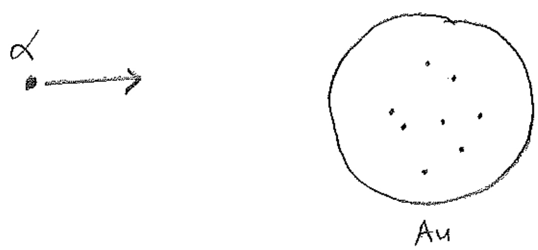
[To explore the structure of atoms, in about

1910 Rutherford, Geiger, Marsden

[shot α particles (^4He nuclei) from radioactive decay
at thin gold foil [to reduce likelihood of multiple collisions]

[They expected small deflections.



← Thomson model of atom
(positive jelly w/ pointlike electrons)
overall electrically neutral

[No effect until α goes inside (because neutral)
Because $m_\alpha \approx 4 m_p$ and $m_e \approx \frac{1}{1800} m_p$ [factor of 7000]

[bowling ball ~ 7 kg, ping pong ball 2.7 g, factor of 2500]

expect little stopping power

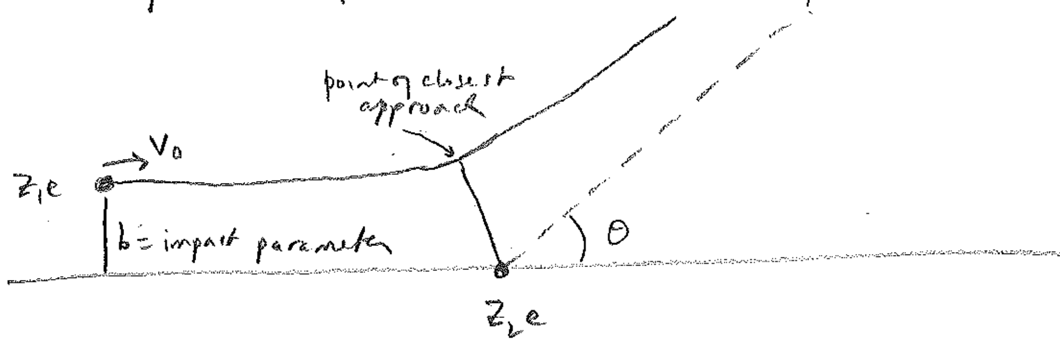
[They measured some large deflections
Rutherford: "It was quite the most incredible event that has ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you" (1936)

Rutherford
concluded mass of gold atom concentrated
in a small nucleus.

Rutherford scattering

[Rutherford calculated]

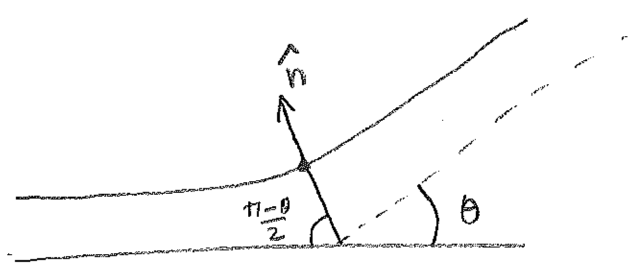
deflection of an α-particle from a fixed, point-like nucleus X



[inverse square $\Rightarrow$ elliptical, parabolic, hyperbolic]

α moves along hyperbolic orbit symmetric w.r.t. point of closest approach

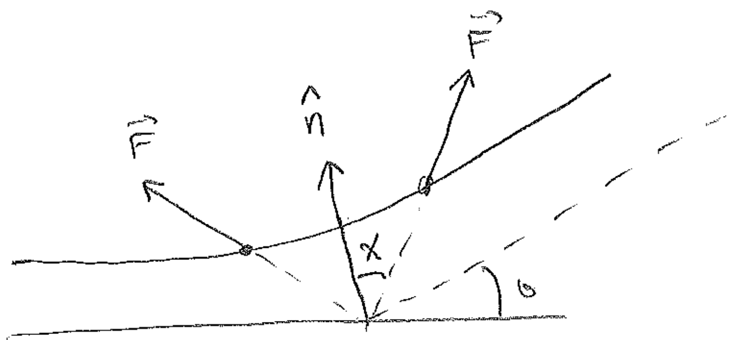
Goal: find θ as a function of v_0 and b



To show: change in momentum $\Delta\vec{p} \parallel \hat{n}$

Energy conservation $\Rightarrow |\vec{p}_f| = |\vec{p}_i|$ [isosceles triangle]

$$\boxed{\Delta p = 2mv_0 \sin\frac{\theta}{2}}$$



$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Delta\vec{p} = \int d\vec{p} = \int \vec{F}\, dt \quad = \text{impulse}$$

Net impulse $\perp \hat{n}$ vanishes

Net impulse $\parallel \hat{n}$

$$\Delta p_n = \hat{n}\cdot\Delta\vec{p} = \int \vec{F}\cdot\hat{n}\, dt = \int F \cos\chi\, dt$$

Assume unshielded Coulomb force

$$F = \frac{K(Z_1 e)(Z_2 e)}{r^2} = \frac{k}{r^2} \qquad k \equiv Z_1 Z_2 K e^2$$

Force is central (radial) $\Rightarrow$ angular momentum is conserved

$$\vec{l} = \vec{r} \times m\vec{v}$$

$$l = r_\perp m v = b m v_0$$

Can also write

$$l = r m v_\perp = r^2 m \frac{d\chi}{dt} = b m v_0$$

$$v_\perp = r \frac{d\chi}{dt} \qquad \Rightarrow \frac{d\chi}{dt} = \frac{b v_0}{r^2}$$

$$dt = \frac{r^2}{b v_0} d\chi$$

$$\Delta p_n = \int \frac{k}{r^2} (\cos\chi) \left(\frac{r^2}{b v_0} d\chi \right) = \frac{k}{b v_0} \int \cos\chi \, d\chi$$

$$= \frac{k}{b v_0} \sin\chi \Big|_{-(\frac{\pi-\theta}{2})}^{\frac{\pi-\theta}{2}} = \frac{2k}{b v_0} \sin\left(\frac{\pi}{2} - \frac{\theta}{2} \right)$$

$$\Delta p_n = \frac{2k}{b v_0} \cos\left(\frac{\theta}{2} \right)$$

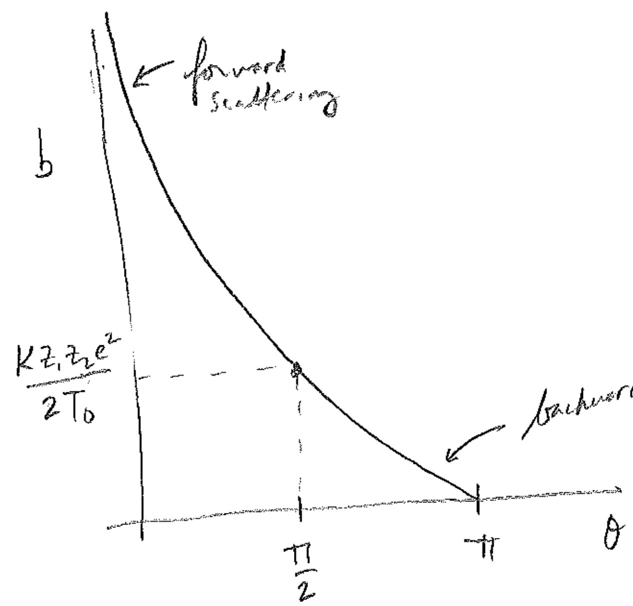
Equating two expressions for Δp:

$$2mv_0 \sin\frac{\theta}{2} = \frac{2k}{v_0 b} \cos\frac{\theta}{2}$$

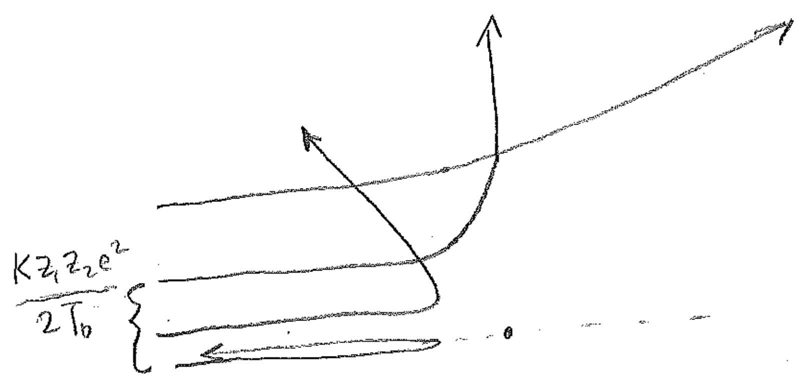
$$\Rightarrow \quad b = \frac{k}{mv_0^2} \cot\frac{\theta}{2}$$

$$\boxed{b = \frac{K Z_1 Z_2 e^2}{2T_0} \cot\frac{\theta}{2}}$$

where $T_0 = \frac{1}{2} m v_0^2$ = kinetic energy of incident α



$$\theta = \frac{\pi}{2} \Rightarrow \cot\left(\frac{\pi}{4}\right) = \frac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)} = 1$$



Scattering cross section

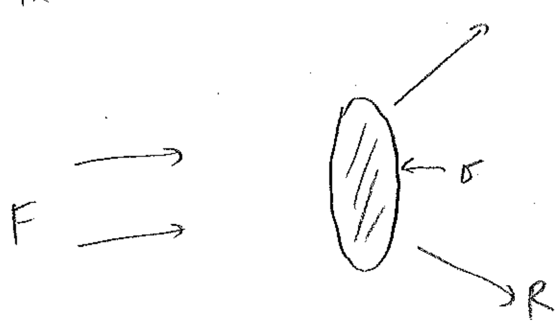
Let F = incident flux of α-particles $= \dfrac{\#\text{ incident pcls}}{\text{sec}\cdot\text{area}}$

Let R = scattering rate of α-particles $= \dfrac{\#\text{ pcls scattered}}{\text{sec}}$

Expect R to be proportional to F

Define $\boxed{\sigma = \dfrac{R}{F}}$

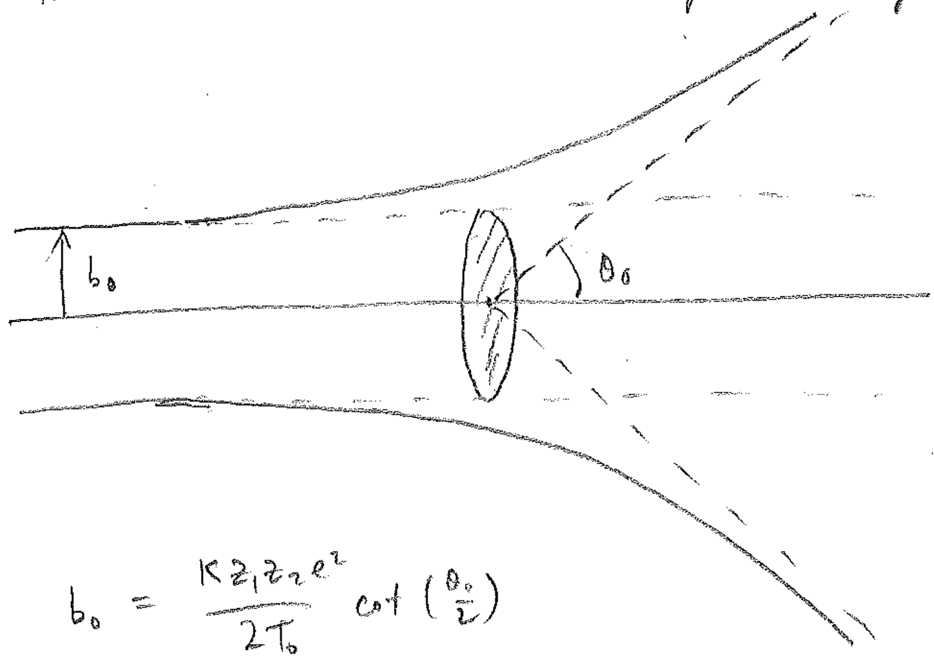
σ has units of area; it is the area of the scatterer intercepted by the incident flux



$$R = F\sigma$$

σ is called the scattering <u>cross-section</u>

Define $\sigma(\theta > \theta_0)$ = cross-section for scattering thru angle $> \theta_0$



$$b_0 = \frac{K z_1 z_2 e^2}{2T_0} \cot\left(\frac{\theta_0}{2}\right)$$

If $b < b_0$ then $\theta > \theta_0$

Any α particle that would have struck a disk of radius b_0 had it not been deflected, will be scattered through $\theta > \theta_0$

$$\sigma(\theta > \theta_0) = \pi b_0^2$$

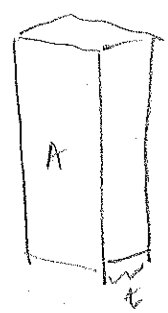
Total cross-section for scattering $\sigma_{tot} = \sigma(\theta > 0)$

is technically infinite because all particles are scattered (deflected) to some extent.

Practically speaking, α particles that pass beyond the electron shells are unscattered (atom is neutral) so effective total cross-section is no larger than πR_{atom}^2.

What fraction of all incident α-particles will be scattered through an angle $> \theta_0$?

Consider a thin foil of thickness t and area A.



Let n = number density of nuclei

Total # of nuclei in foil $N = nAt$

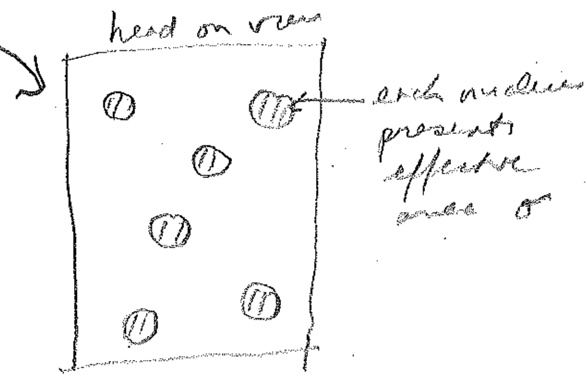
Each nucleus presents a target of size $\sigma(\theta > \theta_0)$.

Total target area $= N\sigma$

Total area $= A$

Fraction occupied by targets

$$f = \frac{N\sigma}{A} = nt\sigma$$



[This is the fraction of α-particles scattered.

$$\boxed{\text{fraction scattered } f = nt\sigma}$$

[Note: advantage of <u>thin</u> foil is to eliminate possibility of multiple interactions, so each α particle scatters only once

Fraction of α-particles scattered through $\theta > \theta_0$

$$f = nt\sigma = nt\pi b_0^2 \quad \text{where } b_0 = \frac{Z_1 Z_2 Ke^2}{2T_0}\cot\left(\frac{\theta_0}{2}\right)$$

(Let's evaluate this.)

- The strength of the electromagnetic force is characterized by the dimensionless fine structure constant, defined by

$$\boxed{\alpha = \frac{Ke^2}{\hbar c} \approx \frac{1}{137}}$$

- A very useful constant to know is

$$\boxed{\hbar c = 197 \text{ MeV}\cdot\text{fm}}$$

- Therefore $Ke^2 = \alpha\hbar c = \frac{197}{137}$ MeV fm ≈ 1.44 MeV fm

- Consider α-particle w/ $T_0 = 5$ MeV incident on gold foil 0.5 microns thick. What fraction is back scattered, ie scattered thru $\theta > 90°$?

$$Z_1 = 2, \quad Z_2 = 79, \quad \theta_0 = \frac{\pi}{2}$$

$$b_0 = \frac{2\cdot(79)\cdot(1.44 \text{ MeV fm})}{2(5 \text{ MeV})} = 23 \text{ fm}$$

$$t = 5\times10^{-7}\text{m}, \quad n = 5.9\times10^{28}\text{ m}^{-3}$$

$$f = nt\pi b_0^2 \approx 5\times10^{-5}$$

Probability is small (1 in 20,000) but measurable

[Geiger & Marsden, 1 in 8,000 (Evans, p. 2)]

Griffiths, Quantum Mechanics, 2nd ed.

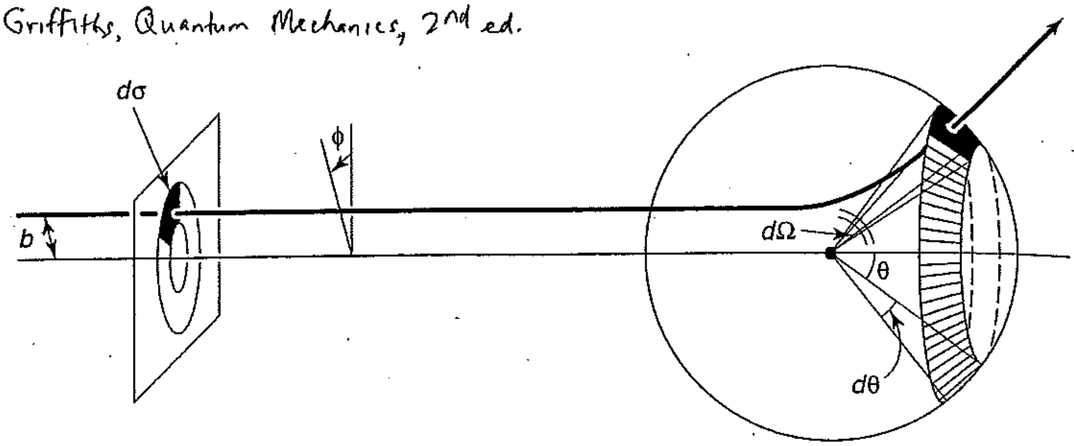
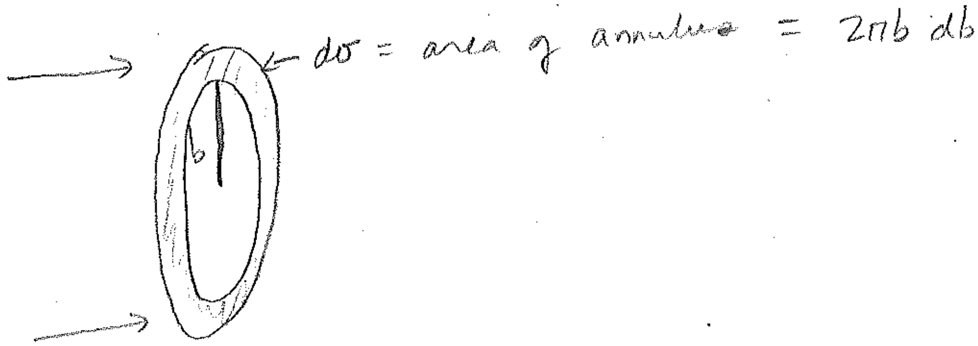


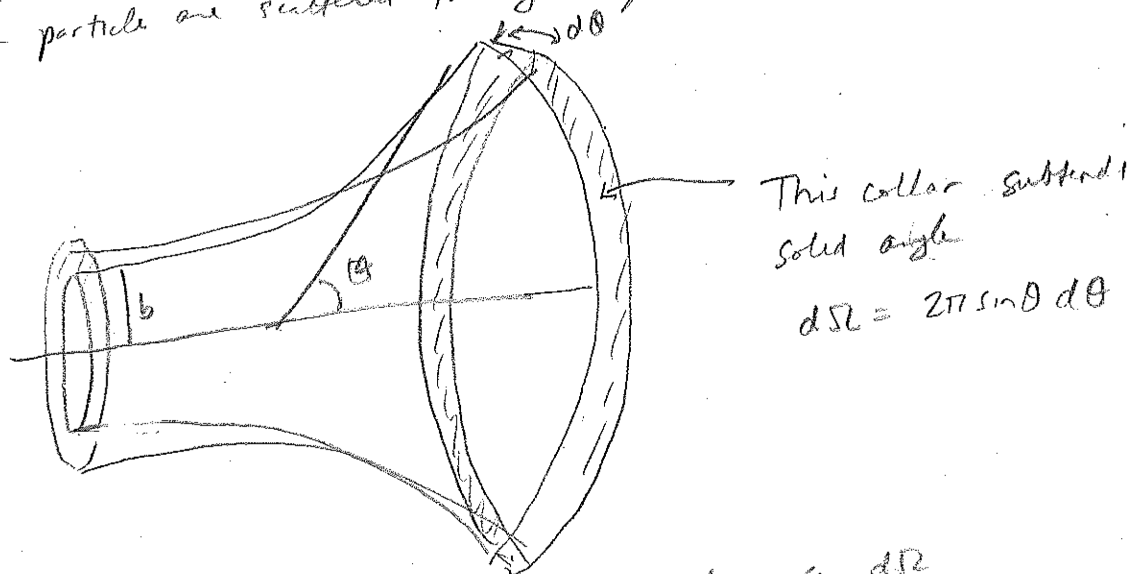
FIGURE 11.3: Particles incident in the area $d\sigma$ scatter into the solid angle $d\Omega$.

Differential cross-section $\frac{d\sigma}{d\Omega}$

Consider incident particles w/ impact parameter between b and $b+db$



All those particles are scattered through angles between θ and $\theta + d\theta$



Rate at which particles are scattered into solid angle $d\Omega$

$\equiv$ Rate at which they pass through the annulus

$$= F\, d\sigma = \underbrace{F\left(\frac{d\sigma}{d\Omega}\right)}_{\text{density (per solid angle) of particles scattered through } \theta} d\Omega$$

$\frac{d\sigma}{d\Omega}$ $\equiv$ differential scattering cross-section

$=$ probability (per incident particle + per solid angle) to scatter thru θ

$$d\sigma = 2\pi b\, db$$

$$d\Omega = 2\pi \sin\theta\, d\theta$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

For Rutherford scattering, $b = B\cot\left(\frac{\theta}{2}\right)$, $B = \frac{Z_1 Z_2 K e^2}{2T_0}$

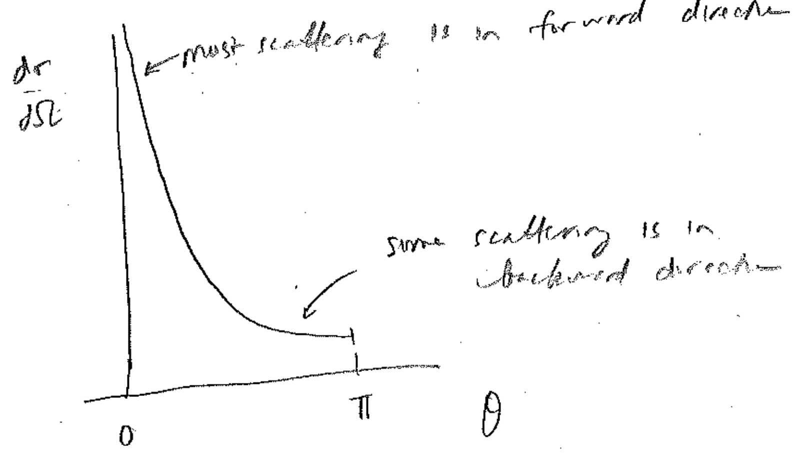
$$\left|\frac{db}{d\theta}\right| = \frac{1}{2} B \csc^2\left(\frac{\theta}{2}\right)$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{2} B^2 \frac{\cot\left(\frac{\theta}{2}\right)}{\sin\theta}\csc^2\left(\frac{\theta}{2}\right)$$

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} B^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}$$

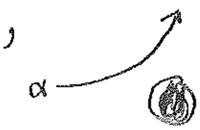
$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{K Z_1 Z_2 e^2}{4T_0}\right)^2 \frac{1}{\sin^4\left(\frac{\theta}{2}\right)}}$$ Rutherford differential cross section



[HW: show $\sigma(\theta > \theta_0) = \int \frac{d\sigma}{d\Omega} d\Omega = \pi b_0^2$]

- Rutherford & colleagues did observe this distribution (w.r.t. angular dependence, Z_2, T_0)

  suggesting ~~that~~ all the positive charge of an atom is concentrated into a very tiny region, thereby acting as a point charge.



- By estimating distance of closest approach of the α-particle one can put an upper bound on size of gold nucleus

  [HW]

- If incident particles get <u>within</u> the radius of the nucleus, departures from Rutherford cross-section are expected
  1. no longer a pt. charge (e.g. electron scattering)
  2. strong force can act

Rutherford did observe departures for lighter elements

1919, Rutherford observed transmutation of elements

$$\alpha + {}^{14}_{7}N \longrightarrow {}^{17}_{8}O + p$$

Various experiments suggested that (e^- scattering)

nuclear radius $R \sim A^{1/3}$

Specifically $R = A^{1/3} r_0$

$r_0 \simeq 1.2 \times 10^{-15}\, m = 1.2$ fm

Thus most nuclei between $1 - 10$ fm

Atomic radii are $\sim 1\,Å \simeq 10^{-10}\, m$, so 4 to 5 orders of magnitude

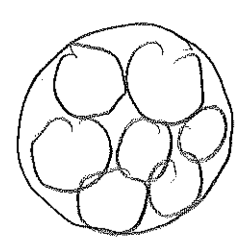
If hydrogen atom were size of this room [$\sim 10\, m$]

then size of nucleus is $\sim \underbrace{0.1\,mm}_{\text{human hair width.}}$ and 1 mm

Volume of nucleus $= \frac{4}{3}\pi R^3 = \left(\frac{4}{3}\pi r_0^3\right) A$

proportional to # nucleons

$\Rightarrow$ nuclei act as a collection of close-packed incompressible nucleons



Bethe & Morrison

size of nuclei determined by

(1) neutron cross-section (for fast but not too fast neutrons)
$\sim 20\,\mathrm{MeV}$

treat neutrons geometrically

nucleus becomes transparent

$$\lambda\!\!\!^{-} = \frac{\hbar}{p} = \frac{\hbar c}{\sqrt{2mc^2 T}} = \frac{200}{\sqrt{2\cdot(20)(940)}} = 1\ \mathrm{fm}$$

(2) α - decay lifetime

(3) nuclear σ for chgd ptcls (must traverse barrier)

(4) mirror nuclei

^{3}H, ^{3}He
^{7}Li, ^{7}Be
^{11}C, ^{11}B
^{13}C, ^{13}N
^{15}N, ^{15}O
^{17}O, ^{17}F
^{29}Si, ^{29}P

(5) semi-empirical

(6) e^- scattering

(7) μ-mesic systems