<u>Massless particles</u>

$$E^2 = (c\vec{p})^2 + \cancel{(mc^2)^2} \to 0$$

$$\Rightarrow \quad E = c|\vec{p}|$$

$$\frac{c\vec{p}}{E} = \frac{\vec{v}}{c} \quad \Rightarrow \quad |\vec{v}| = c$$ always travel at speed of light (photons, gravitons)

$$E = \cancel{mc^2} + T \quad (\to 0)$$

$$\Rightarrow \quad E = T$$ all energy is kinetic (never at rest)

$E = \gamma mc^2$ and $p = \gamma mv$ give indefinite results as $m \to 0$ because $\gamma \to \infty$ as $v \to c$

Photon energy + momentum depend not on speed but on frequency and wavelength

$$E = hf = \hbar\omega$$

$$p = \frac{h}{\lambda} = \hbar k$$

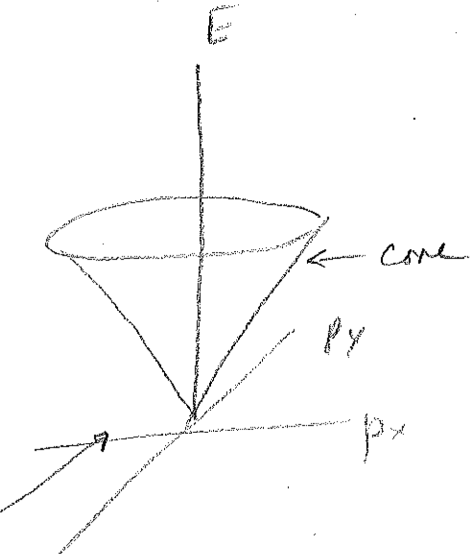
$$E = cp \quad \Rightarrow \quad f = \frac{c}{\lambda} \quad \text{and} \quad \omega = ck$$

Setting $c = 1$

$$E = |\vec{p}|$$

$$p^\mu = (|\vec{p}|, \vec{p})$$

$$p^2 = 0$$



A photon on the mass-shell:

$$E^2 = \vec{p}^{\,2}$$

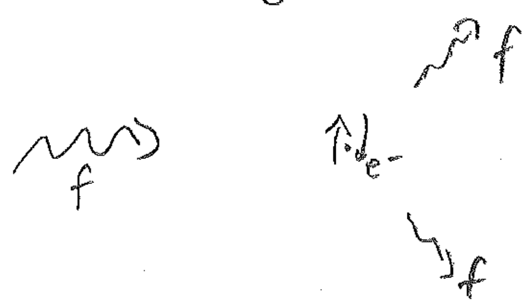
$$E = |\vec{p}|$$

[Physicists were skeptical of the photon concept until Arthur Compton did his expt on X-rays (1922)

X-rays were known to be EM waves

Scattering of X-rays by electrons (Thomson or Compton scattering)

Classical picture: electromagnetic waves of frequency f cause electrons to oscillate w/ frequency f.
These accelerating electrons emit EM waves of frequency f.



Classical cross-section $\sigma = \frac{R}{F}$

R = rate at which energy is radiated by electron

F = flux of incident EM waves

From 1140, $F = \epsilon_0 c |\vec{E}|^2$

From 3120, Larmor formula $R = \frac{e^2 a^2}{6\pi\epsilon_0 c^3}$

$$\vec{a} = \frac{\vec{F}}{m_e} = \frac{e\vec{E}}{m_e} \Rightarrow R = \frac{e^4 |\vec{E}|^2}{6\pi\epsilon_0 m_e^2 c^3} \Rightarrow \sigma =$$

$$\Rightarrow \sigma = \frac{e^4}{6\pi\epsilon_0^2 (m_e c^2)^2} = \frac{8\pi}{3}\left(\frac{ke^2}{m_e c^2}\right)^2 = \frac{8\pi}{3} r_0^2$$

where $r_0 \equiv \frac{ke^2}{m_e c^2}$ = classical electron radius $= \frac{1.44\,\text{MeV fm}}{0.511\,\text{MeV}} = 2.8$ fm

$\sigma = 66.5\ \text{fm}^2 = \frac{2}{3}$ barn = Thomson cross-section

[Thomson used it to measure # electrons in atoms]
Nucleus too massive to radiate much.

Compton, however, measured the wavelength of scattered X-rays and found it was slightly longer than incident X-rays

$\lambda' > \lambda$

Compton shift $\Delta\lambda = \lambda' - \lambda$ is proportional to $\frac{h}{m_e c}$, called the Compton wavelength of the electron

$$\frac{h}{m_e c} = \frac{2\pi\hbar c}{m_e c^2} = \frac{2\pi(197 \text{ MeV fm})}{0.511 \text{ MeV}} = 2400 \text{ fm} \approx 2.4\times10^{-12} \text{ m}$$

~~[illegible]~~ an unnoticeably small shift for visible light $\lambda \sim 5\times10^{-7}$ m

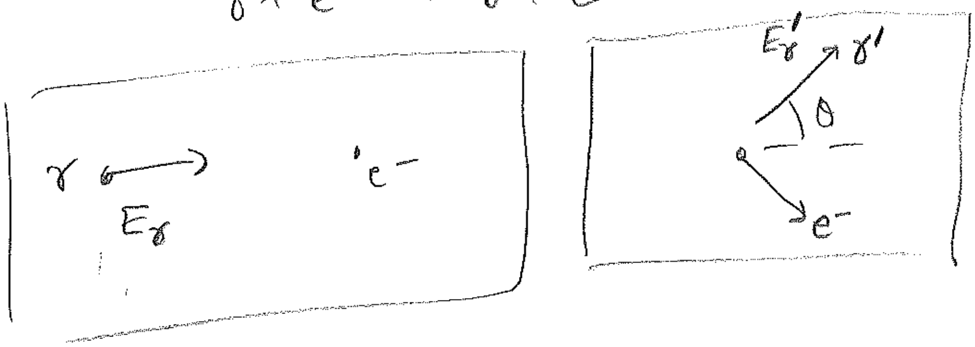
[~~It~~ is still small,] but measurable, for X-rays $\lambda \sim 10^{-10}$ m

Compton found
The shift depends on the scattering angle

$$\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1-\cos\theta)$$

Compton scattering can be understood as a collision of a massless photon & a massive electron

$$\gamma + e^- \to \gamma + e^-$$



Photon gives some of its energy to the electron, so

$E'_\gamma < E_\gamma$

Since $E = hf \Rightarrow f' < f$

Since $f = \frac{c}{\lambda} \Rightarrow \lambda' > \lambda$

One can compute the Compton shift using energy & momentum conservation

This is most elegantly done using 4-momenta

$$p_\gamma = (E_\gamma, 0, 0, E_\gamma) \qquad p_\gamma' = (E_\gamma', E_\gamma' \sin\theta, 0, E_\gamma' \cos\theta)$$

$$p_e = (m_e, 0, 0, 0) \qquad p_e' = (E_e', -|\vec{p}_e'|\sin\phi, 0, |\vec{p}_e'|\cos\phi)$$

$$p_\gamma + p_e = p_\gamma' + p_e'$$

Rewrite this as

$$-p_e' + p_e = p_\gamma' - p_\gamma$$

Consider

$$(-p_e' + p_e)^2 = p_e'^2 - 2p_e' \cdot p_e + p_e^2$$

$$= m_e^2 - 2(E_e' m_e - 0) + m_e^2$$

$$= 2m_e(m_e - E_e') \underset{\substack{\uparrow \\ \text{energy cons.}}}{=} 2m_e(E_\gamma' - E_\gamma)$$

Now consider

$$(p_\gamma' - p_\gamma)^2 = p_\gamma'^2 - 2p_\gamma' \cdot p_\gamma + p_\gamma^2$$

$$= 0 - 2(E_\gamma' E_\gamma - E_\gamma' E_\gamma \cos\theta)$$

$$= -2E_\gamma E_\gamma'(1 - \cos\theta)$$

Equate these

$$-2m_e(E_\gamma - E_\gamma') = -2E_\gamma E_\gamma'(1 - \cos\theta)$$

$$\frac{E_\gamma - E_\gamma'}{E_\gamma E_\gamma'} = \frac{1}{m_e}(1 - \cos\theta)$$

$$\frac{1}{E_\gamma'} - \frac{1}{E_\gamma} = \frac{1}{\underset{\substack{\uparrow \\ \text{restore}}}{m_e c^2}}(1 - \cos\theta)$$

$$E = hf = \frac{hc}{\lambda}$$

$$\Rightarrow \quad \frac{\lambda'}{hc} - \frac{\lambda}{hc} = \frac{1}{m_e c^2}(1-\cos\theta)$$

$$\lambda' - \lambda = \underbrace{\frac{h}{m_e c}}_{\text{Compton wavelength of an electron}}(1-\cos\theta)$$

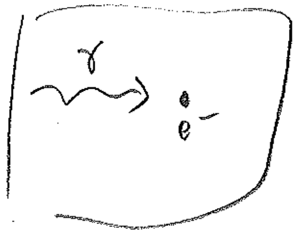
Max shift occurs for back scattered X-rays $(\theta = \pi)$

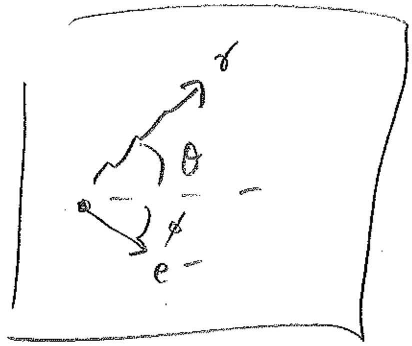
$$(\Delta\lambda)_{max} = \frac{2h}{m_e c} \approx 4.8\times10^{-12}\,m$$

Compton used molybdenum K_α line $\Rightarrow 7\times10^{-11}\,m$
(17 keV)

(7% shift)

Afterwards





Can do this more formally using 4-vectors →

$$\begin{cases} E_\gamma + mc^2 = E_\gamma' + E_e' \\ p_\gamma = p_\gamma' \cos\theta + p_e' \cos\phi \\ 0 = p_\gamma' \sin\theta - p_e' \sin\phi \end{cases}$$

Eliminate ϕ:

$$(p_\gamma - p_\gamma' \cos\theta)^2 + (p_\gamma' \sin\theta)^2 = p_e'^2$$

$$p_\gamma^2 - 2 p_\gamma p_\gamma' \cos\theta + (p_\gamma')^2 = p_e'^2$$

Energy $\Rightarrow$ $(E_\gamma + mc^2 - E_\gamma')^2 = E_e'^2$

$$E_\gamma^2 + E_\gamma'^2 + 2E_\gamma mc^2 - 2E_\gamma' mc^2 - 2E_\gamma E_\gamma' = E_e'^2 - (mc^2)^2$$

$$2E_\gamma mc^2 - 2E_\gamma' mc^2 - 2E_\gamma E_\gamma'(1-\cos\theta) = 0$$ Subtract mom from energy

(a) $$\frac{1}{E_\gamma'} - \frac{1}{E_\gamma} = \frac{1-\cos\theta}{mc^2} \Rightarrow f(\theta) = 1-\cos\theta$$

(b) $$E = \frac{hc}{\lambda} \Rightarrow \lambda' - \lambda = \underbrace{\frac{h}{mc}}_{2.4\times10^{-12}\,m}(1-\cos\theta)$$

$\theta = \pi \Rightarrow \Delta\lambda = 4.8E{-12}\,m$

$$\lambda = \frac{hc}{17\,KeV} = \frac{1.24E{-6}\ eV\cdot m}{1.7E4\ eV} = 7.3\times10^{-11}\,m$$

$$\frac{\Delta\lambda}{\lambda} \sim 0.066$$

Alternative

Use 4-vectors to compute Compton scattering

$$p_e' = p_e + p_\gamma - p_\gamma'$$

$$(p_e')^2 = p_e^2 + p_\gamma^2 + p_\gamma'^2 + 2 p_e \cdot p_\gamma - 2 p_e \cdot p_\gamma' - 2 p_\gamma \cdot p_\gamma'$$

these cancel   these vanish

so

$$p_e \cdot p_\gamma - p_e \cdot p_\gamma' = p_\gamma \cdot p_\gamma'$$

$$p_e = \begin{pmatrix} mc^2 \\ \vec{0} \end{pmatrix}, \quad p_\gamma = \begin{pmatrix} E \\ E\vec{n} \end{pmatrix}, \quad p_\gamma' = \begin{pmatrix} E' \\ E'\vec{n}' \end{pmatrix}$$

$$mc^2 E - mc^2 E' = EE'(1 - \underbrace{\vec{n}\cdot\vec{n}'}_{\cos\theta})$$

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{mc^2}(1 - \cos\theta)$$

Alternatively: $p_e' - p_e = p_\gamma - p_\gamma'$

$$2m^2 - 2 p_e \cdot p_e' = -2 p_\gamma \cdot p_\gamma'$$

$$\underbrace{2m^2 - 2m \underbrace{E_e}_{E - E' + m}}_{-2m(E-E')} = -2EE'(1 - \cos\theta)$$

$$\frac{1}{E'} - \frac{1}{E} = \frac{1}{m}(1 - \cos\theta)$$