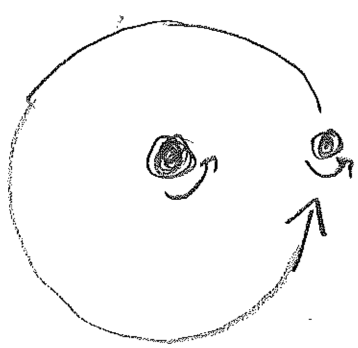
Angular momentum and spin

Invariance of laws of physics under rotations (isotropy)

$\xrightarrow[\text{Noether}]{}$ conservation of angular momentum $\vec{J}$

Earth-sun system



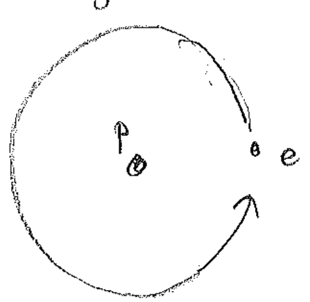
$\vec{L}$ = orbital ang mom

$\vec{S}$ = rotational ang mom

$\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$

[ask: direction of $\vec{L} + \vec{S}$?]

hydrogen atom



$\vec{L}$ = orbital

$\vec{S}$ = spin (intrinsic ang. mom)

↳ [not really rotational!]

$\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$

↓

problem?

Rotational invariance is broken by an external $\vec{B}$ field [because $\vec{B}$ field picks out a preferred direction]

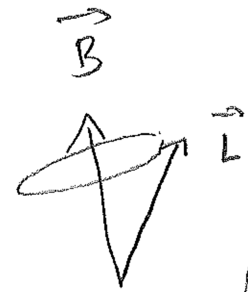
$\Rightarrow$ angular momentum is not conserved.

The change in angular momentum is caused by the torque that the $\vec{B}$ field exerts on the moving charge

$$\frac{d\vec{J}}{dt} = \vec{\tau} = \vec{\mu} \times \vec{B}, \qquad \vec{\mu} = \text{magnetic dipole moment}$$

For an orbiting point charge q w/ mass M (and no spin)

$$\vec{\mu} = \underbrace{\left(\frac{q}{2M}\right)}_{\text{called the gyromagnetic ratio}} \vec{L}$$



$$\Rightarrow \frac{d\vec{L}}{dt} = \frac{q}{2M} \vec{L} \times \vec{B} \quad \text{(gyroscope eqn)}$$

$\vec{L}$ precesses around $\vec{B}$

$$\boxed{\begin{aligned} dA &= \tfrac{1}{2} r^2 d\theta \\ \frac{dA}{dt} &= \tfrac{1}{2} r^2 \omega = \frac{L}{2m} \\ \mu &= IA = \frac{q}{T} A \\ &= q \frac{dA}{dt} = \frac{q}{2M} L \end{aligned}}$$

If $\vec{B} = B\hat{z}$, then L_z is still conserved but L_x + L_y change over time

We can understand that L_z is conserved because $\vec{B}$ field is invariant under rotations about $\hat{z}$

A stationary particle w/ spin $\vec{S}$ also has magnetic moment

$$\vec{\mu} = \underbrace{g}_{\text{dimensionless } g\text{-factor}} \left(\frac{q}{2M}\right)\vec{S}$$

dimensionless <u>g-factor</u> (Landé)

$$\frac{d\vec{S}}{dt} = g \frac{q}{2M} \vec{S} \times \vec{B}$$

Precession rate proportional to g $\Rightarrow$ can experimentally determine g

proton $g \approx 2(2.793\ldots)$ ← [PPB]

neutron $g \approx 2(-1.913\ldots)$ [even though neutral; due to quarks]

[These seem somewhat random.
Due to fact that p + n are composite, made of quarks
By end of course, we'll be able to compute these approximately]

electron $g = 2(1.001\,159\,652\,181\ldots)$

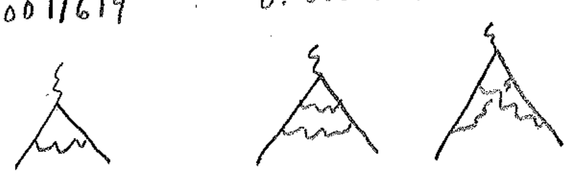
[Much simpler.]

Dirac eqn predicts that fundamental spin-$\frac{1}{2}$ particles have $g = 2$

[proton + neutron not fundamental]

QED predicts small corrections

$$g = 2\Big(1 + \underbrace{\frac{\alpha}{2\pi}}_{0.0011614} - \underbrace{0.328\left(\frac{\alpha}{\pi}\right)^2}_{0.000\,0018} + \ldots\Big)$$



[see fig 8.2 for α^3 corrections. Need α^4 corrections for 12 sig figs]

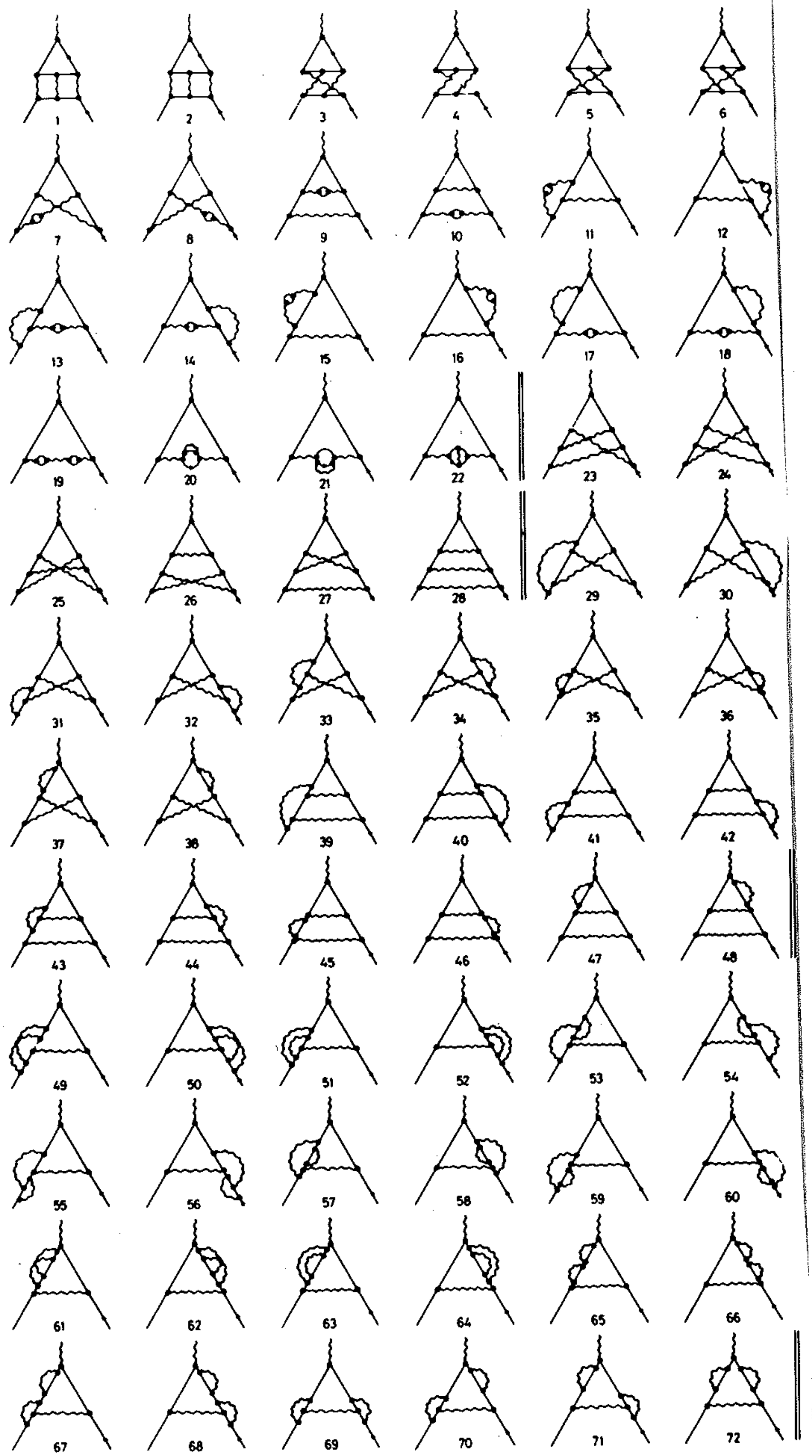


Fig. 8.2 The Feynman graphs which have to be evaluated in computing the α^3 corrections the lepton magnetic moments (after Lautrup *et al.* 1972).

τ – ~~[illegible]~~

1948
[P. Kusch studied Zeeman splitting in In, Ga, Na
$g = 2(1 + 1.19E{-3})$]

exp. ⇒ $g_e = 2.002\ 319\ 304\ 361\ 4(6)$

[experimental result = very accurate: 13 sig figs
agrees w/ theoretical calculation to accuracy measured]
Feynman calls this our "crown jewel"]

usually quoted as fractional difference from Dirac prediction

$$\frac{\mu - \mu_B}{\mu_B} = \frac{g_e - 2}{2} = 0.001\ 159\ 652\ 180\ 76\ (27) \quad \text{(exp)}$$

"electron magnetic moment anomaly"

agrees w/ theoretical prediction [PDB 2012]
$\left[\frac{g-2}{2} = \frac{\alpha}{2\pi} + \ldots,\ \text{Schwinger}\right]$

Currently there is some discrepancy in muon magnetic moment

$$\left(\frac{g_\mu - 2}{2}\right)_{exp.} = 0.001\ 165\ 920\ 9\ (6) \quad \text{(exp)}$$

(Standard model prediction)

$$\left(\frac{g_\mu - 2}{2}\right)_{th} = 0.001\ 165\ 918\ 0\ (6)$$

[RPP 2012, p584]

$$\text{Difference} = 0.000\ 000\ \underbrace{0029}\ (8)$$

$\sim 3\sigma$ effect [new physics? susy?]

$$\left(\frac{g_\tau - 2}{2}\right)_{exp} = 0.$$

Of course ν have no magnetic moment, being neutral

Quantum: angular momentum is quantized
in units of $\hbar$ (Bohr)

[True for each component.]

$L_z = m\hbar$ (m = integer)

[Interested in L_z because it is conserved in $\vec{B} = B\hat{z}$ field.]

Spin is also quantized

$S_z = m\hbar$ (m = integer or half-integer)

For a given type of particle, there is a maximum/minimum
value for m, which we call j (or $-j$)

j = spin of the particle

spin	type	example: fundamental	composite
0	scalar	Higgs boson	scalar meson π
$\frac{1}{2}$	spinor	quarks, leptons	proton & neutron
1	vector	γ, $W^\pm$, Z^0	vector meson ρ
$\frac{3}{2}$	spinor-vector	gravitino?	Δ baryons
2	tensor	graviton	

For a massive particle w/ given spin j,
the possible values of S_z are $m\hbar$ where

$$m = \underbrace{-j,\ -j+1,\ \ldots,\ j-1,\ j}_{2j+1 \text{ possible values}}$$

$2j+1$ possible values (called spin <u>multiplicity</u>)

(For massless particles, only $m = \pm j$ are possible
$\Rightarrow$ 2 photon or graviton polarizations)

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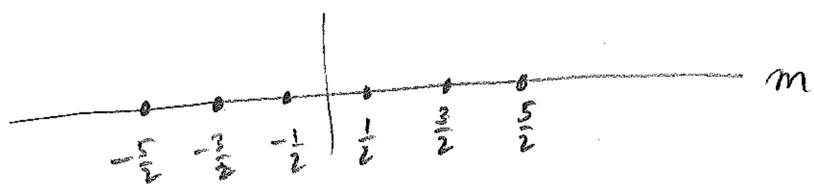
we can represent S_z as points on a "weight diagram"

$j = \frac{5}{2} \Rightarrow$



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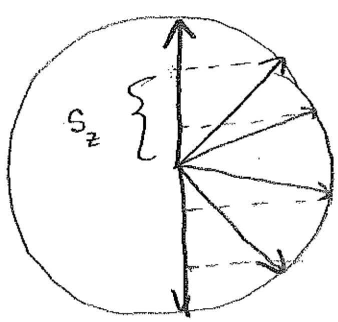
we can visualize these as spin vectors in different directions



don't take too seriously!

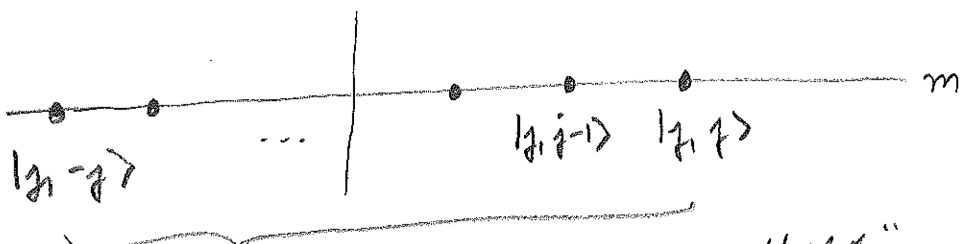
magnitude of spin vector $= j\hbar$ (same in all directions)

In QM, describe the state of a system by a wave function $\psi(x,t)$ or a ket $|\psi\rangle$ which belongs to a vector space (Hilbert space)

Denote the state of a spin-j particle that has $S_z = m\hbar$ by

$$|j,m\rangle$$

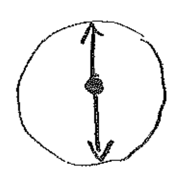
The complete set of states of a spin-j particle particle is



Call this set of states "spin-j multiplet"

Denote it by $(2\underset{\sim}{j}+1)$ where $2j+1$ = multiplicity of the multiplet

An important example will be the spin-$\frac{1}{2}$ multiplet $\underset{\sim}{2}$ (or doublet)



call this $|-\rangle$ or $\downarrow$ (spin down)

call this $|+\rangle$ or $\uparrow$ (spin up)

Quantum mechanical states live in a vector space, so arbitrary linear combinations of states are also states

For spin-$\frac{1}{2}$ particle, states are $|+\rangle$ and $|-\rangle$ (spin up & down)

Most general state is

$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$ where α, β are complex constants

α, β = probability amplitudes

Probability of measuring spin up is $|\alpha|^2$
and spin down is $|\beta|^2$ $\Big\}$ $|\alpha|^2 + |\beta|^2 = 1$

We say: $|\psi\rangle$ lives in a vector space $\underset{\sim}{2}$ spanned by basis states $|+\rangle$ and $|-\rangle$

For spin-j particle, states are $|j, m\rangle$, $m = -j \ldots j$

Most general state is

$$|\psi\rangle = \sum_{m=-j}^{j} c_m |j, m\rangle$$

Probability of measuring $S_z = m\hbar$ is $|c_m|^2$

$$\sum_{m=-j}^{j} |c_m|^2 = 1$$

We say $|\psi\rangle$ lives in a space $(\underset{\sim}{2j+1})$, spanned by $|j, m\rangle$ $m = -j \ldots j$

The different states $|j, m\rangle$ of a spin-j multiplet can be visualized as spin vector pointing in various directions.

In an isotropic (= rotationally-symmetric) environment, these states all have the same energy (energy can't depend on direction in an isotropic environment)

They are <u>degenerate</u>.

Key point: symmetry (rotational) $\Rightarrow$ quantum degeneracy

[recall: classically, symmetry $\Rightarrow$ conservation laws]

A magnetic field $\vec{B}$ breaks the rotational symmetry & $\therefore$ splits the degeneracy among states $|j,m\rangle$
(All directions no longer the same)

Recall $\vec{B}$ exerts torque on a magnet $\vec{\tau} = \vec{\mu} \times \vec{B}$

$$E_{dipole} = -\vec{\mu}\cdot\vec{B}$$

$\vec{B}\uparrow$ $\uparrow\vec{\mu}$ vs $\downarrow\vec{\mu}$

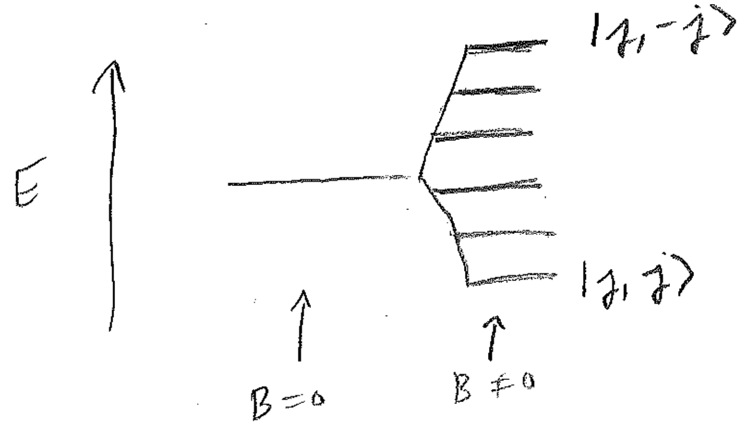
lower energy, higher energy

$$= -g\frac{q}{2M}\vec{S}\cdot\vec{B}$$

Let $\vec{B} = B\hat{z}$

$$E_{dipole} = -\frac{gqB}{2M}S_z$$

$$= -\left(\frac{gqB\hbar}{2M}\right)m \qquad m = -j \ldots j$$



states $|j,m\rangle$ have different energies
[Zeeman effect]

broken symmetry $\rightarrow$ split degeneracy

magnetic moment of
Spin-$\frac{1}{2}$ particles

$$\vec{\mu} = \frac{gq}{2M}\vec{S}$$

$$|\vec{\mu}| = \frac{gq}{2M}\frac{\hbar}{2}$$

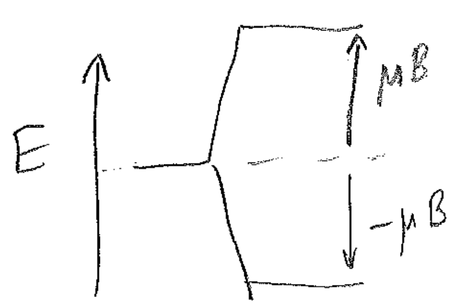
Electron has $g \approx 2 \Rightarrow \mu_e \approx \frac{e\hbar}{2m_e} \equiv \mu_B$ (Bohr magneton)

$\left(\begin{array}{c}\text{Gaussian} \\ \mu_B = \frac{e\hbar}{2m_e c}\end{array}\right)$

Proton has $g = 2(2.793) \Rightarrow \mu_p = 2.793\,\mu_N$

where $\mu_N \equiv \frac{e\hbar}{2m_p}$ (nuclear magneton)

Neutron has $g = 2(-1.913) \Rightarrow \mu_n = -1.913\,\mu_N$



$\mu_B = 5.8 \times 10^{-5} \frac{eV}{\text{Tesla}}$

20T field $\Rightarrow 0.001\,eV$

$\mu_N \approx \frac{\mu_B}{1840}$

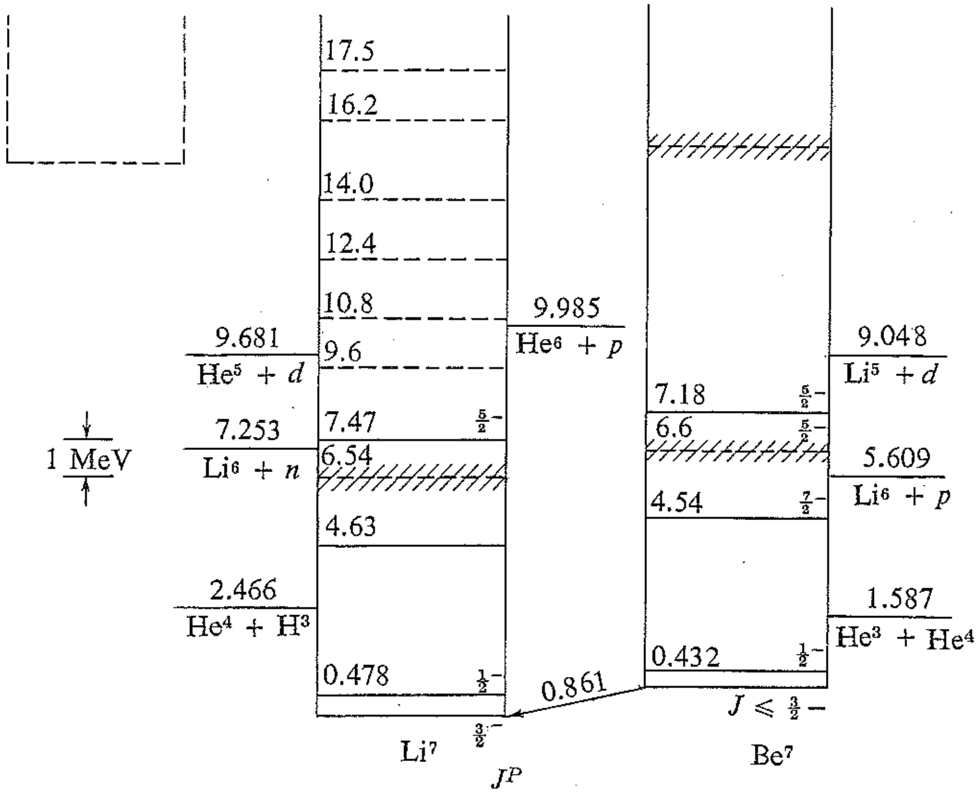
Isospin symmetry

[Consider the proton & neutron.
They have different electric charges,
but nearly the same mass: 938.3 vs 939.6.
Moreover, the strong interaction treats them very similarly: mirror nuclei have similar energy levels
(see energy level diagrams)]

Could this degeneracy be the hint of some symmetry?

[We know that rotational symmetry ⇒ different spin states $|j,m\rangle$ have degenerate energies.
magnetic field breaks the symmetry & splits the degeneracy]

[Heisenberg suggested the existence of a new symmetry called isotopic spin, or isospin.
The strong interaction respects isospin symmetry so that different nuclei have same energies
whereas the electromagnetic interaction breaks the symmetry, leads to some splitting.]


17.5
16.2
14.0
12.4
10.8
9.6
7.47
6.54
4.63
0.478
9.681
He⁵ + d
7.253
Li⁶ + n
2.466
He⁴ + H³
1 MeV
9.985
He⁶ + p
0.861
Li⁷
JP
7.18
6.6
4.54
0.432
J ≤ 3/2 −
Be⁷
9.048
Li⁵ + d
5.609
Li⁶ + p
1.587
He³ + He⁴

Segre, p. 391

We introduce

Isospin $\vec{I} = (I_1, I_2, I_3)$ analogous to $\vec{S} = (S_x, S_y, S_z)$

Isospin is quantized in integer or half-integer units.

Hadronic particles correspond to states

$| I, I_3 \rangle$ analogous to $| j, m \rangle$

$I_3 = -I, \ldots, I$ $\quad$ $m = -j, \ldots, j$

Recall; in a rotationally-symmetric (isotropic) environment

the energy of the state $|j, m\rangle$ does not depend on m

$\Rightarrow (2j+1)$ degenerate states

Turning on a magnetic field splits the degeneracy.

Analogously, if only the strong interaction were present,

the mass of the state $|I, I_3\rangle$ would not depend on I_3

$\Rightarrow (2I+1)$ degenerate particles

Strong force is "isotopically isotropic"

(Weak + electromagnetic interactions do depend on I_3, splitting the degeneracy, but the mass splitting is small because these forces are weak.)

Because p and n are a nearly degenerate doublet we regard them as two states of a single isospin-$\frac{1}{2}$ particle N (nucleon).

$$N \text{ has states } |I, I_3\rangle = \begin{cases} |\tfrac{1}{2}, \tfrac{1}{2}\rangle = p \\ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = n \end{cases}$$

We say N belongs to isospin representation $\underset{\sim}{2}$.

A nucleon also has spin-$\frac{1}{2}$ & so belongs to $\underset{\sim}{2}$ of spin

$$\text{Nucleon belongs to } (\underbrace{\underset{\sim}{2}}_{\text{spin}}, \underbrace{\underset{\sim}{2}}_{\text{isospin}})$$

and has 4 states

$p\uparrow, \quad p\downarrow, \quad n\uparrow, \quad n\downarrow$

[We'll soon see why this way of thinking is useful.]