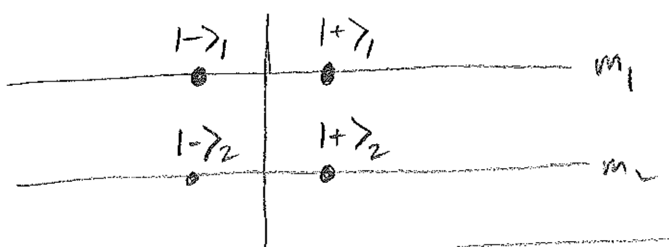
Addition of angular momentum

Consider a system of two spin-$\frac{1}{2}$ particles
say p and n (deuteron) or q and $\bar{q}$ (meson)

Each particle is described by a state belonging to the space

$\underset{\sim}{2}$, spanned by $|\tfrac{1}{2}, \tfrac{1}{2}\rangle = |+\rangle = \uparrow$
and $|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = |-\rangle = \downarrow$



$S_{1z} = m_1 \hbar$

$S_{2z} = m_2 \hbar$

The system is described by a state belonging to the tensor product space $\underset{\sim}{2} \otimes \underset{\sim}{2}$

Spanned by product states

$$|+\rangle \otimes |+\rangle = |+;+\rangle = \uparrow\uparrow$$
$$|+\rangle \otimes |-\rangle = |+;-\rangle = \uparrow\downarrow$$
$$|-\rangle \otimes |+\rangle = |-;+\rangle = \downarrow\uparrow$$
$$|-\rangle \otimes |-\rangle = |-;-\rangle = \downarrow\downarrow$$

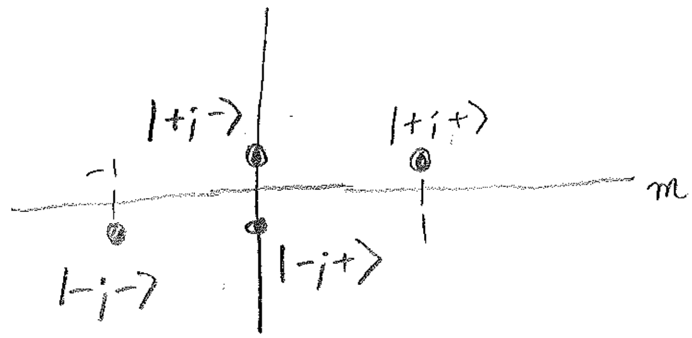
[Note: semicolons]

$\underset{\sim}{2} \otimes \underset{\sim}{2}$ is a 4-dimensional space.

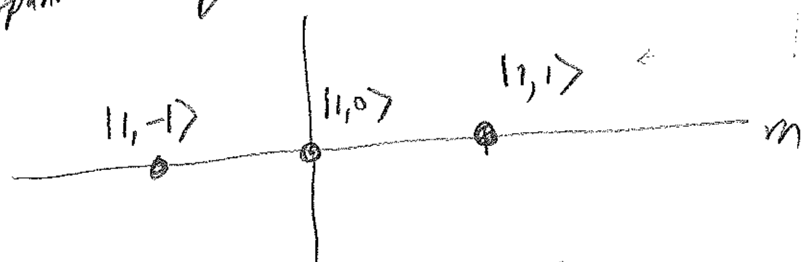
Total spin of the system $S_z = S_{1z} + S_{2z}$

$= (m_1 + m_2)\hbar$

$= m\hbar$



Since max value of m is 1, some of these states belong to a spinol representation, $\underset{\sim}{3}$ spanned by $|j, m\rangle = |1,1\rangle, |1,0\rangle, |1,-1\rangle$



[NB commas vs. semicolons]

[Let's relate these different descriptions.]

$$\text{triplet}\begin{cases} |1,1\rangle = |+;+\rangle \\ |1,-1\rangle = |-;-\rangle \\ |1,0\rangle = \ ? \end{cases}$$

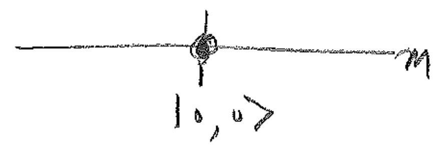
[hard to choose, so split the diff]

$$= \frac{1}{\sqrt{2}}\left[|+;-\rangle + |-;+\rangle\right]$$

[Proved in 3140. Note $\frac{1}{\sqrt{2}} \Rightarrow \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = 1.$]

$\exists$ one remaining state $\frac{1}{\sqrt{2}}\left[|+;-\rangle - |-;+\rangle\right]$ w/ $m=0$

This state belongs to spin-0 rep, $\underset{\sim}{1}$, spanned by $|0,0\rangle$



singlet $\left\{ \; |0,0\rangle = \frac{1}{\sqrt{2}}\left[\,|+;-\rangle - |-;+\rangle\right] \right.$

Thus the full space is a direct sum of spin-1 and spin-0

$\underset{\sim}{3} \oplus \underset{\sim}{1}$, which is 4-dimensional

We write: $\underbrace{\underset{\sim}{2} \otimes \underset{\sim}{2}}_{\text{tensor product}} = \underbrace{\underset{\sim}{3} \oplus \underset{\sim}{1}}_{\text{direct sum}}$

$(\text{spin } \tfrac{1}{2}) \otimes (\text{spin } \tfrac{1}{2}) = (\text{spin } 1) \oplus (\text{spin } 0)$

Observe that spin-1 states are <u>symmetric</u> under exchange of the two spins and spin-0 sts are <u>antisymmetric</u>

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3}_S \oplus \underset{\sim}{1}_A$$

$|1,0\rangle$ and $|0,0\rangle$ are <u>entangled states</u>.

[z-component of spin of neither pcl is well-defined but they are correlated.
"I don't know my spin, but whatever it is, it is opposite of yours."
Enemies: if you are for something, I am against, + vice versa]

Spin-statistics theorem

[G 5.9, p.175]

- Particles w/ half integer spin are fermions (obey Fermi Dirac statistics)

- Particles w/ integer spin are bosons (obey Bose-Einstein statistics)

- The full state (spatial, spin, flavor, color) of two identical fermions / bosons is antisymmetric / symmetric under exchange of particles

$$\psi(r_1, s_1, \dots ; r_2, s_2, \dots) = \mp \psi(r_2, s_2, \dots ; r_1, s_1, \dots)$$

for fermions / bosons

- The ground state of a bound system of two particles has a symmetric spatial wavefunction

So the rest of the state of two identical fermions / bosons must be antisymmetric / symmetric.

If 2 proton or 2 neutrons were in a bound state, the ground state would have to have spin 0.

$$|0,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$$

This is because $|0,0\rangle$ is antisymmetric under exchange + 2 identical fermions must have antisymmetric spin state

Cannot be spin 1 : $\underbrace{\uparrow\uparrow,\ \downarrow\downarrow}_{\text{violate Pauli principle}}$, or $\underbrace{\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)}_{\text{[more subtle] symmetric under exchange}}$

If the fermions are not identical, there is no restriction

Bound state of $p + n$ (deuteron) can have spin 1.
(also, a priori spin 0)

Similarly bound state of u and $\bar{d}$
can have either spin 0 (π^+ = scalar meson)
or spin 1 (ρ^+ = vector meson)

Recall: p and n are different isospin states of N which belongs to isospin representation $\underset{\sim}{2}$.

A system of two nucleons, NN belongs to tensor-product representation of isospin

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3}_S \oplus \underset{\sim}{1}_A$$

isotriplet $\underset{\sim}{3}_S$
$$|I, I_3\rangle = \begin{cases} |1,1\rangle = pp \\ |1,0\rangle = \frac{1}{\sqrt{2}}(pn + np) \\ |1,-1\rangle = nn \end{cases}$$

iso-singlet 1_A
$$|I, I_3\rangle = |0,0\rangle = \frac{1}{\sqrt{2}}(pn - np)$$

Recall strong force is insensitive to I_3, but depends on I.

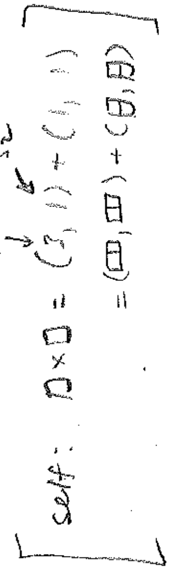
We can see this because $\underbrace{pn}_{^2H}$ forms a bound state, but not $\underbrace{pp}_{^2He}$, nn.

[can blame absence of ^2He on Coulomb; though ^3He exists. Dineutron?]

$I=1$ state is not stable, but $I=0$ is.

Deuteron is an isospin 0 state. [what are the consequences?]

Recall: N belongs to $(\underset{\sim}{2}, \underset{\sim}{2})$ of (spin, isospin)



A system of two nucleons belongs to

$$(2,2) \otimes (2,2)$$

$$= (2 \otimes 2,\ 2 \otimes 2)$$

$$= (3_S \oplus 1_A,\ \cancel{3_S} \oplus 1_A)$$

unstable

$$= \underbrace{(3_S, 1_A)}_{\text{overall antisymmetric}} \oplus \underbrace{(1_A, 1_A)}_{\text{overall symmetric}}$$

$$(1_A, 1_A) = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)\tfrac{1}{\sqrt{2}}(pn - np)$$ ← overall symmetric, not allowed for two identical fermions

$$= \tfrac{1}{2}(p\uparrow n\downarrow + n\downarrow p\uparrow - p\downarrow n\uparrow - n\uparrow p\downarrow)$$

sym sym

Therefore a deuteron <u>must</u> belong to

$$(3_S, 1_A) = \begin{pmatrix} \uparrow\uparrow \\ \tfrac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ \downarrow\downarrow \end{pmatrix} \tfrac{1}{\sqrt{2}}(pn - np) = \begin{pmatrix} \tfrac{1}{\sqrt{2}}(p\uparrow n\uparrow - n\uparrow p\uparrow) \\ \tfrac{1}{2}(p\uparrow n\downarrow - n\downarrow p\uparrow + p\downarrow n\uparrow - n\uparrow p\downarrow) \\ \tfrac{1}{\sqrt{2}}(p\downarrow n\downarrow - n\downarrow p\downarrow) \end{pmatrix}$$

Isospin predicts: Deuteron must have spin 1 (and it does)

General addition of angular momentum

Consider a spin-j_1 particle w/ state belonging to the space $\underline{2j_1+1}$, spanned by $|j_1, m_1\rangle$ $m_1 = -j_1, \dots, j_1$

and a spin-j_2 particle w/ state belonging to the space $\underline{2j_2+1}$, spanned by $|j_2, m_2\rangle$ $m_2 = -j_2, \dots, j_2$

The system of two particles is described by a state belonging to the tensor product space $(\underline{2j_1+1}) \otimes (\underline{2j_2+1})$

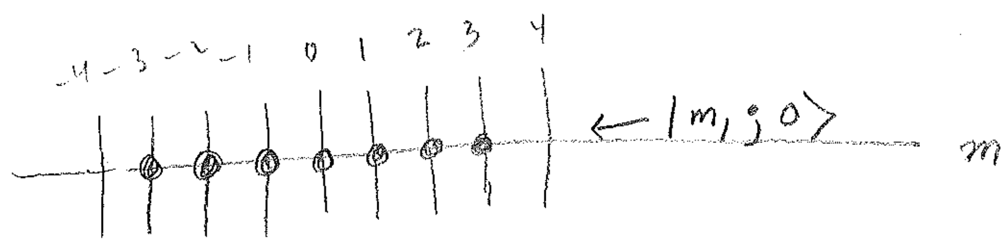
spanned by $(2j_1+1)(2j_2+1)$ product states

$$|j_1, m_1\rangle \otimes |j_2, m_2\rangle \equiv |m_1 ; m_2\rangle$$

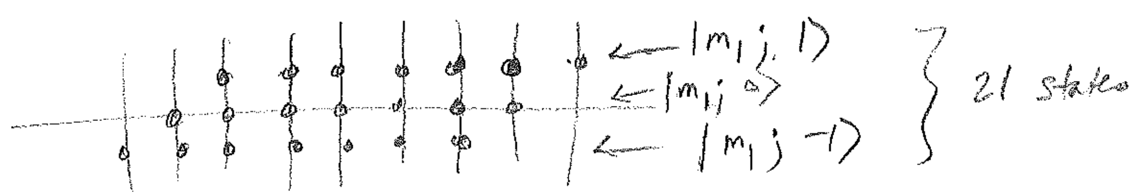
[suppress j_1 & j_2]

As before, the total spin $S_z = S_{1z} + S_{2z} = (\underbrace{m_1 + m_2}_{m})\hbar$

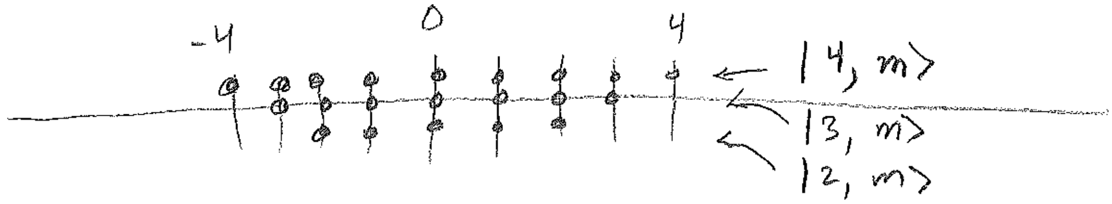
Plot these states on a weight diagram for spin 3 and spin 1: $\underset{\sim}{7} \otimes \underset{\sim}{3}$ $m_1 = -3, -2, \dots, 3$ $m_2 = -1, 0, 1$



Then



Max value of m is 4 so spin-4 rep, $\underset{\sim}{9}$



Also spin-3 rep, $\underset{\sim}{7}$ and spin-2 rep, $\underset{\sim}{5}$

$$\underset{\sim}{7} \otimes \underset{\sim}{3} = \underset{\sim}{9} \oplus \underset{\sim}{7} \oplus \underset{\sim}{5} \qquad \Rightarrow 21 \text{ states}$$

$$(\text{spin } 3) \otimes (\text{spin } 1) = (\text{spin } 4) \oplus (\text{spin } 3) \oplus (\text{spin } 2)$$

In general,

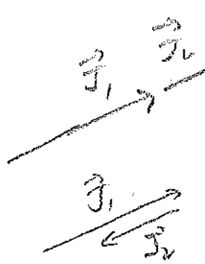
$$(\text{spin } j_1) \otimes (\text{spin } j_2) = \text{spin}(j_1+j_2) \oplus \text{spin}(j_1+j_2-1) \ldots \text{spin}|j_1-j_2|$$

(because max value of $m = j_1+j_2$)

Decomposition of tensor product space into direct sum of spaces

$$\bigoplus_{j=|j_1-j_2|}^{j=j_1+j_2} (\text{spin } j)$$

Picture



j_1+j_2

j_1-j_2

$$(\underset{\sim}{2j_1+1}) \oplus (\underset{\sim}{2j_2+1}) = (\underset{\sim}{2j_1+2j_2+1}) \oplus (\underset{\sim}{2j_1+2j_2-1}) \oplus \ldots \oplus (\underset{\sim}{2|j_1-j_2|+1})$$

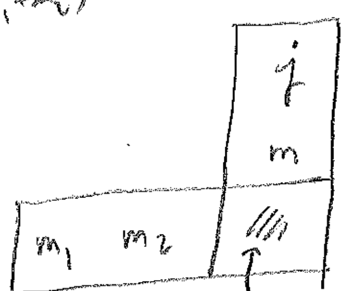
The states $|j, m\rangle$ are linear combinations of the tensor-product states $|m_1; m_2\rangle$, and vice versa.

The coefficients of the linear combinations are called Clebsch-Gordan coefficients $C^{j_1 j_2 j}_{m_1 m_2 m}$

$$|j, m\rangle = \sum_{\substack{m_1, m_2 \\ (m = m_1 + m_2)}} C^{j_1 j_2 j}_{m_1 m_2 m} \, |m_1; m_2\rangle$$

$$|m_1; m_2\rangle = \sum_{\substack{j, m \\ (m = m_1 + m_2)}} C^{j_1 j_2 j}_{m_1 m_2 m} \, |j, m\rangle$$

cf. PPB tables



take $\sqrt{\ }$ to get C
(± outside)

1/2×1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1×1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			

1/2×1/2

m_1	m_2	1, +1
+1/2	+1/2	1

m_1	m_2	1, 0	0, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	1, −1
−1/2	−1/2	1

1×1/2

m_1	m_2	3/2, +3/2
+1	+1/2	1

m_1	m_2	3/2, +1/2	1/2, +1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	3/2, −1/2	1/2, −1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	3/2, −3/2
−1	−1/2	1

2×1/2

m_1	m_2	5/2, +5/2
+2	+1/2	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	5/2, +1/2	3/2, +1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	5/2, −1/2	3/2, −1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	5/2, −3/2	3/2, −3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	5/2, −5/2
−2	−1/2	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

3/2×1/2

m_1	m_2	2, +2
+3/2	+1/2	1

m_1	m_2	2, +1	1, +1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	2, 0	1, 0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	2, −1	1, −1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	2, −2
−3/2	−1/2	1

2×1

m_1	m_2	3, +3
+2	+1	1

m_1	m_2	3, +2	2, +2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	3, +1	2, +1	1, +1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	3, 0	2, 0	1, 0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	3, −1	2, −1	1, −1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	3, −2	2, −2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	3, −3
−2	−1	1

3/2×1

m_1	m_2	5/2, +5/2
+3/2	+1	1

m_1	m_2	5/2, +3/2	3/2, +3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	5/2, −3/2	3/2, −3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	5/2, −5/2
−3/2	−1	1

1×1

m_1	m_2	2, +2
+1	+1	1

m_1	m_2	2, +1	1, +1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	2, 0	1, 0	0, 0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	2, −1	1, −1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	2, −2
−1	−1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

3/2×3/2

m_1	m_2	3, +3
+3/2	+3/2	1

m_1	m_2	3, +2	2, +2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	3, +1	2, +1	1, +1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	3, 0	2, 0	1, 0	0, 0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	3, −1	2, −1	1, −1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	3, −2	2, −2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	3, −3
−3/2	−3/2	1

2×3/2

m_1	m_2	7/2, +7/2
+2	+3/2	1

m_1	m_2	7/2, +5/2	5/2, +5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	7/2, +3/2	5/2, +3/2	3/2, +3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	7/2, +1/2	5/2, +1/2	3/2, +1/2	1/2, +1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	7/2, −1/2	5/2, −1/2	3/2, −1/2	1/2, −1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	7/2, −3/2	5/2, −3/2	3/2, −3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	7/2, −5/2	5/2, −5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	7/2, −7/2
−2	−3/2	1

2×2

m_1	m_2	4, +4
+2	+2	1

m_1	m_2	4, +3	3, +3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	4, +2	3, +2	2, +2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	4, +1	3, +1	2, +1	1, +1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	4, 0	3, 0	2, 0	1, 0	0, 0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	4, −1	3, −1	2, −1	1, −1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	4, −2	3, −2	2, −2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	4, −3	3, −3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	4, −4
−2	−2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

$$(\text{spin } \tfrac{1}{2}) \otimes (\text{spin } \tfrac{1}{2}) = (\text{spin } 1) \oplus (\text{spin } 0)$$

$$\underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\sim}{3}_S \oplus \underset{\sim}{1}_A$$

$$|1,1\rangle = |\tfrac{1}{2};\tfrac{1}{2}\rangle$$

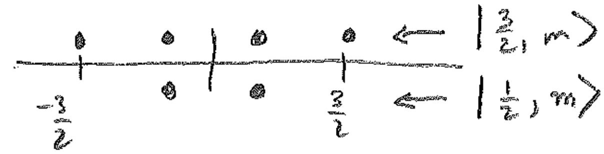
$$|1,0\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2};-\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|-\tfrac{1}{2};\tfrac{1}{2}\rangle$$

$$|1,-1\rangle = |-\tfrac{1}{2};-\tfrac{1}{2}\rangle$$

$$|0,0\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2};-\tfrac{1}{2}\rangle - \frac{1}{\sqrt{2}}|-\tfrac{1}{2};\tfrac{1}{2}\rangle$$

$$(\text{spin } 1) \otimes (\text{spin } \tfrac{1}{2}) = (\text{spin } \tfrac{3}{2}) \oplus (\text{spin } \tfrac{1}{2})$$

$$\underset{\sim}{3} \otimes \underset{\sim}{2} = \underset{\sim}{4} \oplus \underset{\sim}{2}$$



[From table]

$$|\tfrac{3}{2}, \tfrac{3}{2}\rangle = |1; \tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, |1; -\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\, |0; \tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, |0; -\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\, |-1; \tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2}, -\tfrac{3}{2}\rangle = |-1; -\tfrac{1}{2}\rangle$$

[NB: $|\alpha|^2 + |\beta|^2 = 1$]
[in 3140, students derive these]

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\, |1; -\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\, |0; \tfrac{1}{2}\rangle$$

$$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\, |0; -\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\, |-1; \tfrac{1}{2}\rangle$$

Spin state of three spin-$\frac{1}{2}$ particles (e.g. 3 quarks, 3 nucleons)

$$\underset{\sim}{2} \otimes \underset{\sim}{2} \otimes \underset{\sim}{2} = (\underset{\sim}{3}_S \oplus \underset{\sim}{1}_A) \otimes \underset{\sim}{2}$$

refers to sym'y under exchange of first two particles

$$= \underbrace{(\underset{\sim}{3}_S \otimes \underset{\sim}{2})}_{\underset{\sim}{4}_S \oplus \underset{\sim}{2}_S} \oplus \underbrace{(\underset{\sim}{1}_A \otimes \underset{\sim}{2})}_{\underset{\sim}{2}_A}$$

$$\underset{\sim}{2} \otimes \underset{\sim}{2} \otimes \underset{\sim}{2} = \underset{\text{spin-}\frac{3}{2}}{\underset{\sim}{4}_S} \oplus \underset{\text{spin-}\frac{1}{2}}{\underset{\sim}{2}_S} \oplus \underset{\text{spin-}\frac{1}{2}}{\underset{\sim}{2}_A} \qquad (8 = 8)$$

(can skip to AA 15 here)

A composite particle of 3 spin-$\frac{1}{2}$ particle may have total $j = \frac{3}{2}$ or $\frac{1}{2}$

e.g. 3 quarks
- Δ baryon has $j = \frac{3}{2}$
- N baryon has $j = \frac{1}{2}$

e.g. 3 nucleons

$^3He = ppn$
tritium $^3H = nnp$ } have $j = \frac{1}{2}$

In fact, ^3He + ^3H necessarily have spin $j = \frac{1}{2}$
because contain 2 identical fermions, the
spin state of which must be antisymmetric

$\Rightarrow$ state must be $\underset{\sim}{2}_A$

Bound state of nnn not possible (w/ symmetric spatial wavefunction)
because can't have a totally antisymmetric spin state

Spin-$\frac{3}{2}$ states ($\underset{\sim}{4}_{TS}$)

$$|\tfrac{3}{2},\tfrac{3}{2}\rangle = |1;\tfrac{1}{2}\rangle = (\uparrow\uparrow)\uparrow = \uparrow\uparrow\uparrow$$

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|1;-\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|0;\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{1}{3}}\,(\uparrow\uparrow)\downarrow + \sqrt{\tfrac{2}{3}}\sqrt{\tfrac{1}{2}}(\uparrow\downarrow+\downarrow\uparrow)\uparrow$$

$$= \tfrac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$

$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \dots = \tfrac{1}{\sqrt{3}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow + \downarrow\downarrow\uparrow)$$

$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = \downarrow\downarrow\downarrow$$

totally symmetric (TS) under exchange of any two particles

Spin-$\frac{1}{2}$ states ($\underset{\sim}{2}_S$)

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle_S = \sqrt{\tfrac{2}{3}}\,|1;-\tfrac{1}{2}\rangle - \sqrt{\tfrac{1}{3}}\,|0;\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{2}{3}}\,(\uparrow\uparrow)\downarrow - \sqrt{\tfrac{1}{3}}\sqrt{\tfrac{1}{2}}(\uparrow\downarrow+\downarrow\uparrow)\uparrow = \tfrac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle_S = \sqrt{\tfrac{1}{3}}\,|0;-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|-1;\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{1}{3}}\sqrt{\tfrac{1}{2}}(\uparrow\downarrow+\downarrow\uparrow)\downarrow - \sqrt{\tfrac{2}{3}}(\downarrow\downarrow)\uparrow = \tfrac{1}{\sqrt{6}}(\uparrow\downarrow\downarrow + \downarrow\uparrow\downarrow - 2\downarrow\downarrow\uparrow)$$

check: <u>sym'c</u> in exchange of first two

---

spin-$\frac{1}{2}$ state ($\underset{\sim}{2}_A$)

$$|\tfrac{1}{2},\tfrac{1}{2}\rangle_A = |0;\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)\uparrow = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow)$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle_A = |0;-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow)\downarrow = \tfrac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow - \downarrow\uparrow\downarrow)$$

check: <u>anti sym'c</u> in exchange of first two

[seef: $D\times D\times D = (\overset{3/2}{\square\square\square}, \overset{\text{sym}}{\square\square\square}) + (\overset{1/2}{\square}, \square)^{\dim = 2}$]

Magnetic moments of composite particles

$\vec{\mu}$ = vector sum of moments of constituents

Recall $\mu_p = 2.793\,\mu_N$ where μ_N = nuclear magneton $\equiv \frac{e\hbar}{2m_p}$
$\mu_n = -1.913\,\mu_N$

Deuteron $d = pn$ has $J = 1$

Consider $J_z = 1$ state $\frac{1}{\sqrt{2}}(p\uparrow n\uparrow - n\uparrow p\uparrow)$

Spins parallel so $\mu_d = \mu_p + \mu_n = 0.88\,\mu_N$

(expt: $0.857\,\mu_N$)

$^3H = pnn$ has $J = \frac{1}{2}$

Consider $J_z = \frac{1}{2}$ state $\frac{1}{\sqrt{2}}\,p\uparrow(n\uparrow n\downarrow - n\downarrow n\uparrow)$

neutron spins antiparallel $\Rightarrow$ moments cancel

$\mu(^3H) = \mu_p = 2.79\,\mu_N$

(expt. $2.98\,\mu_N$)

$^3He = ppn$ has $J = \frac{1}{2}$

Now proton spins cancel $\Rightarrow$ $\mu(^3He) = \mu_n = -1.913\,\mu_N$

(expt $= -2.13\,\mu_N$)