(1-3-19) ①

Background: time-dep. pert. th. + Fermi golden rule

$$i\hbar \frac{\partial U}{\partial t} = \left[H_0 + V(t)\right] U \qquad \text{where } |\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$$

$$\Rightarrow U(t,t_0) = U_0(t,t_0) - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, U_0(t,t')\, V(t')\, U(t',t_0)$$

where $U_0(t,t_0) = e^{-\frac{iH_0}{\hbar}(t-t')}$

Define $U_I(t,t_0) = U_0(0,t)\,U(t,t_0)\,U_0(t_0,0)$

$$\Rightarrow U_I(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, U_0(0,t')\, V(t')\, U_0(t',0)\, U_I(t',t_0)$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, U_0(0,t')\, V(t')\, U_0(t',0) + \dots$$

$$= 1 - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, e^{\frac{iH_0 t'}{\hbar}}\, V(t')\, e^{-\frac{iH_0 t'}{\hbar}} + \dots$$

Energy basis

$$\langle E_f|U_I(t,t_0)|E_i\rangle = \langle E_f|E_i\rangle - \frac{i}{\hbar}\int_{t_0}^{t} dt'\, e^{\frac{i(E_f-E_i)t'}{\hbar}}\langle E_f|V|E_i\rangle + \dots$$

$$\langle E_f|\underset{\uparrow}{S}|E_i\rangle = \delta_{fi} - \frac{i}{\hbar}\int_{-\infty}^{\infty} dt'\, e^{\frac{i(E_f-E_i)t'}{\hbar}}\langle E_f|V|E_i\rangle + \dots$$

$$\equiv U_I(\infty,-\infty) \qquad = \delta_{fi} - 2\pi i\, \delta(E_f-E_i)\langle E_f|V|E_i\rangle + \dots$$

$$|\langle E_f|S|E_i\rangle|^2 \underset{(f\neq i)}{\overset{}{=}} 2\pi\delta(E_f-E_i)\, \underbrace{2\pi\delta(E_f-E_i)}_{\underbrace{\int \frac{dt'}{\hbar} e^{\frac{i(E_f-E_i)t'}{\hbar}}}_{\frac{t}{\hbar}}} |\langle E_f|V|E_i\rangle|^2$$

$$\text{Rate} = \frac{d}{dt}|\langle E_f|S|E_i\rangle|^2 = \frac{2\pi}{\hbar}\delta(E_f-E_i)\,|\langle E_f|V|E_i\rangle|^2$$

Background

$$\langle p_f | V | p_i \rangle = \int d^3x \, \langle p_f | x \rangle V(x) \langle x | p_i \rangle$$

$$V(x) = \frac{K Z_1 Z_2 e^2}{r}$$

$$\langle x | p_i \rangle = \frac{1}{L^{3/2}} e^{\frac{i \vec{p}_i \cdot \vec{x}}{\hbar}}$$

$$\langle p_f | V | p_i \rangle = \frac{1}{L^3} \int d^3x \, e^{\frac{i(\vec{p}_i - \vec{p}_f)\cdot \vec{x}}{\hbar}} \frac{K Z_1 Z_2 e^2}{r}$$

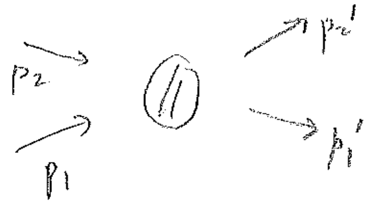
$$= \frac{1}{L^3} \frac{4\pi K Z_1 Z_2 e^2 \hbar^2}{\vec{q}^{\,2}}$$

$$= \frac{\hbar^2}{L^3} (4\pi K) \left(\frac{Z_1 Z_2 e^2}{\vec{q}^{\,2}} \right)$$ ← Feynman diagram

Background

2 → 2 scattering

[cf Holstein, p73-ff]



$$\langle f|V|i\rangle = \langle p_1' p_2'|V|p_1 p_2\rangle$$

$$= \int d^3x_1\, d^3x_2\, \langle p_1' p_2'|x_1 x_2\rangle V(x_1, x_2) \langle x_1 x_2|p_1 p_2\rangle$$

$$= \frac{1}{L^6}\int d^3x_1\, d^3x_2\, e^{\frac{i(\vec{p}_1-\vec{p}_1')\cdot\vec{x}_1}{\hbar}}\, e^{\frac{i(\vec{p}_2-\vec{p}_2')\cdot\vec{x}_2}{\hbar}}\, V(\underbrace{x_1-x_2}_{x})$$

$$\vec{x}_1 = \vec{X}_{cm} + \frac{m_2}{M}\vec{x} \qquad \left|\frac{\partial(x_1\, x_2)}{\partial(x_{cm}, x)}\right| = 1$$
$$\vec{x}_2 = \vec{X}_{cm} - \frac{m_1}{M}\vec{x}$$

$$= \frac{1}{L^6}\underbrace{\int d^3x_{cm}\, e^{\frac{i}{\hbar}(\vec{p}_1+\vec{p}_2-\vec{p}_1'-\vec{p}_2')\cdot\vec{x}_{cm}}}_{(2\pi\hbar)^3\,\delta^{(3)}(\underbrace{p_1+p_2}-p_1'-p_2')}\ \underbrace{\int d^3x\, V(x)\, e^{\frac{i\vec{x}}{M\hbar}\cdot\left[m_2(\vec{p}_1-\vec{p}_1')-m_1(\vec{p}_2-\vec{p}_2')\right]}}_{\underbrace{\int d^3x\, V(x)\, e^{\frac{i\vec{x}}{\hbar}\cdot(\vec{p}-\vec{p}\,')}}_{\hbar^2\frac{(4\pi K)\, Z_1 Z_2 e^2}{(\vec{p}-\vec{p}\,')^2}}}$$

$$\vec{P} - \vec{P}'$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2$$

$$\frac{1}{\mu}\vec{p} \equiv \frac{1}{m_1}\vec{p}_1 - \frac{1}{m_2}\vec{p}_2$$

$$\Rightarrow \vec{p} = \frac{m_2\vec{p}_1}{M} - \frac{m_1\vec{p}_2}{M}$$

if $M_2 = \infty$, $\mu = m_1$ and $\vec{p} = \vec{p}_1$

Background

2→2 scattering

$$R = \left(\frac{L}{h}\right)^3 \int d^3\vec{p}_1{}' \left(\frac{L}{h}\right)^3 \int d^3\vec{p}_2{}' \; \frac{2\pi}{\hbar}\,\delta(\Delta E)\,\left|\langle f|V|i\rangle\right|^2$$

$$\langle f|V|i\rangle = \frac{1}{L^3}\int d^3x_{cm}\, e^{\frac{i}{\hbar}(\vec{P}-\vec{P}')\cdot\vec{x}_{cm}} \underbrace{\left[\frac{1}{L^3}\int d^3x\; V(x)\, e^{\frac{i(\vec{p}-\vec{p}')\cdot\vec{x}}{\hbar}}\right]}$$

This is $\langle f|V|i\rangle$ for 1→1 scattering in presence of a potential

$$\left|\langle f|V|i\rangle_{2\to2}\right|^2 = \frac{1}{L^6}\underbrace{\underbrace{\underbrace{\int d^3x_{cm}\, e^{\frac{i}{\hbar}(\vec{P}-\vec{P}')\cdot\vec{x}_{cm}}}_{h^3\delta^{(3)}(\vec{P}-\vec{P}')}\underbrace{\int d^3x_{cm}\, \underbrace{e^{\frac{i}{\hbar}(\vec{P}-\vec{P}')\cdot\vec{x}_{cm}}}_{1}}_{L^3}}_{(hL)^3\,\delta^{(3)}(\vec{P}-\vec{P}')}}_{\left(\frac{h}{L}\right)^3\delta^{(3)}(\vec{P}-\vec{P}')} \cdot \left|\langle f|V|i\rangle_{1\to1}\right|^2$$

wavefn factors for 2 particles

$\left(\frac{h}{L}\right)^3\delta^{(3)}(\vec{P}-\vec{P}')$ ← this *precisely* cancels out the $\left(\frac{L}{h}\right)^3\int d^3\vec{p}_2{}'$ integral + enforces momentum conservation!

$$\therefore R = \left(\frac{L}{h}\right)^3\int d^3\vec{p}_1{}' \;\frac{2\pi}{\hbar}\,\delta(\Delta E)\,\left|\langle f|V|i\rangle_{1\to1}\right|^2$$

So we omit in this simpler approach

1. $\left(\frac{L}{h}\right)^3\int d^3\vec{p}_1{}'$
2. $(hL)^3\,\delta^{(3)}(\vec{P}-\vec{P}')$
3. two factors of $\frac{1}{L^3}$ from 1 initial + 1 final (heavy) particle

And this now has just the normalization factors for p_1 + p_1' (not p_2 + p_2') (namely $\left(\frac{1}{L^3}\right)^2 = \frac{1}{L^6}$)

so $R \sim \frac{1}{L^3}$ ← (cancels our $\frac{1}{L^3}$ in Flux.)

Counting factors of L^3 in the full approach

$1 \to n$ decay

$2 \to n$ scattering

- Final state wavefunction gives factor of $\left(\frac{1}{L^{3/2}}\right)^n$ to amplitude

  or $\left(\frac{1}{L^3}\right)^n$ to $|\text{amplitude}|^2$

- Final state phase space gives $(L^3)^n \int \prod_1^n d^3 p$

  so these cancel

- For decay, initial state gives $\left(\frac{1}{L^3}\right)$ to $|\text{ampl}|^2$
- $\delta^{(3)}(\Delta \vec{P})$ gives L^3 to decay rate

$\Big\}$ cancel

- For scattering, initial state gives $\left(\frac{1}{L^3}\right)^2$ to $|\text{ampl}|^2$
- $\delta^{(3)}(\Delta \vec{P})$ gives L^3 to ~~cross~~ scattering rate
- ~~flux gives~~ Dividing by flux to get σ gives L^3

$\Big\}$ cancel

Counting factors of L^3 in simpler approach (used in my class notes)

We omit $\delta^{(3)}(\Delta\vec{p})$ factor of L^3

and we also omit one of the final state integrations.. This leads to 2 fewer factors of (L^3)

[You see $L^3 \int d^3p \; L^3 \delta^{(3)}(\vec{p}) = L^6$, not 1 !]

However, ~~tacking~~ omitting 2 of the (heavy) particles also omits two wavefunction factors.

Thus it all works out the same for these processes

eg see Segrè app. B who employs this simpler approach

$$\boxed{\underset{\substack{\uparrow \\ \text{omit}}}{e^-p} \to e^- \underset{\substack{\uparrow \\ \text{omit}}}{p}}$$

$$\underset{\substack{\uparrow \\ \text{omit}}}{\gamma e^-} \to \gamma \underset{\substack{\uparrow \\ \text{omit}}}{e^-} \quad \text{in QED}$$

$$\underset{\substack{\uparrow \\ \text{omit}}}{\mu} \to \nu_\mu \bar{\nu}_e \underset{\substack{\uparrow \\ \text{omit}}}{e} \qquad \nu \underset{\substack{\uparrow \\ \text{omit}}}{X} \to \underset{\substack{\uparrow \\ \text{omit}}}{Y} e$$

The only dodgy one is

for this ← $W \to e\nu$

where there is only one heavy one. (one might expect need $\frac{1}{L^3}$ for each of e or ν) Here we actually use absence of L^3 dependence to determine the answer (+ it works!) → (but to justify, should really use full approach!)

Scattering amplitude

Fermi golden rule

$$R = \left[\prod_{i=1}^{n_f} \left(\frac{L}{h}\right)^3 \int d\vec{p}_i^{\,\prime}\right] \frac{2\pi}{\hbar}\,\delta(\Delta E)\,\left|\langle p_1' \cdots p_{n_f}' \,|\, H_{int} \,|\, p_1 \cdots p_{n_i}\rangle\right|^2$$

Now $\langle p_1' \cdots p_{n_f}' | H_{int} | p_1 \cdots p_{n_i}\rangle = \prod \int dx \, \langle p'|x'\rangle \cdots \langle x' \cdots | H | x \cdots\rangle \langle x|p\rangle \cdots$

$$= \frac{1}{L^{n/2}} \int \prod dx \; e^{i\sum \frac{p\cdot x - p'\cdot x'}{\hbar}} \underbrace{\langle x' \cdots | H | x \cdots\rangle}_{\text{invt under translations}}$$

i.e. indep of x_{cm}

$x = x_{cm} + \tilde{x}$ same

$$= \frac{1}{L^{n/2}} \underbrace{\int dx_{cm}\, e^{i\frac{(P_{init} - P_{fin})x_{cm}}{\hbar}}}_{(2\pi\hbar)^3\,\delta(\Delta P)} \int d\tilde{x}\, e^{i\sum \frac{p\cdot\tilde{x} - p'\cdot\tilde{x}'}{\hbar}} \langle \tilde{x}' \cdots | H | \tilde{x} \cdots\rangle$$

can do this integral because $\langle x|H|x\rangle$ indep of x_{cm}

$$= (2\pi\hbar)^3\,\delta(\Delta P) \underbrace{\frac{1}{L^{n/2}} \int d\tilde{x}\, e^{i\sum\frac{p\tilde{x} - p'\tilde{x}'}{\hbar}} \langle x' \cdots | H | x \cdots\rangle}_{\text{call this } A}$$

$$|\langle p' \cdots | H | p \cdots\rangle|^2 = (2\pi\hbar)^3\,\delta(\Delta P) \underbrace{\int dx_{cm}\, e^{i\frac{(\Delta P)x_{cm}}{\hbar}}}_{L^3} |A|^2$$

$$= (hL)^3\,\delta(\Delta P)\,|A|^2$$

In my class notes I use $|A|^2$ for what is here $L^6|A|^2$ so A in my notes has units of energy

$$R = \prod^{n_f} \left(\frac{L}{h}\right)^3 \int d\vec{p}_i^{\,\prime} \; (hL)^3\,\delta(\Delta P)\; \frac{2\pi}{\hbar}\,\delta(\Delta E)\,|A|^2$$

$$\boxed{R = \prod^{n_f - 1} \left(\frac{L}{h}\right)^3 \int d\vec{p}_i^{\,\prime} \cdot L^6 \cdot \frac{2\pi}{\hbar}\,\delta(\Delta E)\,|A|^2}$$

← use this as starting point.

Since $R \sim \frac{1}{(\text{time})} \Rightarrow L^6|A|^2 \sim (\text{energy})^2 \Rightarrow \boxed{A \sim \frac{\text{energy}}{(\text{length})^3}}$ ← use this!

2-particle final state phase space

$$R = \left(\frac{L}{h}\right)^3 \int d^3p_1 \ \frac{2\pi}{\hbar} \delta(\Delta E) \ L^6 \, |A|^2$$

$$= \frac{L^9}{(2\pi\hbar)^3} \left(\frac{2\pi}{\hbar}\right) \int d\Omega \ p_1^2 \, dp_1 \ \delta(E_0 - E_1 - E_2) \, |A|^2$$

cm frame w/ total energy E_0

$$E_0 = \sqrt{(cp)^2 + (m_1c^2)^2} + \sqrt{(cp)^2 + (m_2c^2)^2}$$

$$\frac{\partial E_0}{\partial p} = \underbrace{\underbrace{\frac{c^2 p}{E_1}}_{|v_1|} + \underbrace{\frac{c^2 p}{E_2}}_{|v_2|}}_{v_f} = \frac{c^2 p E_0}{E_1 E_2}$$

Since $\vec{v}_1$ + $\vec{v}_2$ are in opposite directions $\underbrace{|\vec{v}_1 - \vec{v}_2|}_{\text{call this } v_f} = |\vec{v}_1| + |\vec{v}_2|$

$$R = \frac{L^9}{(2\pi)^2} \frac{1}{\hbar^4} \int d\Omega \ p_1^2 \underbrace{\frac{dp_1}{dE_0}}_{\frac{1}{v_f}} \delta(E_0 - E_1 - E_2) \, dE_0 \ |A|^2$$

$$R = \frac{L^9}{(2\pi)^2} \frac{1}{\hbar^4} \frac{p_1^2}{v_f} \int d\Omega \, |A|^2$$

$$= \frac{L^9}{(2\pi)^2} \frac{1}{\hbar^4} \frac{p_1 E_1 E_2}{c^2 E_0} \int d\Omega \, |A|^2$$

$$= \frac{L^9}{(2\pi)^2} \frac{1}{\hbar} \frac{1}{(\hbar c)^3} \frac{(cp) E_1 E_2}{E_0} \int d\Omega \, |A|^2$$

$\boxed{1 \to 2 \text{ decay}}$

(set $c=1$ for now)

$$E_0 = E_1 + E_2$$

$$E_0 - E_1 = E_2 = \sqrt{p^2+m_2^2} = \sqrt{E_1^2 - m_1^2 + m_2^2}$$

$$E_0^2 - 2E_0E_1 = -m_1^2 + m_2^2$$

$$2E_0E_1 = E_0^2 + m_1^2 - m_2^2 = (E_0+m_1+m_2)(E_0+m_1-m_2)$$

$$2E_0E_2 = (E_0+m_1+m_2)(E_0-m_1+m_2)$$

Griffiths p. 112 $\Rightarrow$ $2E_0p = \left[(E_0+m_1+m_2)(E_0+m_1-m_2)(E_0-m_1+m_2)(E_0-m_1-m_2)\right]^{\frac{1}{2}}$

$$\therefore \quad R = \frac{1}{(2\pi)^2}\frac{1}{\hbar(\hbar c)^3}\frac{cp\,E_1E_2}{E_0}\int d\Omega\,|A|^2L^9 = \frac{1}{(2\pi)^2\hbar^4}\frac{p^2}{v_f}\int d\Omega|A|^2L^9$$

$$= \frac{1}{(2\pi)^2\hbar(\hbar c)^3}\frac{(2E_0cp)(2E_0E_1)(2E_0E_2)}{8E_0^3\cdot E_0}\int d\Omega\,|A|^2L^9$$

$$= \frac{E_0^2}{8(2\pi)^2\hbar(\hbar c)^3}\Bigg[\underbrace{\frac{2E_0cp}{E_0^2}\;\frac{2E_0E_1}{E_0^2}\;\frac{2E_0E_2}{E_0^2}}_{f}\Bigg]\int d\Omega\,|A|^2L^9$$

$$\boxed{\begin{aligned} f &\equiv \frac{(E_0+m_1+m_2)^{5/2}(E_0+m_1-m_2)^{3/2}(E_0-m_1+m_2)^{3/2}(E_0-m_1-m_2)^{1/2}}{E_0^6} \\ &\xrightarrow[m_1=0]{} \frac{(E_0+m_2)^4(E_0-m_2)^2}{E_0^6} \\ &\xrightarrow[m_1,m_2\to 0]{} 1 \end{aligned}}$$

(if isotropic decay)

See relativistic expressions later

$$R = \frac{f\cdot E_0^2}{32\pi^2\hbar(\hbar c)^3}\int d\Omega\,|A|^2L^9 = \frac{f\cdot E_0^2}{8\pi\hbar(\hbar c)^3}|A|^2L^9$$

[but not too useful since $|A|$ may have dependence on E_1, E_2, p as well.]

$$[|A|^2L^9] = \frac{(\hbar c)^3}{E}$$

$$\Rightarrow [R] = \frac{E}{\hbar}$$

$2 \to 2$ scattering

$$R = \frac{L^9}{(2\pi)^2 \hbar^4} \frac{p_{1f}^2}{v_f} \int d\Omega \, |A|^2 \qquad \text{(in cm frame)}$$

↳ or lab frame if one particle is massive

$$= \frac{L^9}{(2\pi)^2 \hbar (\hbar c)^3} \frac{(c p_{1f}) E_{1f} E_{2f}}{E_0} \int d\Omega \, |A|^2$$

$$F = \frac{v_i}{L^3} \qquad \text{where } v_i = v_{1i} + v_{2i}$$

$$\sigma = \frac{R}{F} = \frac{1}{(2\pi)^2 \hbar^4} \frac{p_{1f}^2}{v_f v_i} \int d\Omega \, |A|^2 L^{12}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2} \frac{1}{\hbar^4} \frac{p_{1f}^2}{v_f v_i} L^{12} |A|^2$$

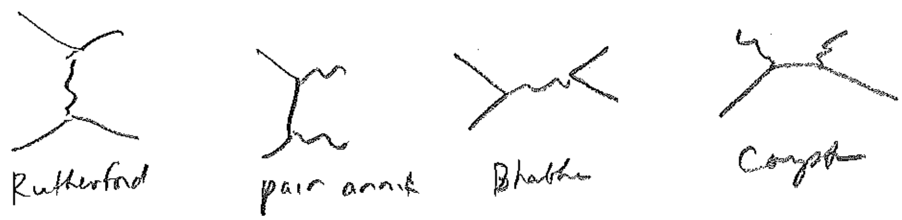
$$= \frac{1}{(2\pi)^2} \cdot \frac{1}{(\hbar c)^4} \frac{(c p_{1f}) E_{1f} E_{2f}}{\left(\frac{v_i}{c}\right) E_0} |A|^2 L^{12}$$

See relativistic expression later

$$[|A|] = \frac{(\hbar c)^3}{L^6 E^2} \qquad \text{Since } [|A|] = \frac{\text{energy}}{(\text{length})^3}$$

$$\left[\frac{d\sigma}{d\Omega}\right] = \frac{E^2}{(\hbar c)^4} \frac{(\hbar c)^6}{E^4} = \left(\frac{\hbar c}{E}\right)^2 \checkmark$$

2 → 2 QED processes



All contain $e^2 = 4\pi K e^2 = 4\pi\alpha\,\hbar c$ and a propagator

$$A = \frac{1}{L^6}\,\frac{4\pi\alpha\,\hbar c}{p_I^2 - (m_I c)^2}\cdot\ ?$$

Now $[L^6|A|] = \frac{(\hbar c)^3}{E^2}$ so $[?] = \hbar^2$

$$A = \frac{1}{L^6}\,\frac{4\pi\alpha(\hbar c)^3}{(c\,p_I)^2 - (m_I c^2)^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2}\,\frac{1}{(\hbar c)^4}\,\frac{(cp_{if})^2}{\left(\frac{v_i}{c}\right)\left(\frac{v_f}{c}\right)}\,\frac{(4\pi\alpha)^2(\hbar c)^6}{[(cp_I)^2 - (m_I c^2)^2]^2}$$

$$= \frac{1}{\left(\frac{v_i}{c}\right)\left(\frac{v_f}{c}\right)}\left[\frac{2\alpha\,\hbar c\,(cp_{if})}{(cp_I)^2 - (m_I c^2)^2}\right]^2$$

$$= \frac{1}{\left(\frac{v_i}{c}\right)\left(\frac{v_f}{c}\right)}\left[\frac{2Ke^2\,(cp_{if})}{(cp_I)^2 - (m_I c^2)^2}\right]^2$$

Rutherford

Heavy nucleus (Z_2) $\Rightarrow$ lab frame = cm frame

$$p_{1f} = p_{1i} = p_1$$

$$v_f = v_{1f} = v_{1i} = v_i = v_1$$

$$(cp_{\mathrm{I}})^2 = -c^2|\Delta\vec{p}\,|^2 = -c^2\left(2p_1 \sin\frac{\theta}{2}\right)^2$$

$$= -4c^2p_1^2 \sin^2\frac{\theta}{2}$$

$$\frac{d\sigma}{d\Omega} = \left[\frac{2Ke^2\ cp_1\ Z_1Z_2}{\frac{v_1}{c}\left(-4c^2p_1^2\sin^2\frac{\theta}{2}\right)}\right]^2$$

$$= \left[\frac{Z_1Z_2Ke^2}{2v_1p_1\sin^2\frac{\theta}{2}}\right]^2$$

Now $v_1p_1 = \dfrac{c^2p_1^2}{E_1}$

$$\xrightarrow{\text{nonrel}} \frac{c^2p_1^2}{m_1c^2} = 2T_1$$

$$\xrightarrow{\text{rel}} E_1$$

$$\frac{d\sigma}{d\Omega} = \xrightarrow{\text{nonrel}} \left(\frac{KZ_1Z_2e^2}{4T_1\sin^2\frac{\theta}{2}}\right)^2$$

$$\xrightarrow{\text{rel}} \left(\frac{KZ_1Z_2e^2}{2E_1\sin^2\frac{\theta}{2}}\right)^2$$

Møller

$$e^-e^- \to e^-e^-$$

equal mass particles in cm frame

This differs from Rutherford only in that

$$v_f = v_{1f} + v_{2f} = 2v_{1f} = 2v_{1i}$$

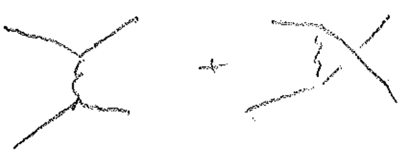
$$v_i = v_{1i} + v_{2i} = 2v_i$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{2Ke^2\, c p_1}{\frac{2v_1}{c}\left(-4c^2 p_1^2 \sin^2\frac{\theta}{2}\right)} \right)^2$$

$$= \left(\frac{Ke^2}{4 v_1 p_1 \sin^2\frac{\theta}{2}} \right)^2$$

$$\rightarrow \begin{cases} \text{nonrel}: & \left(\frac{Ke^2}{8T_1 \sin^2\frac{\theta}{2}} \right)^2 = \left(\frac{Ke^2}{4T_{cm} \sin^2\frac{\theta}{2}} \right)^2 \\ \text{rel}: & \left(\frac{Ke^2}{4E_1 \sin^2\frac{\theta}{2}} \right)^2 = \left(\frac{Ke^2}{2E_{cm} \sin^2\frac{\theta}{2}} \right)^2 \end{cases}$$

But we've neglected exchange diagram



$e^- e^+ \to \mu^- \mu^+$

$$s = E_{cm}^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{\left(\frac{v_f}{c}\right)\left(\frac{v_i}{c}\right)} \left[\frac{2 K e^2 \, c p_{if}}{E_{cm}^2} \right]^2$$

If all particles relativistic: $v_f = v_i = c$

$$c p_{if} = \frac{E_{cm}}{2}$$

$$\frac{d\sigma}{d\Omega} = \left[\frac{2 K e^2 \left(\frac{E_{cm}}{2}\right)}{2 E_{cm}^2} \right]^2$$

$\hbar c = 1$

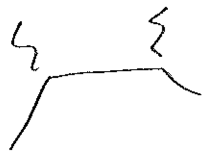
$$= \left(\frac{K e^2}{2 E_{cm}} \right)^2 = \frac{\alpha^2}{4s}$$

cf Perkins PA (1?1)

$$\sigma = (4\pi) \frac{\alpha^2}{4s} = \frac{\pi \alpha^2}{s}$$ actual result $\frac{4}{3}\left(\frac{\pi \alpha^2}{s}\right)$

Perkins PA, p 20

Thomson $\gamma e \to \gamma e$



$$\frac{1}{s-(m_e c^2)^2}$$

$$\frac{1}{u-(m_e c^2)^2}$$

$$s-(m_e c^2)^2 = 2 m_e c^2 E_\gamma \leftarrow E_\gamma \ll m_e c^2$$

$v_f = v_i = c$ (assume non relativistic electron)

$p_f = \frac{E_\gamma}{c}$

$$\frac{d\sigma}{d\Omega} = \left[\frac{2Ke^2 E_\gamma}{2 m_e c^2 E_\gamma}\right]^2$$

$$= \left(\frac{Ke^2}{m_e c^2}\right)^2 = r_0^2$$

$\sigma = 4\pi r_0^2$ actual: $\frac{8\pi}{3} r_0^2$

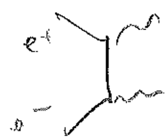
Hi energy $\sigma(\gamma e \to \gamma e) = \frac{2\pi\alpha^2}{s}\left[\log\left(\frac{s}{m^2}\right) + \frac{1}{2}\right]$

Perkins PA
(1.26c)

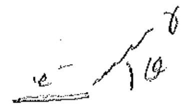
needs more thought

$e^+e^- \to \gamma\gamma$

16



$$\mathcal{A} = \frac{(4\pi K)e^2\hbar^2}{t - m_e^2}\frac{1}{L^6}$$



Relativistic electrons: $t = -|\Delta p|^2 = -4p^2 \sin^2\frac{\theta}{2}$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2}\frac{1}{\hbar^4}\frac{p_f^2}{v_f v_i}|A|^2 L^{12}$$

All particles relativistic

$v_i = v_f = 2c$

$p_f = \frac{1}{2}\left(\frac{E_{cm}}{c}\right)$

$$\frac{d\sigma}{d\Omega} = \frac{1}{\cancel{(2\pi)^2}}\frac{1}{\cancel{\hbar^4}}\frac{p_f^2}{\cancel{4}c^2}\frac{\cancel{(4\pi)^2}(Ke^2)^2\cancel{\hbar^4}}{16\, p^4 \sin^2\frac{\theta}{2}}$$

$16p^2 = 4E_{cm}$

$$= \frac{(Ke^2)^2}{16\underbrace{c^2p^2}_{\frac{E_{cm}^2}{4}}\sin^4\frac{\theta}{2}}$$

$$= \left(\frac{Ke^2}{2E_{cm}\sin^2\frac{\theta}{2}}\right)^2 \cong \frac{\alpha^2}{4s(\sin^4\frac{\theta}{2})}$$

← however this is based on simplistic assumption that $\mathcal{M}$ doesn't depend on anything other than propagator

In fact cf Griffiths p. 260

low energy → $\mathcal{M} \sim e^2$

Perkins PA (high energy)

$$\sigma = \frac{2\pi\alpha^2}{s}\left[\ln\left(\frac{s}{m^2}\right) - 1\right]$$

cutoff

(3-21-19)

Connect our calculation to relativistic QFT

$$R = \left[\prod_{i=1}^{n_f} \left(\frac{L}{h}\right)^3 \int d\vec{p}_i\right] (hL)^3\, \delta(\Delta p)\, \frac{2\pi}{\hbar}\, \delta(\Delta E)\, |A|^2$$

$$= \frac{(\hbar L)^3}{\hbar} \left[\prod_{i=1}^{n_f} \left(\frac{L}{\hbar}\right)^3 \int \frac{d\vec{p}_i}{(2\pi)^3}\right] (2\pi)^4\, \delta^4(\Delta p^\mu)\, |A|^2$$

A contains normalization factors $\frac{1}{(2EL^3)^{1/2}}$ for each particle

Define $A = \prod_{i=1}^{n} \frac{1}{(2EL^3)^{1/2}} \mathcal{M}$, then

[perhaps could also include factors of $\hbar$ & $m c$]

$$R = \frac{\hbar^2 L^3}{\prod_1^{n_i} (2EL^3)} \left[\prod_{i=1}^{n_f} \frac{1}{\hbar} \int \frac{d\vec{p}_i}{(2\pi)^3\, 2E}\right] (2\pi)^4\, \delta^4(\Delta p^\mu)\, |\mathcal{M}|^2$$

---

units of A are $\frac{\text{energy}}{(\text{length})^3}$ $\Rightarrow$ units of $\mathcal{M}$ are $\frac{E}{\ell^3}(E\ell^3)^{\frac{n}{2}} = E^{\frac{n}{2}+1}\, \ell^{\frac{3n}{2}-3}$

Since $\ell \sim \frac{\hbar}{p} \sim \frac{\hbar c}{E}$ this is same as $E^{\frac{n}{2}+1}\left(\frac{\hbar c}{E}\right)^{\frac{3n}{2}-3} = (\hbar c)^{\frac{3}{2}n-3}\, E^{4-n}$

1→2 Decay ($n=3$) $\Rightarrow$ $[\mathcal{M}] = (\hbar c)^{\frac{3}{2}}\, E$

2→2 Scatter ($n=4$) $\Rightarrow$ $[\mathcal{M}] = (\hbar c)^3$.

2 → 1 decay (relativistic)

Earlier we found $R = \frac{1}{(2\pi)^2} \frac{1}{\hbar (\hbar c)^3} \, cp \, \frac{E_1 E_2}{E_0} \int d\Omega \, |A|^2 L^9$

$$= \frac{E_0^{\,2}}{32\pi^2 \hbar^4 c^3} \, f \int d\Omega \, |A|^2 L^9$$

where $f = \dfrac{(E_0+m_1+m_2)^{3/2}(E_0+m_1-m_2)^{3/2}(E_0-m_2+m_1)^{3/2}(E_0-m_1-m_2)^{1/2}}{E_0^6}$

But $A = \prod_{i=1}^{n} \frac{1}{(2EL^3)^{1/2}} \mathcal{M} \Rightarrow L^9 |A|^2 = \frac{|\mathcal{M}|^2}{(2E_0)(2E_1)(2E_2)}$

$\Rightarrow R = \frac{1}{(2\pi)^2} \frac{1}{\hbar (\hbar c)^3} \frac{cp}{8E_0^2} \int d\Omega \, |\mathcal{M}|^2$ → agrees w/ Griffiths (6.35)

(From p. 9
$2E_0 cp = \sqrt{(\ )(\ )(\ )(\ )}$)

$$= \frac{1}{64\pi^2 \hbar (\hbar c)^3 E_0} \underbrace{\left[\frac{(E_0+m_1+m_2)(E_0+m_1-m_2)(E_0-m_1+m_2)(E_0-m_1-m_2)}{E_0^4}\right]^{\frac{1}{2}}}_{f} \int d\Omega \, |\mathcal{M}|^2$$

As before, we have to know dependence of $|\mathcal{M}|$ on the energies & masses

$[|\mathcal{M}|^2] = (\hbar c)^3 E^2 \Rightarrow [R] = [\frac{E}{\hbar}]$ ✓

2 → 2 scattering (relativistic)

Earlier we found, in the CM frame

$$R = \frac{L^9}{(2\pi)^2 \hbar (\hbar c)^3} \frac{c p_{1f} E_{1f} E_{2f}}{E_0} \int d\Omega |A|^2$$

$$F = \frac{v_i}{L^3} = \frac{|\vec{v}_{1i} - \vec{v}_{2i}|}{L^3}$$

$$|A|^2 = \frac{|M|^2}{(2E_{1f})(2E_{2f})(2E_{1i})(2E_{2i}) L^{12}}$$

$$\Rightarrow \sigma = \frac{R}{F} = \frac{1}{(2\pi)^2 \hbar (\hbar c)^3} \frac{c p_{1f}}{v_i (16 E_{1i} E_{2i}) E_0} \int d\Omega |M|^2$$

$$= \frac{1}{64\pi^2 (\hbar c)^4} \frac{c p_{1f}}{\left(\frac{v_i}{c}\right) E_{1i} E_{2i} E_0} \int d\Omega |M|^2$$

[Units: $[M] = (\hbar c)^3 \Rightarrow [\sigma] = \frac{(\hbar c)^2}{E^2}$ ✓]

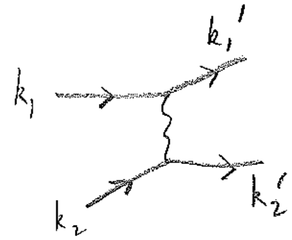
[If heavy target then $\frac{v_i}{c} = \frac{c p_{1i}}{E_{1i}}$, $E_{2i} = Mc^2 \approx E_0$

$$\Rightarrow \frac{1}{64\pi^2 (\hbar c)^4} \left(\frac{p_{1f}}{p_{1i}}\right) \frac{1}{E_0^2} \int d\Omega |M|^2$$]

(1-21-19) Rutherford scattering in cm frame

Relativistic vs nonrelativistic approach

4



$$Z_1 Z_2 e^2 \frac{(k_1+k_1')\cdot(k_2+k_2')}{(k_1-k_1')^2}$$

Treat particles as charged scalars

$$k_1 + k_2 = k_1' + k_2'$$

$$\Rightarrow (k_1+k_1')\cdot(k_2+k_2') = (k_1+k_1')\cdot(k_1 - k_1' + 2k_2)$$

$$= \underbrace{k_1^2 - k_1'^2}_{0} + \underbrace{2\,k_1\cdot k_2}_{s-m_1^2-m_2^2} + \underbrace{2k_1'\cdot k_2}_{m_1^2+m_2^2-u} = \underline{\underline{s-u}}$$

$$s = (k_1+k_2)^2 = m_1^2 + m_2^2 + 2k_1\cdot k_2$$

$$u = (k_1' - k_2)^2 = m_1^2 + m_2^2 - 2k_1'\cdot k_2 = m_1^2 + m_2^2 - 2k_1\cdot k_2'$$

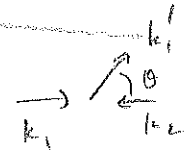
$$t = (k_1 - k_1')^2 = 2m_1^2 - 2k_1\cdot k_1'$$

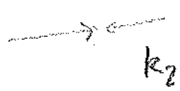
$$s+t+u = 4m_1^2 + 2m_2^2 + 2k_1\cdot(\underbrace{k_2 - k_2' - k_1'}_{-k_1}) = 2m_1^2 + 2m_2^2 \checkmark$$

$$\mathcal{M} = Z_1 Z_2 e^2 \frac{s-u}{t}$$

$$\begin{pmatrix} k_1 = (E_1, 0, 0, p_1) \\ k_2 = (E_2, 0, 0, -p_1) \end{pmatrix}$$

$$k_1' = (E_1, p_1\sin\theta, 0, p_1\cos\theta)$$





$$s = (E_1+E_2)^2$$

$$u = (E_1 - E_2)^2 - p_1^2\sin^2\theta - p_1^2(\cos\theta+1)^2$$

$$= (E_1-E_2)^2 - p_1^2 - 2p_1^2\cos\theta - p_1^2$$

$$= (E_1-E_2)^2 - 2p_1^2(\cos\theta+1)$$

$$t = -p_1^2\sin^2\theta - p_1^2(\cos\theta-1)^2$$

$$= -2p_1^2 + 2p_1^2\cos\theta$$

$$= 2p_1^2(\cos\theta - 1)$$

$$\frac{s-u}{t} = \frac{(E_1+E_2)^2 - (E_1-E_2)^2 + 2p_1^2(\cos\theta+1)}{2p_1^2(\cos\theta-1)}$$

$$= \frac{4E_1E_2 + 2p_1^2(\cos\theta+1)}{2p^2(\cos\theta-1)}$$

cancels the $\frac{1}{2E_1}\frac{1}{2E_2}$ normalization factors!