[To describe scattering and decay processes, we use]

quantum field theory (QFT)

In this framework, <u>fields</u> are the fundamental entities of the universe
one field for each type of fundamental particle
Particles are understood as quantum excitations of the field
For example

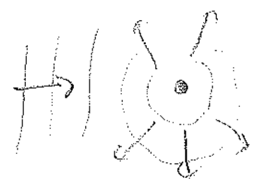
electromagnetic field $\longrightarrow$ photons

Dirac fields $\longrightarrow$ electrons, neutrinos, quarks

Higgs field $\longrightarrow$ Higgs boson classical physics = deterministic

In a classical process,
the final state is completely determined by
the initial state + the laws of physics.

e.g. in a collision of α-particle w/ nucleus,
given p_i and P_i (and impact parameter) one can
determine p_f and P_f



quantum physics: probabilistic

In a quantum process,
the final state is not completely determined.
one can compute only probabilities of various final states
that are allowed by the laws of physics

The probability of a given final state is given by the square of modulus of the amplitude $|A|^2$

The rate of a process (e.g. scattering, decays) is then obtained by adding up the probabilities of all possible final states.

$$\text{Rate} = \sum_{\substack{\text{final}\\\text{states}}} |A|^2$$

How fast a process proceeds :: depends on 2 things:

① Amplitude A, represented by a sum of Feynman diagrams

- [depends on strength of the interactions and other factors]

② Final state phase space [how many final states there are]

- [In an inelastic process, this depends on amt. of kinetic energy released, Q]

Generally, larger Q ⇒ more phase space

⇒ process is more likely

(larger cross-section, shorter lifetime)

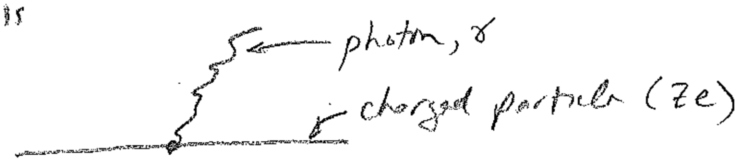
Amplitude = infinite sum of Feynman diagrams

- Not all contribute equally.
- If the strength of the interaction is weak, we can keep only a few diagrams & neglect the rest.
- The more accuracy we need, the more diagrams we need to calculate (perturbation theory, Taylor series)
- Works well for EM & for the weak interaction (If the interaction is strong, the perturbative approach breaks down.)

Feynman diagrams constructed from <u>vertices</u> and <u>lines</u>

QFT for the EM field is called QED (quantum electrodynamics)
[cf popular book by Feynman]

The QED vertex is



Vertex involves the <u>creation</u> of a new particle, the photon

[classically, particles are not created or destroyed.
accel. a charge generates an EM ~~field~~ wave
but Einstein's idea means photons are created

The strength of the vertex is given by the charge of the particle: Ze

We use "rationalized" particle physics units (Heaviside-Lorentz) in which $\epsilon_0 = 1$

Recall dimensionless fine structure constant $\alpha \equiv \frac{ke^2}{\hbar c} = \frac{e^2}{4\pi\epsilon_0 \hbar c} \simeq \frac{1}{137}$

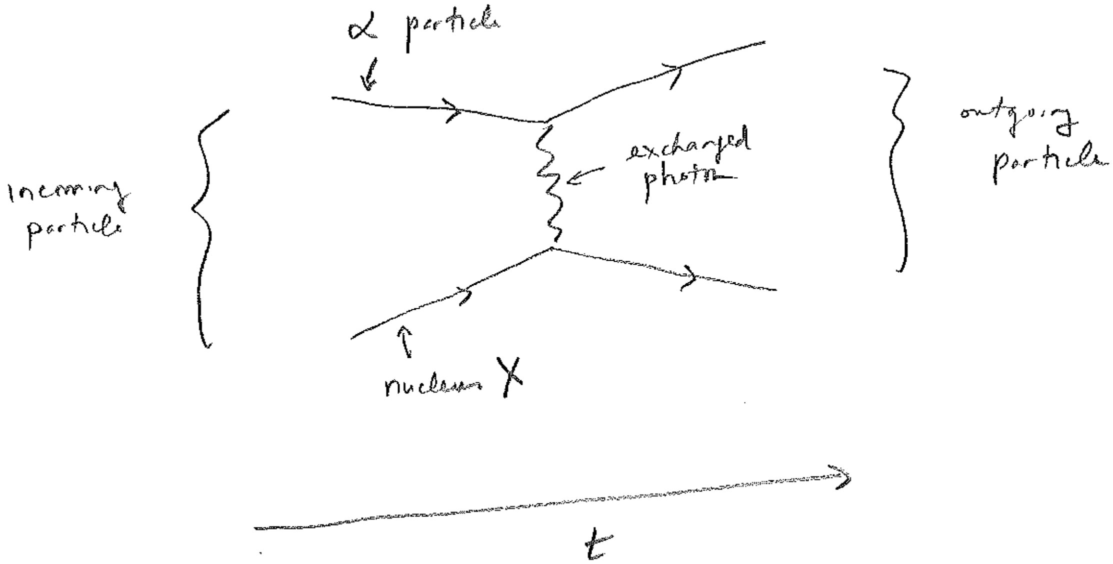
In rationalized units $e^2 = 4\pi\hbar c\,\alpha$,

so vertex strength is $Z\sqrt{4\pi\hbar c\,\alpha}$

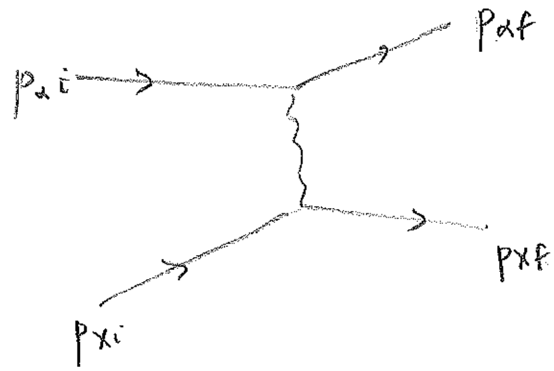
Since $\alpha \ll 1$, the QED interaction is <u>weak</u>.

Vertex also depends on other factors
e.g. the spin of the charged particle.
We'll mostly ignore these complications in this course.

The Rutherford scattering process is primarily described by a Feynman diagram assembled from two QED vertices

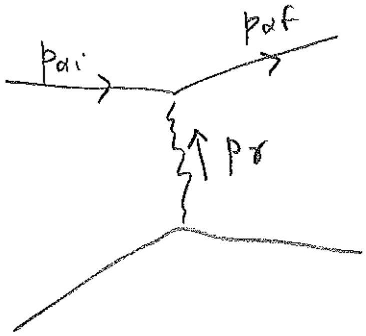


The incoming & outgoing particles have well defined 4-momenta



Four-momentum conservation $\Rightarrow$ $p_{\alpha i} + p_{Xi} = p_{\alpha f} + p_{Xf}$

Let p_γ = 4-momentum of the exchanged photon



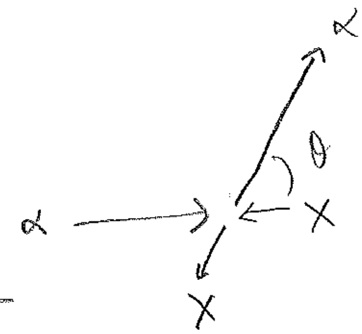
4-momentum is conserved <u>at each vertex</u> in a Feynman diag.

$$p_{\alpha i} + p_\gamma = p_{\alpha f}$$

$$\Rightarrow p_\gamma = p_{\alpha f} - p_{\alpha i} = \Delta p_\alpha$$

$$\Rightarrow \begin{cases} E_\gamma = \Delta E_\alpha \\ \vec{p}_\gamma = \Delta \vec{p}_\alpha \end{cases}$$

Evaluate this in cm frame



$\Delta \vec{p}_\alpha \neq 0$ because α changes direction

Elastic collision $\Rightarrow \Delta E_\alpha = 0$ (i.e. $|\vec{p}_f| = |\vec{p}_i|$)

$p_\gamma = (0, \Delta \vec{p}_\alpha)$ photon has spacelike 4-momentum.

$$p_\gamma^2 = E_\gamma^2 - \vec{p}_\gamma^{\,2} < 0$$

Photon is <u>off-shell</u> ($E_\gamma \neq |p_\gamma|$)

We call an off-shell particle <u>virtual</u>

It is not physical, only exists for a <u>short time</u>.

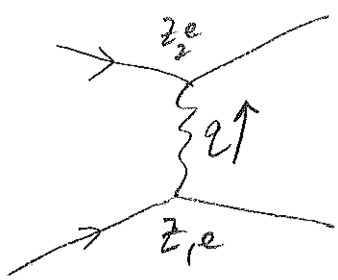
Associated w/ each <u>internal line</u> in a Feynman diagram is a "propagator".

For a photon, the propagator is given by $\frac{1}{p_\gamma^2}$

N.B. if the photon were on-shell, the propagator $\rightarrow \infty$

But for a virtual particle it is finite.

The amplitude for the process is the product of vertex factors and propagators in the Feynman diag.



$$A = \frac{(Z_1 e)(Z_2 e)}{p_\gamma^2} M$$

M = matrix element, contains normalization factors, & other details due to spin etc

$$\text{Rate} = \sum_{\text{final states}} |A|^2$$

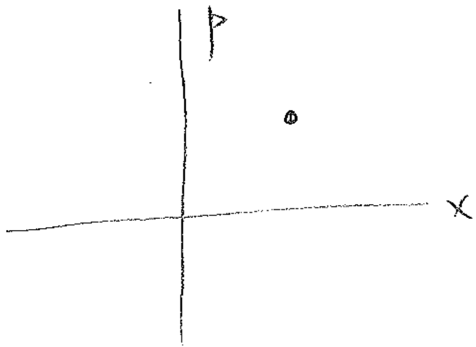
[Note to self: this is not the relativistic M but rather is precisely $\tilde{V}(q)$.]

[How do we sum over final states?]

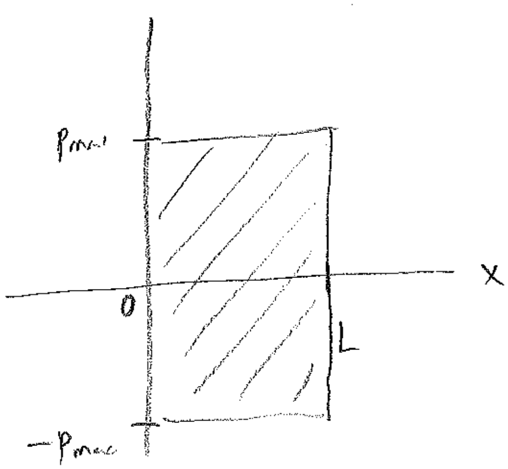
Phase space

First consider a particle moving in one dimension

Classically, the state of the particle is described by a point in the x-p plane (phase space)



Suppose particle is confined to a box $0 < x < L$ and has momentum $|p| \leq p_{max}$



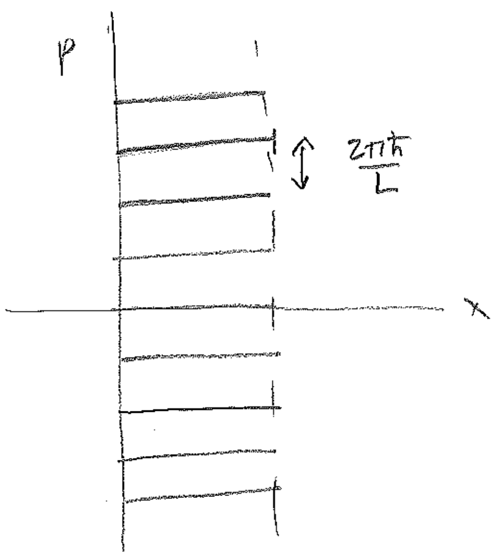
Phase space volume of the particle is $2 p_{max} L$.

Quantum mechanically, a particle confined to a box $0<x<L$ can only have discrete momenta.

Describing it by wavefunction $\psi(x) \sim e^{\frac{ipx}{\hbar}}$

with *periodic* boundary condition $\psi(x+L) = \psi(x)$

we have $p = \frac{2\pi\hbar}{L} \cdot n = \frac{hn}{L}$



Thus the number of quantum states* with $|p| < p_{max}$ is

$$N = \frac{L}{h} p_{max} = \frac{2L\, p_{max}}{2\pi\hbar}$$

The phase space volume of one quantum state is

$$2\pi\hbar = h$$

Planck's constant h is the volume of a quantum state in phase space!

* we're now ignoring spin multiplicity

Sum over quantum states

To sum over the discrete quantum states of a particle, integrate over its phase space, dividing by the volume of a single state

[states are very closely spaced as $L \to \infty$]

$$\sum_{\text{states}} = \int \frac{dx\,dp}{h}$$

Check that this gives the correct # of states

$$\sum_{\text{states}} 1 = \frac{1}{h} \int_0^L dx \int_{-p_{max}}^{p_{max}} dp = \frac{2L\,p_{max}}{h} \quad \checkmark$$

For a particle in 3 dimensions confined to a cube

$$\sum_{\text{states}} = \int \frac{dx\,dy\,dz\;dp_x\,dp_y\,dp_z}{h^3}$$

$$= \left(\frac{L}{h}\right)^3 \int d^3\vec{p}$$

use $\vec{p}$ to indicate 3-mom. rather than 4-mom.

[as we saw earlier in 1D model]

External states (incoming, outgoing) are physical & ∴ on-shell. Thus given $\vec{p}$, E is determined. Thus we do not integrate also over the energies of the external states

If there are several particles in the final state, we integrate over the phase space of each of them

$$\sum_{\substack{\text{final}\\\text{states}}} = \left(\frac{L}{h}\right)^{3n_f} \prod_{j=1}^{n_f} \int d^3\vec{p}_j \qquad (n_f = \#\text{ of final state particles})$$

However, the final state momenta $\vec{p}_j$ are not independent.

Recall the external momenta obey 4-momentum conservation

$$\sum_{j=1}^{n_f} p_j \equiv P_f^{tot} = P_i^{tot}$$

This means 3-momenta obey

$$\sum_1^{n_f} \vec{p}_j = \vec{P}_i^{tot}$$

This can be used to solve for one of the final state momenta in terms of the others.

We need only integrate over phase space of $n_f - 1$ final state particles.

We also have energy conservation

$$\sum_{j=1}^{n_f} E_j \equiv E_f^{tot} = E_i^{tot}$$

We impose this by include a dirac delta function

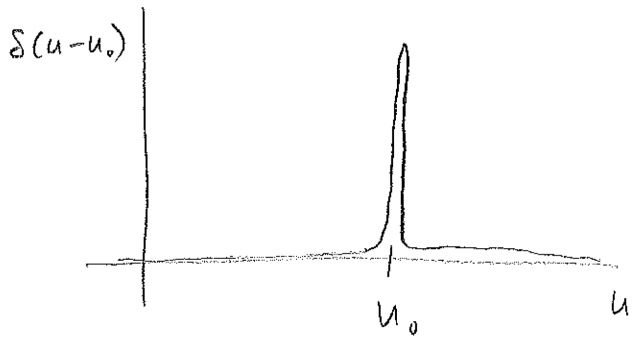
$$\text{Rate } R = \sum_{\substack{\text{final}\\ \text{state}}} \frac{2\pi}{\hbar} \delta(E_f^{tot} - E_i^{tot}) |A|^2$$

$$= \left(\frac{L}{h}\right)^{3(n_f-1)} \prod_j^{n_f-1} \int d^3\vec{p}_j \, \frac{2\pi}{\hbar} \delta(E_f^{tot} - E_i^{tot}) |A|^2$$

Fermi's golden rule

δ-function is really a distribution.

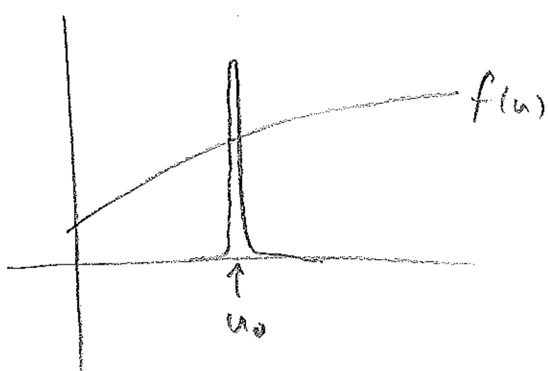
It only appears inside an integral



$\delta(u-u_0)$ vanishes whenever $u - u_0 \neq 0$

The integral $\int du\, f(u)\, \delta(u-u_0)$

is defined to have the value $f(u_0)$,

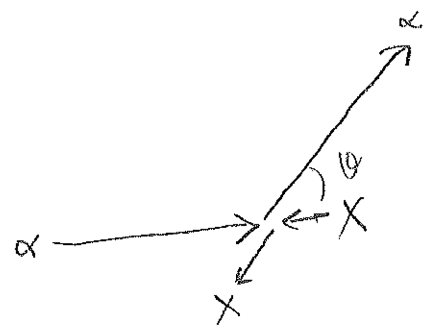


← only the value of $f(u)$ at $u = u_0$ matters.

We say integral has "support" only at $u = u_0$

Rutherford scattering of α from nucleus X

c.m. frame



3-momentum conservation

$$\vec{p}_{\alpha f} + \vec{p}_{Xf} = \vec{P}_f^{tot} = 0$$

$$\Rightarrow \vec{p}_{Xf} = -\vec{p}_{\alpha f}$$

we need only integrate over final momentum of the α particle

What about energy conservation?

$$E_i^{tot} = m_\alpha + T_\alpha + m_X + T_X$$

For massive nuclei we can ignore T_X (treat it as fixed)

(Nonrelativistically $T = \frac{p^2}{2m}$.
So if $|\vec{p}_X| = |\vec{p}_\alpha|$ and $m_X \gg m_\alpha$ then $T_X \ll T_\alpha$.)

$$E_f^{tot} - E_i^{tot} \cong T_{\alpha f} - T_{\alpha i} \qquad \text{(masses cancel)}$$

Scattering rate

$$R = \left(\frac{L}{h}\right)^3 \int d^3\vec{p}_{\alpha f} \, \frac{2\pi}{\hbar} \, \delta(T_{\alpha f} - T_{\alpha i}) \, |A|^2$$

Write $\vec{p}_{\alpha f}$ in spherical coordinates.

θ = angle between $\vec{p}_{\alpha f}$ and $\vec{p}_{\alpha i}$

$$d^3\vec{p}_{\alpha f} = p^2\, dp\, d\Omega$$

$$d\Omega = \sin\theta\, d\theta\, d\phi$$

Relativistically [let's be more careful w/ factors of c now]

$$E^2 = (cp)^2 + (mc^2)^2$$

$$E\,dE = c^2 p\, dp$$

Also

$$E = mc^2 + T$$

$$dE = dT \qquad \Rightarrow d^3 p_{\alpha f} = \frac{pE}{c^2}\, dT\, d\Omega$$

$$R = \underbrace{\left(\frac{L}{h}\right)^3 \int d\Omega \frac{|\vec{p}_{\alpha f}|\, E_{\alpha f}}{c^2}\, dT_{\alpha f}}\ \frac{2\pi}{\hbar}\, \delta(T_{\alpha f} - T_{\alpha i})\, |A|^2$$

[for self: $\rho(E) = \frac{dn}{dE}$, phase space density]

Elastic $\Rightarrow E_{\alpha f} = E_{\alpha i}$ and $|\vec{p}_{\alpha f}| = |\vec{p}_{\alpha i}|$

$$R = \underbrace{\left(\frac{L}{h}\right)^3 \left(\frac{2\pi}{\hbar}\right)}_{\frac{L^3}{(2\pi\hbar^2)^2}} \frac{p_{\alpha i} E_{\alpha i}}{c^2} \int d\Omega\, |A|^2$$

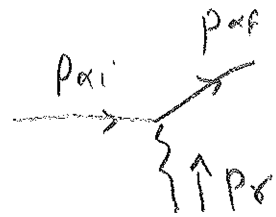
evaluated w/ $|\vec{p}_{\alpha f}| = |\vec{p}_{\alpha i}|$ due to δ-function

NB $|\vec{p}_{\alpha f}|$ fixed by momentum conservation but angle of scatter undetermined quantum mechanically

(classically, it is determined by impact parameter)

Recall $A = \frac{Z_1 Z_2 e^2}{p_\gamma^2} M$

Mom. cons. $\Rightarrow$ $p_\gamma = \Delta p_\alpha$



$$p_\gamma^2 = (\Delta E_\alpha)^2 - (\Delta \vec{p}_\alpha)^2$$

$\Delta E_\alpha = 0$ (elastic)

$$|\Delta \vec{p}_\alpha| = 2 |\vec{p}_{\alpha i}| \sin \frac{\theta}{2}$$

$$A = - \frac{Z_1 Z_2 e^2}{4 \vec{p}_{\alpha i}^{\,2} \sin^2 \frac{\theta}{2}} M$$

We will assume M is as simple as possible but we must get the units correct.

Initial and final state α particles are described by wavefunctions normalized in a cube of volume L^3

$$\psi = \frac{1}{L^{3/2}}\, e^{\frac{i\vec{p}\cdot\vec{x}}{\hbar}} \quad \Rightarrow \quad \int |\psi|^2\, dx = 1.$$

$\therefore$ the vertex  $\sim \langle \psi_f | Ze\gamma^\mu | \psi_i \rangle$

Dirac gamma matrix

contributes a factor of $\frac{1}{L^3}$ to M

We're using conventions in which $c = 1$
and also $\epsilon_0 = 1$.
Particle physicists also set $\hbar = 1$.
We must restore appropriate factors of ϵ_0, $\hbar$ and c.

Force between charges $K e^2 \sim \frac{e^2}{4\pi\epsilon_0}$

Since $A \sim e^2$ we must include $\frac{1}{\epsilon_0}$ in M

So far $M \sim \frac{1}{\epsilon_0 L^3}$

Let's write $M = \frac{?}{\epsilon_0 L^3}$

What about factors of $\hbar$ and c?

We determine these by requiring
A to have units of <u>energy</u>.

Why is this?

Proof that A must have units of energy

$$R = \sum_{\text{final}} \frac{2\pi}{\hbar} \delta(E_f - E_i) \, |A|^2$$

$\delta(E)$ has units of $(\text{energy})^{-1}$ because $\int dE \, \delta(E) = 1$

$\hbar$ has units of (energy) (time)

R has units of $(\text{time})^{-1}$

$\Rightarrow |A|^2$ has units of $(\text{energy})^2$

$$A = -\frac{Z_1 Z_2 e^2}{4|\vec{p}|^2 \sin^2\frac{\theta}{2}} \; \frac{?}{\epsilon_0 L^3}$$

Just consider units

$$[A] = \frac{e^2}{\epsilon_0} \frac{?}{\vec{p}^2 L^3} = \left(\frac{Ke^2}{L}\right) \frac{?}{(pL)^2} \sim (\text{energy})\left(\frac{?}{\hbar^2}\right)$$

$$\Rightarrow \; ? = \hbar^2$$

$$M = \frac{\hbar^2}{\epsilon_0 L^3} = \frac{4\pi K \hbar^2}{L^3}$$

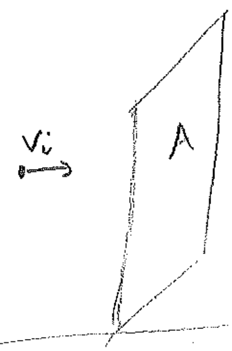
$$A = -\frac{Z_1 Z_2 e^2}{4|\vec{p}|^2 \sin^2\frac{\theta}{2}} \; \frac{4\pi K \hbar^2}{L^3} = -\frac{2\pi\hbar^2}{L^3} \; \frac{Z_1 Z_2 K e^2}{2|\vec{p}_{\alpha i}|^2 \sin^2\frac{\theta}{2}}$$

$$R = \frac{L^3}{(2\pi\hbar^2)^2} |\vec{p}_{\alpha i}| \frac{E_{\alpha i}}{c^2} \int d\Omega \, |A|^2$$

$$= \frac{1}{L^3} \frac{|p_{\alpha i}| E_{\alpha i}}{c^2} \int d\Omega \left(\frac{Z_1 Z_2 K e^2}{2|p_{\alpha i}|^2 \sin^2\frac{\theta}{2}} \right)^2$$

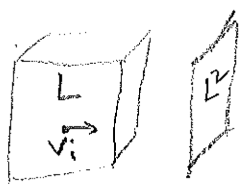
Recall scattering cross-section $\sigma = \frac{R}{F}$

F = incident particle flux $= \frac{\#\text{ particles}}{\text{sec} \cdot \text{area}}$



$$\#\text{particles} = F \cdot t \cdot A$$

1 quantum particle in a cube of size L



time for cube to pass through screen $t = \frac{L}{v_i}$

$$\therefore \quad 1 \text{ particle} = F \cdot \left(\frac{L}{v_i}\right) L^2$$

$$F = \frac{v_i}{L^3}$$

Recall $\frac{v}{c} = \frac{c|p|}{E}$

$$\Rightarrow F = \frac{c^2 |\vec{p}_{\alpha i}|}{E_{\alpha i} L^3}$$

$$\sigma = \frac{R}{F} = \frac{E_{\alpha i}^2}{c^2}\int d\Omega \left(\frac{z_1 z_2 K e^2}{2|p_{\alpha i}|\sin^2\frac{\theta}{2}}\right)^2$$ [Indep of L!]

$$= \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)$$

where $$\frac{d\sigma}{d\Omega} = \left(\frac{E_{\alpha i}\, K z_1 z_2 e^2}{2c^2 |\vec{p}_{\alpha i}|^2 \sin^2\frac{\theta}{2}}\right)^2$$

This is valid even for relativistic particles.

In nonrelativistic limit $E_{\alpha i} \simeq mc^2$, $|\vec{p}_{\alpha i}| = mv$

$$\frac{d\sigma}{d\Omega} = \left(\frac{K z_1 z_2 e^2}{2mv^2 \sin^2\frac{\theta}{2}}\right)^2$$

exactly agrees w/ classical Rutherford differential cross-section
(This only works $\frac{1}{r^2}$ forces)

Coulomb force is caused by exchange of a massless photon.

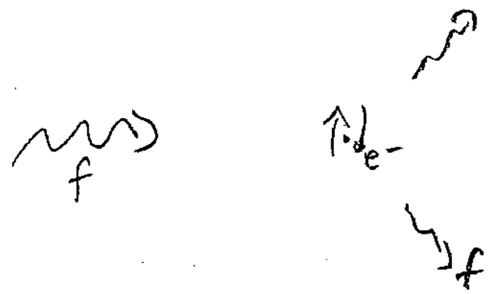
(c.f. quantum corrections from including other subleading Feynman diagrams.)

If we include spin, we'll get instead the Mott cross-section

~~[Physicists were skeptical of the photon concept until Arthur Compton did his expts on X-rays (1922)~~
~~X-rays were known to be EM waves~~

Scattering of X-rays by electrons (Thomson or Compton scattering)

Classical picture: electromagnetic waves of frequency f cause electrons to oscillate w/ frequency f.
Those accelerating electrons emit EM wave of frequency f.



Classical cross-section $\sigma = \frac{R}{F}$
R = rate at which energy is radiated by electron
F = flux of incident EM waves

From 1140, $F = \epsilon_0 c |\vec{E}|^2$
From 3120, Larmor formula $R = \frac{e^2 a^2}{6\pi\epsilon_0 c^3}$

$$\vec{a} = \frac{\vec{F}}{m_e} = \frac{e\vec{E}}{m_e} \Rightarrow R = \frac{e^4 |\vec{E}|^2}{6\pi\epsilon_0 m_e^2 c^3} \Rightarrow \sigma =$$

$$\Rightarrow \sigma = \frac{e^4}{6\pi\epsilon_0^2 (m_e c^2)^2} = \frac{8\pi}{3}\left(\frac{ke^2}{m_e c^2}\right)^2 = \frac{8\pi}{3} r_0^2$$

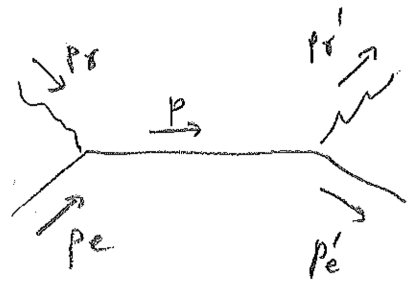
where $r_0 \equiv \frac{ke^2}{m_e c^2}$ = classical electron radius $= \frac{1.44\,\text{MeV fm}}{0.511\,\text{MeV}} = 2.8$ fm

$\sigma = 66.5\ \text{fm}^2 = \frac{2}{3}$ barn = Thomson cross-section

[Thomson used it to measure # electrons in atoms]
Nucleus too massive to radiate much.

[Now let's look at this as a quantum process]

Compton scattering: $\gamma + e^- \rightarrow e^- + \gamma$



$$p_e + p_\gamma = p_e' + p_\gamma'$$

$$p = p_e + p_\gamma$$

$$p^2 = \underbrace{p_e^2}_{m_e^2} + 2 p_e \cdot p_\gamma + \underbrace{p_\gamma^2}_{0}$$

Assume initial electron at rest: $p_e \cdot p_\gamma = m_e E_\gamma$

$$p^2 = m_e^2 + 2 m_e E_\gamma > m_e^2$$ so intermediate electron is off-shell (virtual)

$$\text{Electron propagator} = \frac{1}{p^2 - m_e^2}$$

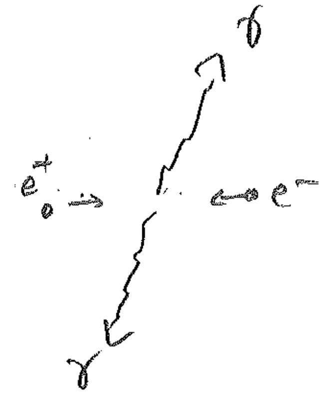
$$\Rightarrow A = \frac{e^2}{p^2 - m_e^2} = \frac{e^2}{2 m_e E_\gamma}$$

[HW: compute $\sigma = 4\pi \left(\frac{e^2}{m_e c^2}\right)^2$]

New material

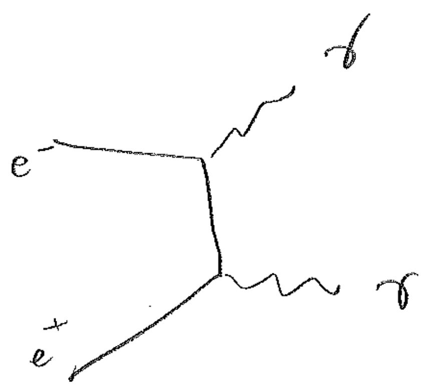
e^+e^- annihilation

Physical picture



$\Rightarrow E_\gamma \sim m_e c^2$

Feynman

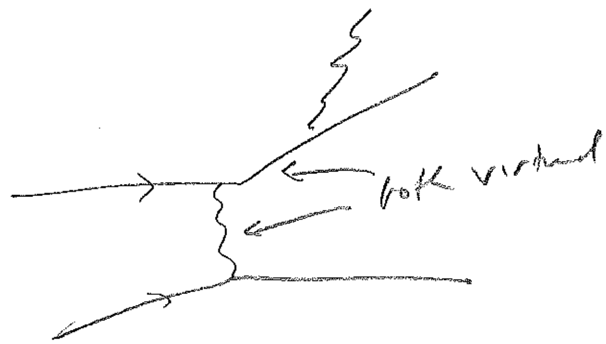


New

Bremsstrahlung

Physical

Feynman



New

Pair production

Physical

Feynman

not possible in isolation