle p_1' p_2' | V | p_1 p_2 \rangle \qquad \langle x | p \rangle = \frac{1}{L^{3/2}} e^{\frac{i \vec{p} \cdot \vec{x}}{\hbar}}$$

$$= \int d^3x_1 d^3x_2 \langle p_1' p_2' | x_1 x_2 \rangle V(x_1 - x_2) \langle x_1 x_2 | p_1 p_2 \rangle$$

$$= \frac{1}{L^6} \int d^3x_1 d^3x_2 \, e^{\frac{i}{\hbar}(\vec{p}_1 - \vec{p}_1') \cdot \vec{x}_1 + \frac{i}{\hbar}(\vec{p}_2 - \vec{p}_2') \cdot \vec{x}_2} \, V(x_1 - x_2)$$

Transform to cm coordinates

$$\vec{x}_1 = \vec{x}_{cm} + \frac{m_2}{M} \vec{x} \qquad \vec{P} = \vec{p}_1 + \vec{p}_2 \qquad \left| \frac{\partial(\vec{x}_1, \vec{x}_2)}{\partial(\vec{x}_{cm}, \vec{x})} \right| = 1$$

$$\vec{x}_2 = \vec{x}_{cm} - \frac{m_1}{M} \vec{x} \qquad \vec{p} = \frac{m_2}{M} \vec{p}_1 - \frac{m_1}{M} \vec{p}_2$$

$$\langle E_f | V | E_i \rangle = \frac{1}{L^6} \int d^3x_{cm} \, e^{\frac{i}{\hbar}(\vec{p}_1 + \vec{p}_2 - \vec{p}_1' - \vec{p}_2') \cdot \vec{x}_{cm}} \int d^3x \, e^{\frac{i}{\hbar M}[m_2(\vec{p}_1 - \vec{p}_1') - m_1(\vec{p}_2 - \vec{p}_2')] \cdot \vec{x}} \, V(x)$$

$$= \frac{1}{L^6} \underbrace{\int d^3x_{cm} \, e^{\frac{i}{\hbar}(\vec{P} - \vec{P}') \cdot \vec{x}_{cm}}}_{\text{enforces 3-momentum cons.}} \; \underbrace{\int d^3x \, e^{\frac{i}{\hbar}(\vec{p} - \vec{p}')} \, V(x)}_{\equiv \tilde{V}(\vec{q}) \text{ where } \vec{q} = \vec{p} - \vec{p}'}$$

Scattering rate and cross-section for 2→2 potential scattering

$$\langle E_f|V|E_i\rangle = \frac{1}{L^6}\int d^3x_{cm}\, e^{\frac{i}{\hbar}(\vec{P}_f-\vec{P}_i)\cdot\vec{x}_{cm}}\,\tilde{V}(\vec{q})$$

$$|\langle E_f|V|E_i\rangle|^2 = \frac{1}{L^{12}}\underbrace{\int d^3x_{cm}\, e^{\frac{i}{\hbar}(\vec{P}_f-\vec{P}_i)\cdot\vec{x}_{cm}}}_{(2\pi\hbar)^3\,\delta(\vec{P}_f-\vec{P}_i)}\underbrace{\int d^3x_{cm}}_{L^3}\underbrace{e^{\frac{i}{\hbar}(\vec{P}_f-\vec{P}_i)\cdot\vec{x}_{cm}}}_{1}\,|\tilde{V}(\vec{q})|^2$$

$$= \frac{(2\pi\hbar)^3}{L^9}\,\delta(\vec{P}_f-\vec{P}_i)\,|\tilde{V}(\vec{q})|^2$$

Integrate the Fermi golden rule over final state phase space

$$R = \left(\frac{L}{2\pi\hbar}\right)^6\int d^3\vec{p}_1{}'\, d^3\vec{p}_2{}'\,\frac{2\pi}{\hbar}\,\delta(cP_f^0-cP_i^0)\,\frac{(2\pi\hbar)^3}{L^9}\,\delta(\vec{P}_f-\vec{P}_i)\,|\tilde{V}(\vec{q})|^2$$

$$= \frac{1}{L^3}\frac{1}{\hbar^4}\frac{1}{c}\int\frac{d^3\vec{p}_1{}'}{(2\pi)^3}\frac{d^3\vec{p}_2{}'}{(2\pi)^3}\,(2\pi)^4\,\delta(P_f^\mu-P_i^\mu)\,|\tilde{V}(\vec{q})|^2$$

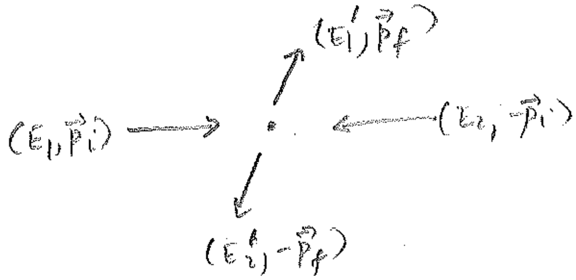
Incident flux $F = \dfrac{v_{ri}}{L^3}$ where $v_{ri} = |\vec{v}_1-\vec{v}_2|$ = "relative velocity"

$$\sigma = \frac{R}{F} = \frac{1}{\hbar^4 c\, v_{ri}}\int\frac{d^3\vec{p}_1{}'}{(2\pi)^3}\frac{d^3\vec{p}_2{}'}{(2\pi)^3}\,(2\pi)^4\,\delta(P_f^\mu-P_i^\mu)\,|\tilde{V}(\vec{q})|^2$$

Independent of L, as expected

Henceforth, assume CM frame

$$\vec{p}_i = \vec{p}_1 = -\vec{p}_2$$
$$\vec{p}_f = \vec{p}_1{}' = -\vec{p}_2{}'$$



$$E_0 = E_1 + E_2 = E_1' + E_2'$$
$$= c\sqrt{p^2 + (m_1c)^2} + c\sqrt{p^2 + (m_2c)^2}$$

$$\frac{\partial E_0}{\partial p_f} = \frac{cp}{\sqrt{p_f^2 + (m_1c)^2}} + \frac{cp}{\sqrt{p_f^2 + (m_2c)^2}} = \frac{c^2 p_f}{E_1'} + \frac{c^2 p_f}{E_2'} = \frac{c^2 p_f E_0}{E_1' E_2'}$$

But $v_1' = \frac{c^2 p_f}{E_1'}$ etc so

$$\frac{\partial E_0}{\partial p_f} = v_1' + v_2' = v_{rf}$$ since $v_{rf} = |\vec{v}_1{}' - \vec{v}_2{}'| = v_1' + v_2'$ since $\vec{v}_1{}', \vec{v}_2{}'$ are antiparallel

From this we also see that $v_{ri} = v_1 + v_2 = \frac{c^2 p_i}{E_1} + \frac{c^2 p_i}{E_2} = \frac{c^2 p_i E_0}{E_1 E_2}$

$$\sigma = \frac{E_1 E_2}{\hbar^4 c^3 E_0 p_i} \int \frac{d^3\vec{p}_1{}'}{(2\pi)^3} \frac{d^3\vec{p}_2{}'}{(2\pi)^3} (2\pi)^4 \delta(p_f^\mu - p_i^\mu) \left|\tilde{V}(\vec{q})\right|^2$$

Use the $\delta^{(3)}(\vec{P}_f - \vec{P}_i)$ to eliminate $\vec{p}_2{}'$

$$\sigma = \frac{1}{\hbar^4 v_{ri}} \frac{1}{(2\pi)^2} \int d^3\vec{p}_1{}' \, \delta(\underbrace{E_f}_{E_0 - E_1 - E_2} - E_i) \left| \tilde{V}(q) \right|^2$$

$$\int d^3 \vec{p}_1{}' = \int d\Omega \, p_f^2 \frac{\partial p_f}{\partial E_0} dE_0 = \int d\Omega \frac{p_f^2}{v_{rf}} dE_0$$

$$\sigma = \frac{1}{(2\pi)^2} \frac{1}{\hbar^4 v_{ri}} \int d\Omega \frac{p_f^2}{v_{rf}} dE_0 \, \delta(E_0 - E_1 - E_2) \left| \tilde{V}(q) \right|^2$$

$$= \frac{1}{(2\pi)^2} \frac{p_f^2}{\hbar^4 v_{ri} v_{rf}} \int d\Omega \left| \tilde{V}(q) \right|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2} \frac{p_f^2}{\hbar^4 v_{ri} v_{rf}} \left| \tilde{V}(q) \right|^2$$

Can use $v_r = \dfrac{c^2 E_0 p}{E_1 E_2}$ to write this as

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2} \frac{E_1 E_2 E_1{}' E_2{}'}{(\hbar c)^4 E_0^2} \frac{p_f}{p_i} \left| \tilde{V}(q) \right|^2$$

Specializations

If scattering is elastic then $E_1' = E_1$, $E_2' = E_2$, $p_f = p_i$, $v_{ri} = v_{rf}$

$$\sigma = \frac{1}{(2\pi)^2} \frac{p^2}{\hbar^4 v_r^2} \left| \tilde{V}(q) \right|^2$$

$$= \frac{1}{(2\pi)^2} \frac{E_1^2 E_2^2}{\hbar^4 c^4 (E_1+E_2)^2} \left| \tilde{V}(q) \right|^2$$

If target is much heavier than projectile ($m_2 \gg m_1$)
then $v_r \approx v_1$ and $E_2 \gg E_1$

$$\sigma = \frac{1}{(2\pi)^2} \frac{p^2}{\hbar^4 v_1^2} \left| \tilde{V}(q) \right|^2$$

$$= \frac{1}{(2\pi)^2} \frac{E_1^2}{\hbar^4 c^4} \left| \tilde{V}(q) \right|^2$$

Finally, if projectile is non relativistic $p = m_1 v_1$, $E_1 = m_1 c^2$

$$\sigma = \left(\frac{m_1}{2\pi\hbar^2} \right)^2 \left| \tilde{V}(q) \right|^2$$

Coulomb scattering

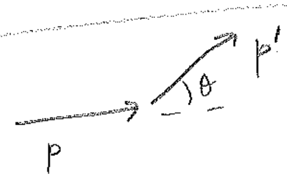
$$V(x) = \frac{Z_1 Z_2 K e^2}{x}$$

$$\tilde{V}(\vec{q}) = \int d^3x \; e^{\frac{i\vec{q}\cdot\vec{x}}{\hbar}} \; V(x)$$

$$= 4\pi \frac{Z_1 Z_2 K e^2}{\left(\frac{q}{\hbar}\right)^2}$$

$$= \hbar^2 (4\pi K) \frac{Z_1 Z_2 e^2}{q^2}$$

In cm frame



$$q^2 = \left(2p \sin\frac{\theta}{2}\right)^2$$

Coulomb scattering is elastic so

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2} \frac{p^2}{\hbar^4 v_r^2} \left|\tilde{V}(q)\right|^2$$

$$= \left[\frac{p}{2\pi v_r} \frac{4\pi K \, Z Z_2 e^2}{\left(2p \sin\frac{\theta}{2}\right)^2} \right]^2$$

$$= \left[\frac{Z_1 Z_2 K e^2}{2 p v_r \sin^2\frac{\theta}{2}} \right]^2 = \text{classical cross-section}$$

Return to expression on p. 4

$$\sigma = \frac{E_1 E_2}{\hbar^4 c^3 \; E_0 \, p_i} \int \frac{d^3\vec{p}_1{}'}{(2\pi)^3} \frac{d^3\vec{p}_2{}'}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(p_f^\mu - p_i^\mu) \, |\tilde{V}(q)|^2$$

Griffiths (2e.) (6.38) writes

$$\sigma = \underbrace{\frac{\hbar^2}{4\sqrt{(p_1\cdot p_2)^2 - (m_1 m_2 c^2)^2}}}_{\text{dimns of area}} \underbrace{\int \frac{d^3\vec{p}_1{}'}{(2\pi)^3 \, 2p_1^{0\prime}} \frac{d^3\vec{p}_2{}'}{(2\pi)^3 \, 2p_2^{0\prime}} (2\pi)^4 \delta^{(4)}(p_f^\mu - p_i^\mu)}_{\text{dimensionless}} \underbrace{|\mathcal{M}|^2}_{\text{dimensionless}}$$

$$\frac{\hbar^2 c}{4 E_0 \, p_i} \text{ in cm frame (eq 6.41)}$$

$$= \frac{\hbar^2 c^3}{16 E_1' E_2' \; E_0 \, p_i} \int \frac{d^3\vec{p}_1{}'}{(2\pi)^3} \frac{d^3\vec{p}_2{}'}{(2\pi)^3} (2\pi)^4 \delta^{(4)}(p_f^\mu - p_i^\mu) \, |\mathcal{M}|^2$$

Comparing these we have

$$|\mathcal{M}|^2 = \frac{(2E_1)(2E_2)(2E_1')(2E_2')}{(\hbar c)^6} |\tilde{V}(q)|^2$$

Expression on p. 5 becomes

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{\hbar c}{E_0}\right)^2 \frac{p_f}{p_i} |\mathcal{M}|^2$$

For Coulomb scattering (elastic $\Rightarrow E_1' = E_1,\ E_2' = E_2$)

$$|\mathcal{M}| = \frac{(2E_1)(2E_2)}{(\hbar c)^3} |\tilde{V}(\vec{q})|$$

$$= \frac{(2E_1)(2E_2)}{c^2(\hbar^3 c)} \frac{\hbar^2 (4\pi K) Z_1 Z_2 e^2}{q^2}$$

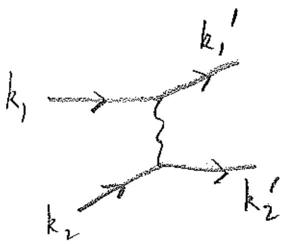
$$= (2p_1^0)(2p_2^0) \frac{4\pi K e^2}{\hbar c} \frac{Z_1 Z_2}{q^2}$$

$$= (2p_1^0)(2p_2^0)\ 4\pi\alpha\ \frac{Z_1 Z_2}{q^2} \quad = \text{dimensionless!}$$

See next page how factor of $(2E_1)(2E_2)$ arises from Feynman rules for scalar particles coupling to a photon

(1-21-19) Rutherford scattering in cm frame

Relativistic vs nonrelativistic approx



$$Z_1 Z_2 e^2 \frac{(k_1+k_1')\cdot(k_2+k_2')}{(k_1-k_1')^2}$$

[e^2 here means $\frac{(4\pi K)e^2}{\hbar c} = 4\pi\alpha$]

Treat particles as charged scalars

$$k_1 + k_2 = k_1' + k_2'$$

$$\Rightarrow (k_1+k_1')\cdot(k_2+k_2') = (k_1+k_1')\cdot(k_1-k_1'+2k_2)$$

$$= \underbrace{k_1^2 - k_1'^2}_{0} + \underbrace{2\,k_1\cdot k_2}_{s-m_1^2-m_2^2} + \underbrace{2\,k_1'\cdot k_2}_{m_1^2+m_2^2-u} = \underline{s-u}$$

$$s = (k_1+k_2)^2 = m_1^2 + m_2^2 + 2k_1\cdot k_2$$

$$u = (k_1'-k_2)^2 = m_1^2 + m_2^2 - 2k_1'\cdot k_2 = m_1^2+m_2^2 - 2k_1\cdot k_2'$$

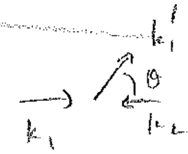
$$t = (k_1-k_1')^2 = 2m_1^2 - 2k_1\cdot k_1'$$

$$s+t+u = 4m_1^2 + 2m_2^2 + 2k_1\cdot(\underbrace{k_2-k_2'-k_1'}_{-k_1}) = 2m_1^2+2m_2^2 \checkmark$$

$$\mathcal{M} = Z_1 Z_2 e^2 \frac{s-u}{t}$$

$$\begin{pmatrix} k_1 = (E_1, 0, 0, p_1) \\ k_2 = (E_2, 0, 0, -p_1) \end{pmatrix}$$

$$k_1' = (E_1, p_1\sin\theta, 0, p_1\cos\theta)$$



$\longrightarrow \longleftarrow$

k_2

$$s = (E_1+E_2)^2$$

$$u = (E_1-E_2)^2 - p_1^2\sin^2\theta - p_1^2(\cos\theta+1)^2$$

$$= (E_1-E_2)^2 - p_1^2 - 2p_1^2\cos\theta - p_1^2$$

$$= (E_1-E_2)^2 - 2p_1^2(\cos\theta+1)$$

$$t = -p_1^2\sin^2\theta - p_1^2(\cos\theta-1)^2$$

$$= -2p_1^2 + 2p_1^2\cos\theta$$

$$= 2p_1^2(\cos\theta-1)$$

$$\frac{s-u}{t} = \frac{(E_1+E_2)^2-(E_1-E_2)^2+2p_1^2(\cos\theta+1)}{2p_1^2(\cos\theta-1)}$$

$$= \frac{4E_1E_2 + 2p_1^2(\cos\theta+1)}{2p^2(\cos\theta-1)}$$

cancels the $\frac{1}{2E_1}\frac{1}{2E_2}$ normalization factors!