Momentum

$$E = mc^2 + T$$

$$E^2 = (c\vec{p})^2 + (mc^2)^2 \qquad \vec{p}^2 \equiv \vec{p}\cdot\vec{p} = |\vec{p}|^2$$

As before, we'll often omit the c's

$$E = m + T$$

$$E^2 = \vec{p}^2 + m^2$$

and express $E, \vec{p}, m$ all in terms of energy units (MeV)

When we say $|\vec{p}| = 5$ MeV or $m = 0.5$ MeV
we mean $|c\vec{p}| = 5$ MeV and $mc^2 = 0.5$ MeV

Conservation of energy and 3-momentum

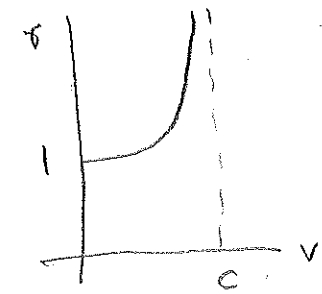
$$E^{sys}_{init} = E^{sys}_{final}$$

$$\vec{P}^{sys}_{init} = \vec{P}^{sys}_{final}$$

If we need to express $E, \vec{p}$ in terms of $\vec{v}$, use

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

gamma factor
time-dilation factor



$$E = \gamma m c^2$$

$$T = (\gamma - 1) m c^2$$

$$\vec{p} = \gamma m \vec{v}$$

$$\frac{c\vec{p}}{E} = \frac{\vec{v}}{c}$$

omitting c's:

$$\frac{\vec{p}}{E} = \vec{v}$$

where $p, E \sim$ MeV and $v \sim$ dimensionless ($v = 1 \Rightarrow$ speed of light)

often not necessary or helpful to use these

when applying energy + mom conservation

Non relativistic limits

$$\gamma = 1 + \frac{1}{2}v^2 + \dots$$

$$\vec{p} = \gamma m\vec{v} = m\vec{v} + \dots$$

$$E = \gamma m = m + \frac{1}{2}mv^2 + \dots$$

$$E = \sqrt{p^2+m^2} = m\sqrt{1+\frac{p^2}{m^2}} = m + \frac{p^2}{2m} + \dots$$

Ultrarelativistic limit

$$\gamma = \frac{1}{\sqrt{1-v^2}}$$

$$v = \sqrt{1-\frac{1}{\gamma^2}} = 1 - \frac{1}{2\gamma^2} + \dots$$

$$E = \sqrt{p^2+m^2} = p\sqrt{1+\frac{m^2}{p^2}} = p + \frac{m^2}{2p} + \dots$$

[proton in LHC has $E = 7$ TeV, $\gamma = 7000$, $v = 1-(10^{-8}) = 0.\underbrace{999\ 999\ 99}_{8\ 9\text{'s}}$]

4-vector $u^\mu = (u^0, u^1, u^2, u^3)$

Transforms as $u^\mu \to \Lambda^\mu{}_\nu u^\nu$ under Lorentz transformation

We'll often omit the index, simply writing u
but it is implied by context

To distinguish 4-vectors from 3-vectors: u vs $\vec{u}$

Primarily we'll be concerned w/

4-momentum $p = (E, \vec{p})$

Conservation of 4-momentum

$$P^{sys}_{init} = P^{sys}_{final}$$

Given two 4-vectors u and w

define Lorentz-invariant scalar product

$$u \cdot w = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} u^\mu w^\nu$$

where Minkowski metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \leftarrow$$

particle physics convention "mostly minus metric"

$$u \cdot w = u^0 w^0 - \vec{u} \cdot \vec{w}$$

$$u^2 = (u^0)^2 - \vec{u}^{\,2}$$

Use arrow to distinguish square of 3-vector vs 4-vector

[4-momenta & Lorentz scalars are a powerful tool.

For a single particle

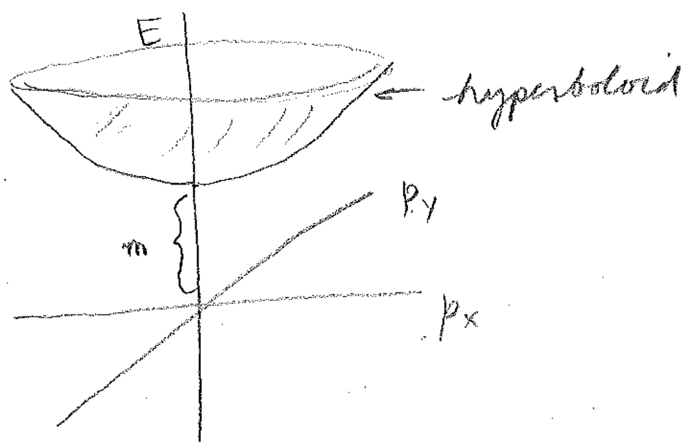
$p = (E, \vec{p})$

$p^2 = E^2 - \vec{p}^2 = m^2$

$\boxed{p^2 = m^2}$

If a particle obeys $p^2 = m^2$ we say it is "on the mass shell"

$E^2 = \vec{p}^2 + m^2$



This is true for physical particles, but not virtual ones

Two particle decay (e.g. α-decay)

$$X \to Y + \alpha$$

Energy + momentum conservation complete determine the kinematics

X at rest $\Rightarrow$ 

momentum: $\vec{p}_Y = -\vec{p}_\alpha$

energy: $E_Y + E_\alpha = m_X$

mass-shell condition $\Rightarrow$ 2 eq., 2 unknowns

Elegant solution using 4-momenta

$$p_X = p_Y + p_\alpha$$

$$p_X - p_\alpha = p_Y$$

"square" both sides

$$p_X^2 - 2p_X \cdot p_\alpha + p_\alpha^2 = p_Y^2$$

$$p_X \cdot p_\alpha = E_X E_\alpha - \vec{p}_X \cdot \vec{p}_\alpha = m_X E_\alpha - 0$$

$p_X^2 = m_X^2$ etc.

$$E_\alpha = \frac{m_X^2 + m_\alpha^2 - m_Y^2}{2m_X}$$

Similarly: $$E_Y = \frac{m_X^2 + m_Y^2 - m_\alpha^2}{2m_X}$$

check: $E_\alpha + E_Y = m_X$ ✓