Liquid drop model of nucleus

Semi-empirical binding energy formula
(Bethe-Weizsäcker)

(Fits data well for $A \gtrsim 15$)

$$B = c_1 A - c_2 A^{2/3} - c_3 \frac{Z^2}{A^{1/3}} - c_4 \frac{(A-2Z)^2}{A} + \delta_{pair}$$

A fit to the empirical data gives

[Krane]

$$\begin{cases} c_1 = 15.5 \text{ MeV} \\ c_2 = 16.8 \text{ MeV} \\ c_3 = 0.72 \text{ MeV} \\ c_4 = 23 \text{ MeV} \\ c_5 = 34 \text{ MeV} \end{cases}$$

$$\delta_{pair} = \begin{cases} c_5 A^{-3/4} & Z, N \text{ even} \\ 0 & Z+N \text{ odd} \\ -c_5 A^{-3/4} & Z, N \text{ odd} \end{cases}$$

c_1 due to strong interaction (volume term) and kinetic energy

c_2 due to strong interaction (surface term)

c_3 due to EM. int. (Coulomb repulsion) $\Rightarrow$ acts to reduce Z

c_4 due to kinetic energy + strong force $\Rightarrow$ acts to keep $Z \approx N$

c_5 due to spin pairing $\Rightarrow$ favors even-even nuclei

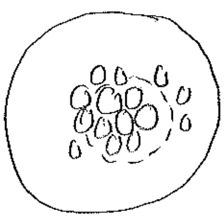
Recall $B = -(\sum T + \sum V)$

[First think about potential energies due to forces

(Force primarily responsible for existence of nucleus is

<u>strong nuclear force</u>

- acts between pairs of nucleons (pp, pn, nn)
- short range) range $\ll$ size of nucleus R



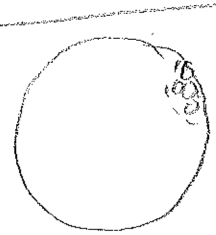
Suppose each nucleon acts on r others for some r

total # of pair ("bonds") is $\frac{1}{2} r A$

Potential energy $= -$(# pairs)(energy of "bond")

$\Rightarrow B \approx c_1 A$ (volume term)

Then $\frac{B}{A} \approx c_1$ explains why binding energy per nucleon is $\approx$ const (due to short range nature of force, else $B \sim A^2$)



Nucleons on surface form fewer "bonds"

"missing bonds" $\sim$ # of nucleons on surface $\sim 4\pi R^2 \sim A^{2/3}$

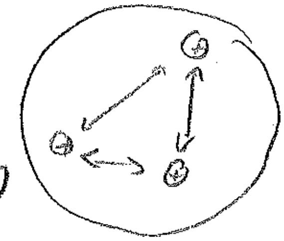
$B \approx c_1 A - c_2 A^{2/3}$ (surface term)

Surface term causes nucleus to minimize surface area $\Rightarrow$ sphere

Liquid drop model of nucleus [Bohr]

Electromagnetic potential energy

- acts between pairs of protons
- long range: affects all protons in nucleus
- repulsive $\Rightarrow$ $V > 0$ $\Rightarrow$ reduces binding energy



Treat protons as uniformly distributed charge $q = Ze$

Potential energy of sphere of charge $V = \frac{3}{5}\frac{Kq^2}{R} = \underbrace{\left(\frac{3Ke^2}{5r_0}\right)}_{c_3}\frac{Z^2}{A^{1/3}}$

Estimate $c_3 = \frac{3Ke^3}{5r_0} = \frac{3\alpha\hbar c}{5r_0} = \frac{(0.6)(1.44\text{ MeV}\cdot\text{fm})}{(1.2\text{ fm})} = \underline{\underline{0.72\text{ MeV}}}$

[Krane, p. 67]

$$B \approx c_1 A - c_2 A^{2/3} - c_3 \frac{Z^2}{A^{1/3}} + \dots$$

$$\frac{B}{A} \approx c_1 - \frac{c_2}{A^{1/3}} - c_3 \frac{Z^2}{A^{4/3}}$$

causes curve to bend down for small A

[hydrogen has B = 0]

causes curve to bend down for large Z & A & eventually nucleus becomes unstable

[EM acts on all protons, strong force only on nearby nucleons]

$$B = -(\Sigma T) - (\Sigma V)$$

Binding energy depends, not only on potential energy but also on the kinetic energy of the <u>nucleons</u> (not the nucleus itself, which is at rest)

Nucleons must have kinetic energy because they are confined to a small space, so by Heisenberg uncertainty principle cannot have zero momentum

We will treat the nucleons as particles confined to a box, but within the box they are <u>noninteracting</u> (not very realistic).

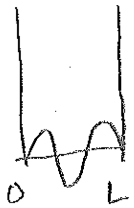
Box is really a sphere of radius $R = A^{1/3} r_0$ but we'll treat it as a cube of the same volume

$$L^3 = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi A r_0^3$$

Boundary conditions don't make much difference

First question: what are the allowed momenta?

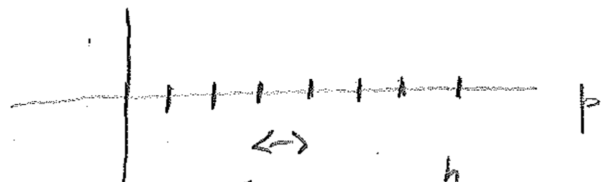
Consider a 1-dimensional box of length L.



If we impose Dirichlet b.c. on the particle (i.e. $\psi(x)$ vanishes at edges of box)

then de Broglie wavelength $\lambda = \frac{2L}{n}$

then $|\vec{p}| = \frac{h}{\lambda} = \frac{hn}{2L}$, $n = 1, 2, 3, \ldots$

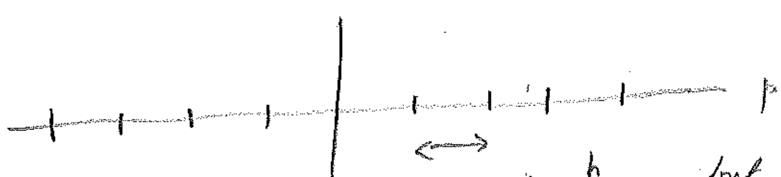


momentum spacing is $\frac{h}{2L}$

Alternatively one can impose <u>periodic</u> b.c. $\psi(x+L) = \psi(x)$

Then $\psi = e^{\frac{ipx}{\hbar}} \Rightarrow e^{\frac{ipL}{\hbar}} = 1 \Rightarrow \frac{pL}{\hbar} = 2\pi n$

so $p = \frac{hn}{L}$, $n = \ldots, -3, -2, -1, 0, 1, 2, 3 \ldots$



spacing is $\frac{h}{L}$ but <u>negative</u> p are allowed

These give equivalent results because both give same # of states for $|\vec{p}| \leq p_{max}$,

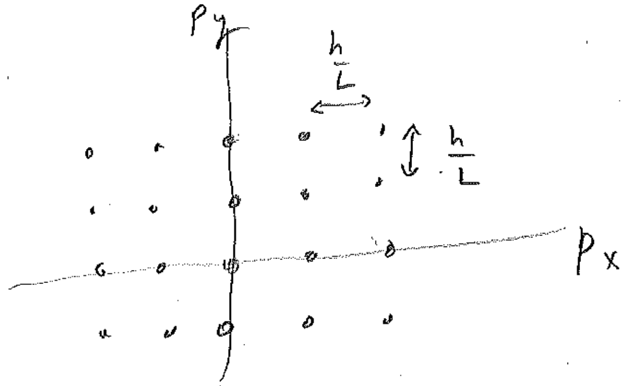
namely $N_{states} = \frac{p_{max}}{\left(\frac{h}{2L}\right)} = \frac{2\, p_{max}}{\left(\frac{h}{L}\right)}$

For a particle in a cube w/ periodic boundary cond.

$$\vec{p} = (p_x, p_y, p_z) = \frac{h}{L}(n_x, n_y, n_z) = \frac{h}{L}\vec{n}$$

$\Rightarrow$ <u>Momenta</u> live on a 3d lattice w/ spacing $\frac{h}{L}$

in "momentum space"



[If we used Dirichlet b.c., lattice points would be more closely spaced — 8 times more dense — but we would only consider points in the first octant.]

If we want to sum over points on the lattice

$$\sum_{\vec{n}} \approx \int d^3\vec{n} = \int dn_x\, dn_y\, dn_z$$

$$= \int \left(\frac{L}{h}dp_x\right)\left(\frac{L}{h}dp_y\right)\left(\frac{L}{h}dp_z\right)$$

$\vec{n} = \frac{L}{h}\vec{p}$

$$= \left(\frac{L}{h}\right)^3 \int d^3\vec{p}$$

$\Rightarrow$ just integrate over $\vec{p}$-space w/ measure $\left(\frac{L}{h}\right)^3$

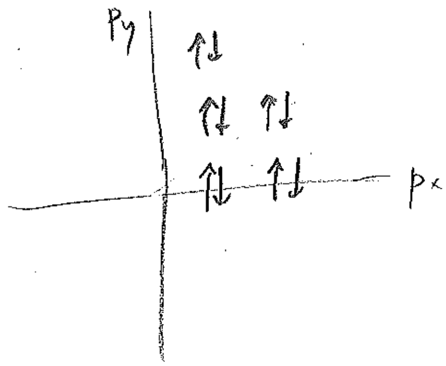
[to account for density of states]

Protons and neutrons are both spin-$\frac{1}{2}$ fermions.

Spin-$\frac{1}{2}$ particles have 2 spin states, $\uparrow$ and $\downarrow$.

The quantum state of a nucleon is determined by its momentum and its spin.

$\Rightarrow \exists$ 2 distinct quantum states for each lattice point in momentum space



Let's count the number of distinct quantum states w/ $|\vec{p}|$ less than a certain value, call it p_F.

$$N_{states} = \sum_{\vec{n}} \underset{\text{spin multiplicity}}{2} = 2\left(\frac{L}{h}\right)^3 \int d^3\vec{p}.$$

sum over lattice points

$$= 2\left(\frac{L}{h}\right)^3 \underbrace{\int_0^{p_F} 4\pi p^2\, dp}_{\frac{4\pi}{3} p_F^3}$$

volume of sphere

$$= \frac{8\pi}{3}\left(\frac{L p_F}{h}\right)^3$$

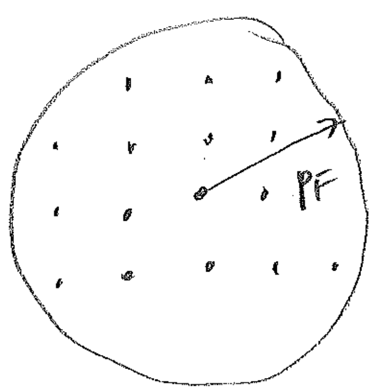
$$\Rightarrow p_F = \frac{h}{L}\left(\frac{3}{8\pi} N_{states}\right)^{1/3}$$

No two identical fermions can occupy the same quantum state, so a collection of identical fermions must occupy lattice points of larger & larger momentum.

At zero temperature, N neutrons will occupy the N quantum states of <u>lowest energy</u> (& $\therefore$ smallest momentum $|\vec{p}|$) up to some maximum momentum $p_F = \frac{h}{L}\left(\frac{3N}{8\pi}\right)^{\frac{1}{3}}$.

This is called a <u>degenerate fermi gas</u> and p_F is the <u>fermi momentum</u>.

Z protons occupy their own degenerate fermi gas w/ their own fermi momentum $p_F = \frac{h}{L}\left(\frac{3Z}{8\pi}\right)^{\frac{1}{3}}$



What is the kinetic energy of this degenerate fermi gas?

$$T^{tot} = \underset{\substack{\uparrow \\ \text{spin multiplicity}}}{2} \sum_{\vec{n}} T_{\vec{n}} = 2\left(\frac{L}{h}\right)^3 \int (4\pi p^2\, dp)\, T(p)$$

For nonrelativistic particles ($v \ll c$), $\quad T \approx \frac{p^2}{2m}$

For ultrarelativistic particles ($v \approx c$), $\quad T \approx cp$

[HW] Show that average kinetic energy $\bar{T} = \frac{3}{5}\epsilon_F$

where $\epsilon_F = \frac{p_F^2}{2m}$ is the nonrelativistic fermi energy

The <u>total</u> kinetic energy of a degenerate gas of N nonrelativistic neutrons is then

$$T^{tot} = N\bar{T} = \frac{3}{10}\frac{p_F^2}{m} N$$

[HW] Calculate the average kinetic energy of a degenerate gas of ultrarelativistic particles

Recall fermi momentum of a degenerate gas of N neutrons

$$p_F = \frac{h}{L}\left(\frac{3N}{8\pi}\right)^{\frac{1}{3}} = \hbar\left(\frac{3\pi^2 N}{L^3}\right)^{\frac{1}{3}}$$

Replace cube volume L^3 by sphere volume $\frac{4\pi}{3} A r_0^3$

$$p_F = \frac{\hbar}{r_0}\left(\frac{9\pi}{4}\frac{N}{A}\right)^{\frac{1}{3}}$$

Degenerate gas is <u>nonrelativistic</u> if $\frac{v}{c} \ll 1 \Rightarrow \frac{p}{mc} \ll 1$

The max. momentum of the gas is p_F

$$\frac{p_F}{mc} = \frac{\hbar c}{mc^2 r_0}\underbrace{\left(\frac{9\pi}{4}\frac{N}{A}\right)^{\frac{1}{3}}}_{\text{of order } 1} \sim \frac{(200\ \text{MeV}\cdot\text{fm})}{(940\ \text{MeV})(1.2\ \text{fm})} \sim \frac{1}{5} \quad \checkmark$$

we can use nonrelativistic kinetic energy

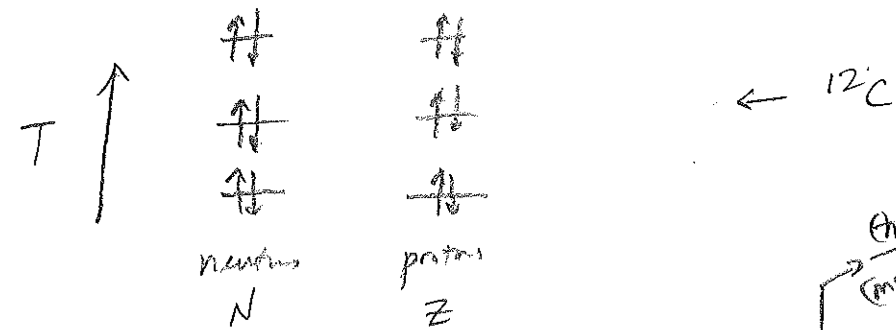
$$\Rightarrow T^{tot} = \frac{3}{10}\frac{p_F^2}{m}\frac{N}{A}\cdot A = \underbrace{\frac{3}{10}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}}_{\sim 1.105}\frac{\hbar^2}{mr_0^2}\left(\frac{N}{A}\right)^{\frac{5}{3}}\cdot A$$

Total kinetic energy of neutron and proton gas is

$$T = 1.105\ \frac{\hbar^2}{mr_0^2}\left[\left(\frac{N}{A}\right)^{\frac{5}{3}} + \left(\frac{Z}{A}\right)^{\frac{5}{3}}\right] A$$

(masses about the same)

First, consider nuclei w/ $Z \approx N \approx \frac{1}{2}A$



$$\frac{(\hbar c)^2}{(mc^2)\,r_0^2} = \frac{(197)^2}{(940)(1.2)^2} = 29\ \text{MeV}$$

$$T^{tot} = \underbrace{1.105\left[\left(\tfrac{1}{2}\right)^{5/3} + \left(\tfrac{1}{2}\right)^{5/3}\right]}_{\approx\, 0.696} \underbrace{\frac{\hbar^2}{m r_0^2}}_{29\ \text{MeV}} A$$

$$\approx (20\ \text{MeV})\, A$$

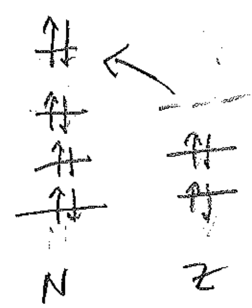
[For deuteron, we found $T = 130$ MeV]

This is a contribution to the binding energy <u>volume term</u>

Suggests that potential energy per nucleon is ~ -35 MeV

$$B = -T^{tot} - V^{tot} \approx (15\ \text{MeV})\, A$$

If $N \neq Z$, kinetic energy increases. Why?

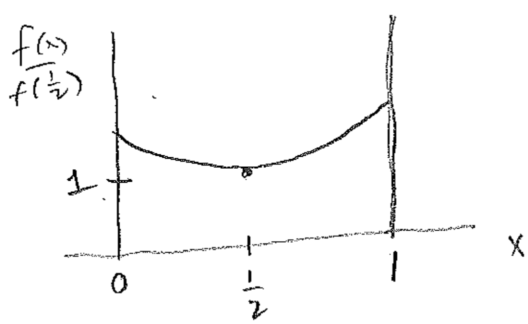


Let's estimate change in T^{tot}.

Let $x = \frac{Z}{A}$

Then $T^{tot} = (20\,\text{MeV})\,A \cdot \frac{f(x)}{f(\frac{1}{2})}$

where $f(x) = x^{5/3} + (1-x)^{5/3}$ $\qquad f(\frac{1}{2}) \cong 0.63$



Near $x = \frac{1}{2}$, curve is $\approx$ parabolic

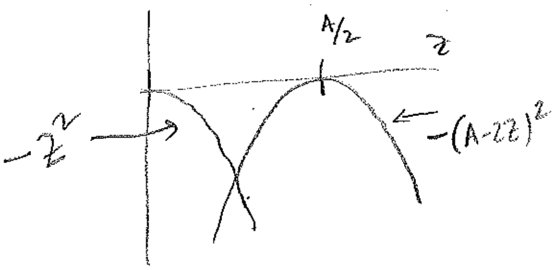
$$\frac{f(x)}{f(\frac{1}{2})} = 1 + \#\left(x - \tfrac{1}{2}\right)^2 + \ldots \qquad \left[\# = \frac{20}{9}\right]$$

HW: calculate # + use it estimate c_4 in semi-empirical

Hint: it doesn't agree too well, but is right order of mag

[$c_4 = 11$ MeV, whereas actual $c_4 = 23$ MeV]

$$B = c_1 A - c_2 A^{2/3} - \underbrace{c_3 \frac{Z^2}{A^{1/3}}} - \underbrace{c_4 \frac{(A-2Z)^2}{A}} + \delta_{pair}$$



favors $Z \ll N$

more effective at large A

⇓

heavy nuclei have fewer protons than neutrons

favors $Z = \frac{A}{2}$

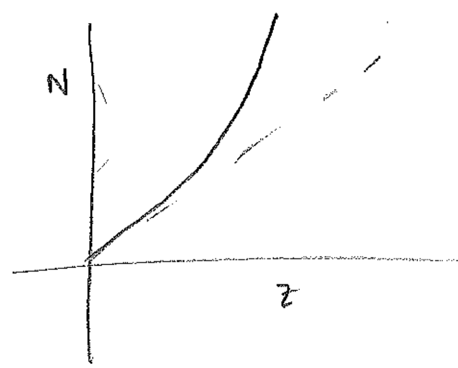
or $Z = N$

more effective at small A

⇓

light nuclei have $Z \approx N$

^4He, ^{12}C, ^{16}O



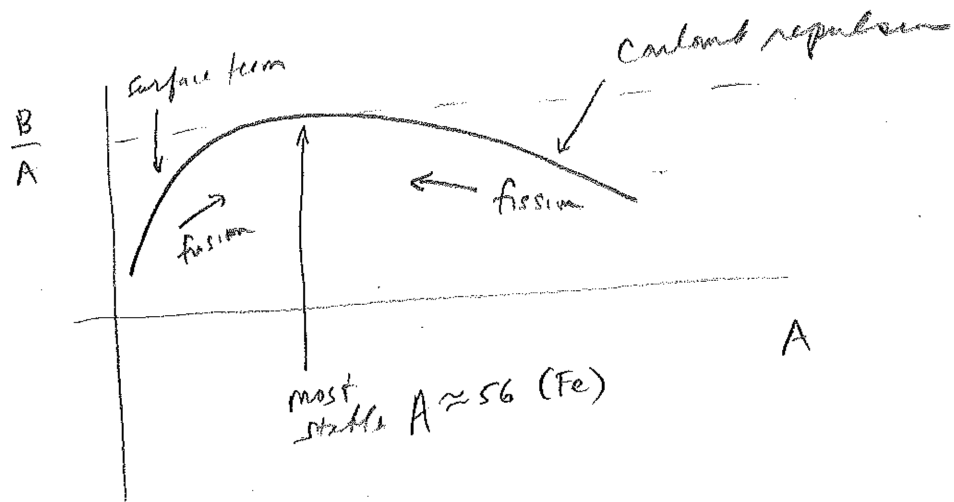
[problem: solve for Z as a fcn of A that minimizes Δ (not B)]

Binding energy per nucleon

$$\frac{B}{A} = c_1 - c_2 A^{-1/3} - c_3 \frac{Z^2}{A^{4/3}} - c_4 \frac{(A-2Z)^2}{A^2} + \frac{\delta_{pair}}{A}$$

assume $Z \approx N \approx \frac{A}{2}$

$$\frac{B}{A} = c_1 - \frac{c_2}{A^{1/3}} - c_3 A^{2/3}$$



$$X \to Y + Z \quad \text{iff} \quad m_X > m_Y + m_Z$$

A nucleus is stable only if its mass is less than the sum of masses of nuclei into which it could decay

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Calc of asymmetry term in semi-empirical formula

Kinetic energy of nucleus

$$\vec{k} = \frac{2\pi}{L}\vec{n}$$

$$\frac{4\pi}{3} n_{max}^3 = \frac{N}{2}$$

$$\epsilon_F = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 \left(\frac{3N}{8\pi}\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$$

$$U = \frac{3}{5} N \epsilon_F$$

$r_0 = 1.2$ fm (Serway)
$= 1.5$ fm (Fermi)

For a nucleus: $V = A V_0$, $V_0 = \frac{4}{3}\pi r_0^3$

$$\epsilon_F = \underbrace{\frac{\hbar^2 c^2}{2mc^2 r_0^2}\left(\frac{9\pi}{4}\right)^{2/3}}_{C_0} \left(\frac{N}{A}\right)^{2/3}$$

53 MeV (Serway)
34 MeV (Fermi)

$\to N_p = N_n = \frac{A}{2} \Rightarrow \epsilon_F \approx 21$ MeV/fe

if only neutrons, $\epsilon_F = \begin{Bmatrix} 53 \\ 34 \end{Bmatrix}$

27 MeV
say K+K
$r_0 = 1.3$ fm

$$U_{Nuc} = \frac{3}{5} N_p C_0 \left(\frac{N_p}{A}\right)^{2/3} + \frac{3}{5} N_n C_0 \left(\frac{N_n}{A}\right)^{2/3}$$

$$= \frac{3}{5}\frac{C_0}{A^{2/3}}\left[\underbrace{Z^{5/3} + (A-Z)^{5/3}}_{f(Z,A)}\right]$$

$$\rho = Z - \frac{A}{2}$$

$$f(Z,A) = \left(\frac{A}{2}+\rho\right)^{5/3} + \left(\frac{A}{2}-\rho\right)^{5/3}$$

$$= \frac{A^{5/3}}{2^{3/2}}\left[1 + \frac{20}{9}\frac{\rho^2}{A^2}\right] = (.63)A^{5/3} + (.35)A^{-1/3}(A-2Z)^2$$

$$U_{Nuc} = \underbrace{(.38)C_0}_{} A + \underbrace{(.21)C_0}_{} \frac{(A-2Z)^2}{A}$$

20 MeV (Serway)
13 MeV (Fermi)

11 MeV (Serway)
7 MeV (Fermi)

If all neutrons, $(A = N_n, Z = 0)$ $U_{Nuc} = \frac{3}{5} C_0 A = \begin{Bmatrix} 32 \text{ MeV} \\ 20 \text{ MeV} \end{Bmatrix} A$,