Four-momentum

recall $x^\mu = (ct, \vec{x})$
$k^\mu = (\frac{\omega}{c}, \vec{k})$
$p^\mu = (p^0, \vec{p})$

Conservation of momentum (for isolated systems)

$\sum_{init} \vec{p} \quad \sum_{final} \vec{p}$

is invariant under rotations because $\vec{p}$ tx as a 3-vector

Conservation of 4-momentum

$\sum_{init} p^\mu = \sum_{final} p^\mu$

is invariant under boosts as well provided p^μ tx as a 4-vector

[Q: what is zeroth component p^0, + why is it conserved?]
[Q: what is the definition of p^μ for a single object?]

Nonrelativistic momentum $\vec{p} = m\vec{v} = m \frac{d\vec{x}}{dt}$

$\vec{p}$ tx as a 3-vector because $\vec{x}$ is a 3-vector &
t is a scalar under rotation

Can we simply say $p^\mu = m \frac{dx^\mu}{dt}$?

No, because although dx^μ is a " vector
dt is not a Lorentz scalar

However, proper time $d\tau$ is a Lorentz scalar (time as measured in frame of moving object)

$$\Rightarrow \quad p^\mu = m \frac{dx^\mu}{d\tau}$$ transforms as a 4 vector

Recall $d\tau = \frac{dt}{\gamma}$ so $p^\mu = m\gamma \frac{dx^\mu}{dt}$

ie $\boxed{\vec{p} = \gamma m \vec{v}}$ (relativistic 3-momentum for an object of mass m & speed v)

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \dots$$

$$\Rightarrow \quad \vec{p} = \underbrace{m\vec{v}}_{\text{normal approx}} + \underbrace{\frac{1}{2} m \frac{v^2}{c^2}\vec{v} + \frac{3}{8} m \frac{v^4}{c^4}\vec{v} + \dots}_{\text{corrections that are very tiny at low speeds}}$$

High speed experiments verify that $\sum \gamma m \vec{v}$ not $\sum m\vec{v}$ is conserved for isolated systems

$$p^0 = m\frac{dx^0}{d\tau} = m\gamma\frac{d}{dt}(ct) = mc\gamma$$

$$= mc\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \dots\right)$$

This will look more familiar if we multiply both sides by c:

$$cp^0 = \underbrace{\underbrace{mc^2}_{\text{rest energy}} + \underbrace{\frac{1}{2}mv^2}_{\text{nonrelativistic kinetic energy}} + \underbrace{\frac{3}{8}m\frac{v^4}{c^2} + \dots}_{\text{corrections to kinetic energy tiny at low speed}}}_{\text{total energy}}$$

$$\boxed{E = cp^0 = \gamma mc^2}$$ relativistic energy of a moving object

$$E = E_{rest} + K$$

$$E_{rest} = mc^2$$

$$K = (\gamma - 1)mc^2$$ (relativistic kinetic energy)

$$|\vec{p}| = \gamma mv$$ (relativistic momentum)

[These are not the same thing!]

Recall $p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$

p^μ is a 4-vector $\Rightarrow$ $p'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \, p^\nu$

$$\begin{pmatrix} \frac{E'}{c} \\ p_x' \\ p_y' \\ p_z' \end{pmatrix} = \begin{pmatrix} \gamma_0 & -\gamma_0\beta_0 & 0 & 0 \\ -\gamma_0\beta_0 & \gamma_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

under a boost in x direction with velocity v_0

$$E' = \gamma_0 (E - \beta_0 c p_x)$$
$$p_x' = \gamma_0 \left(p_x - \beta_0 \frac{E}{c} \right)$$
$$p_y' = p_y$$
$$p_z' = p_z$$

Consider a particle <u>at</u> <u>rest</u> in S'

$$E' = mc^2, \quad p_x' = p_y' = p_z' = 0$$

Let S' be moving at speed v_0 in $+x$ direction wrt. S

In S, particle is moving at speed v_0 in $+x$ direction

$$E = \gamma_0 (E' + \beta_0 c p_x') = \gamma_0 (mc^2 + 0) = \gamma_0 mc^2 \quad ✓$$
$$p_x = \gamma_0 \left(p_x' + \beta_0 \frac{E'}{c}\right) = \gamma_0 \left(0 + \beta_0 \frac{mc^2}{c}\right) = \gamma_0 m v_0 \quad ✓$$
$$p_y = p_y' = 0$$
$$p_z = p_z' = 0$$

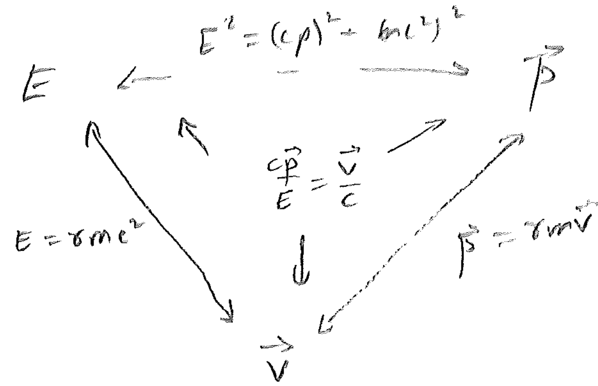
A particle of mass m & speed $\vec{v}$ has

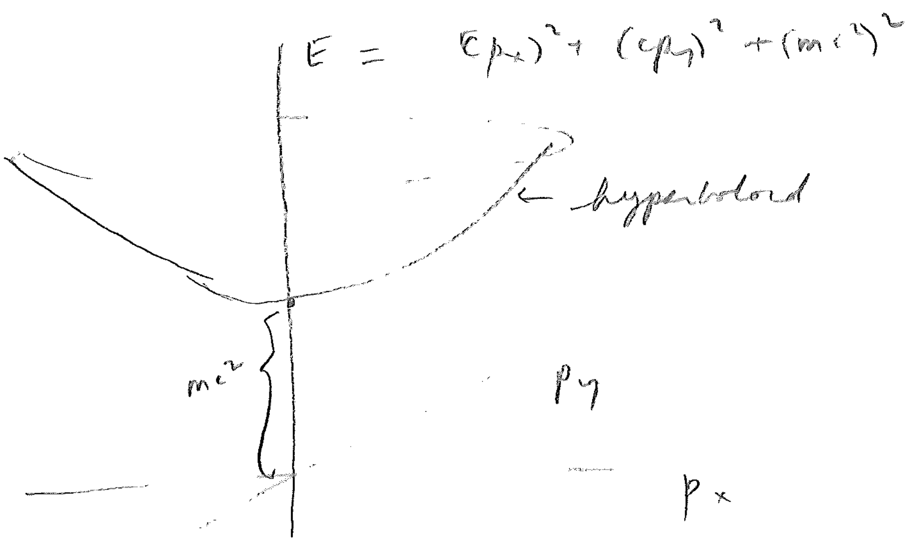
$$\vec{p} = \gamma m \vec{v}$$
$$E = \gamma m c^2$$

Observe $\quad \dfrac{\vec{p}}{E} = \dfrac{\vec{v}}{c^2} \implies \vec{v} = \dfrac{c^2 \vec{p}}{E}$

Also (exercise)

$$E^2 = (c\vec{p})^2 + (mc^2)^2$$





In nature, $\exists$ particles of no mass

$$m = 0 \implies E^2 = (c\vec{p})^2$$

$$E = cp$$

Recall $\frac{v}{c} = \frac{cp}{E} = 1$ i.e. $v = c$

massless particles always move at light speed
(photons, gravitons)

They have no rest energy, only kinetic energy.

N.B. formulas $E = \gamma mc^2$ and $\vec{p} = \gamma m\vec{v}$
do not apply because $m \to 0$, $\gamma \to \infty$

There is no reference frame in which
they are at rest.