A Lorentz boost in the x-direction

$$ct' = \gamma(ct - \beta x) \qquad \beta = \frac{v_0}{c}$$

$$x' = \gamma(x - \beta \cdot ct) \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$y' = y$$

$$z' = z$$

can be expressed in matrix form

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

This suggests we define a <u>four-vector</u>

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \text{where} \begin{cases} x^0 = ct \\ x^1 = x \\ x^2 = y \\ x^3 = z \end{cases}$$

[Greek index $\mu = 0, 1, 2, 3$]

which transforms under a <u>general</u> Lorentz transformation as

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \underbrace{\begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix}}_{\Lambda = \text{Lorentz transformation matrix}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

This can be written more compactly in index notation

$$x^{\mu\prime} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \text{where } \Lambda^{\mu}{}_{\nu} = \text{matrix element of } \Lambda$$

(row) (column)

($\mu = 0, 1, 2, 3$)

Def: A four-vector is any object $A^{\mu} = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$

that transforms as $A^{\mu\prime} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} A^{\nu}$

under a Lorentz transformation.

examples

$$x^{\mu} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$ ← all components have same units

wave four-vector

$$k^{\mu} = \begin{pmatrix} k^0 \\ k^1 \\ k^2 \\ k^3 \end{pmatrix} = \begin{pmatrix} \frac{\omega}{c} \\ k_x \\ k_y \\ k_z \end{pmatrix}$$ ← all components have same units

A Lorentz transformation matrix Λ must obey

$$\Lambda^T \eta \Lambda = \eta$$

where

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{Minkowski metric}$$

Lorentz transformations include <u>boosts</u> and <u>rotations</u>

e.g. Boost in x-direction $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Rotation in xy-plane $\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$

[ex. show these obey $\Lambda^T \eta \Lambda = \eta$]

"The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

H. Minkowski, "Space and Time," 1908

Def: <u>Lorentz scalar</u> = a quantity invariant under Lorentz transformations

e.g. scalar product of two 4-vectors

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} A^\mu \eta_{\mu\nu} B^\nu$$

$$= A^0 \underbrace{\eta_{00}}_{-1} B^0 + A^0 \underbrace{\eta_{01}}_{0} B^1 + \dots + A^1 \underbrace{\eta_{11}}_{1} B^1 + \dots$$

$$= -A^0 B^0 + \underbrace{A^1 B^1 + A^2 B^2 + A^3 B^3}_{\vec{A}\cdot\vec{B}}$$

This is analogous to a dot product of two 3-vectors, but w/ relative minus sign

~~example~~

$$\sum\sum k^\mu \eta_{\mu\nu} k^\nu = -\left(\frac{\omega}{c}\right)\left(\frac{\omega}{c}\right) + \vec{k}\cdot\vec{k} = -\frac{1}{c^2}\underbrace{(\omega^2 - c^2 k^2)}_{=0}$$

$$\sum\sum k^\mu \eta_{\mu\nu} x^\nu = -\left(\frac{\omega}{c}\right)(ct) + \vec{k}\cdot\vec{x} = \underbrace{\vec{k}\vec{x} - \omega t}_{\text{phase of travelling wave}}$$

invariant : nodes are nodes in any frame

A Lorentz scalar is necessarily a scalar under rotations

But not vice versa

Proof that scalar product is invariant (similar to earlier proof) under Lorentz transformation.

$$\sum\sum A^{\mu\prime}\eta_{\mu\nu}B^{\nu\prime} = \underbrace{(A^{0\prime}\ A^{1\prime}\ A^{2\prime}\ A^{3\prime})}_{(A^0\,A^1\,A^2\,A^3)\Lambda^T}\ \eta\ \underbrace{\begin{pmatrix} B^{0\prime} \\ B^{1\prime} \\ B^{2\prime} \\ B^{3\prime} \end{pmatrix}}_{\Lambda\begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix}}$$

$$= (A^0\ A^1\ A^2\ A^3)\ \underbrace{\Lambda^T\eta\Lambda}_{\eta}\begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix}$$

$$= \sum\sum A^{\mu}\eta_{\mu\nu}B^{\nu}$$

Einstein's
Principle of relativity

just talk about this

All the laws of physics are invariant under (Lorentz) boosts

ie

- Eqns of physics have the same form in reference frames related by Lorentz transformations.

They must be expressed in terms of quantities covariant under Lorentz transformations

Lorentz-scalar = Lorentz scalar
4-vector = 4-vector
tensor = tensor

probably skip

eg wave equation

$$-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} = 0 \quad \left(\begin{array}{c}\text{invariant}\\ \text{under rotations}\end{array}\right)$$

Although not invariant under Galilean boosts, it is invariant under Lorentz boosts (see 3rd class)

Proof: Define 4-gradient $\partial^\mu = \begin{cases} -\frac{1}{c}\frac{\partial}{\partial t} & \mu = 0 \\ \frac{\partial}{\partial x^i} & \mu = 1,2,3 \end{cases}$

[minus sign necessary so that ∂^μ txs as 4-vector]

This transforms as a 4-vector

wave eqn $\Rightarrow$ $\left(-\partial^0\partial^0 + \partial^1\partial^1 + \partial^2\partial^2 + \partial^3\partial^3\right)A = 0$

$$\Big(\underbrace{\sum_{\mu,\nu=0}^{3} \partial^\mu \eta_{\mu\nu} \partial^\nu}_{\text{Lorentz scalar}}\Big) A = 0$$

[called d'Alembertian]

$\hookrightarrow$ invariant under LT txs

Let two events A + B have spacetime coordinates x^μ_A + x^μ_B

Let $\Delta x^\mu = x^\mu_B - x^\mu_A = \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$ be the spacetime <u>separation</u> between events

Δx^μ is a 4-vector

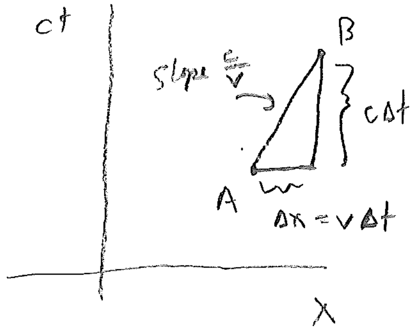
Define the <u>spacetime interval</u> Δs^2 between A and B as

$$\Delta s^2 = \sum_\mu \sum_\nu \Delta x^\mu \eta_{\mu\nu} \Delta x^\nu = -(c\,\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Since Δs^2 is a Lorentz scalar (same in all IRFs)

we may evaluate it in any convenient frame

Consider an object travelling from event A to event B at const velocity v $\Rightarrow$ $\Delta x = v\,\Delta t$



Space time interval between events is

$$\Delta s^2 = -(c\,\Delta t)^2 + (\Delta x)^2$$
$$= -c^2\Delta t^2 + v^2\Delta t^2$$
$$= -c^2(\Delta t)^2\left(1-\frac{v^2}{c^2}\right)$$
$$= -\frac{c^2(\Delta t)^2}{\gamma^2}$$

Define the <u>proper</u> time between events $\Delta\tau$ as the time measured in the frame S' (if it exists) in which they occur in the same place (i.e the rest frame of the moving object)

$$\Delta s^2 = -c(\Delta t')^2 + (\Delta x')^2$$
$$= -c^2(\Delta\tau)^2 + 0$$

Since Δs^2 invariant $\Rightarrow$ $(\Delta\tau)^2 = \frac{(\Delta t)^2}{\gamma^2}$ or $\Delta t = \gamma\,\Delta\tau$
(time dilation)

Proper time is <u>less</u> than the time between events in any other frame (since $\gamma \geq 1$)

[can't do length contraction similarly because can't go btw events at $v<c$]

NOT DONE IN CLASS

Why is $\partial^0 = -\partial_0$?

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

$$\Lambda^{\mu}{}_{\nu}\, \eta^{\nu\lambda} \frac{\partial A}{\partial x^{\lambda}} = \Lambda^{\mu}{}_{\nu}\, \eta^{\nu\lambda} \frac{\partial A}{\partial x'^{\kappa}} \frac{\partial x'^{\kappa}}{\partial x^{\lambda}}$$

$$= \underbrace{\Lambda^{\mu}{}_{\nu}\, \eta^{\nu\lambda} \Lambda^{\kappa}{}_{\lambda}}_{\eta^{\mu\kappa}} \frac{\partial A}{\partial x'^{\kappa}}$$

ie $\Lambda^{\mu}{}_{\nu} \left(\eta^{\nu\lambda} \partial_{\lambda} A\right) = \eta^{\mu\kappa} (\partial_{\kappa} A)'$

ie $\Lambda^{\mu}{}_{\nu}\, \partial^{\nu} A = \partial^{\kappa} A'$

ie ∂^{μ} is a 4-vector

4-gradient $$\frac{\partial\psi}{\partial x^\mu} = \begin{pmatrix} \frac{\partial\psi}{\partial x^0} \\ \frac{\partial\psi}{\partial x^1} \\ \frac{\partial\psi}{\partial x^2} \\ \frac{\partial\psi}{\partial x^3} \end{pmatrix} = \begin{pmatrix} \frac{1}{c}\frac{\partial\psi}{\partial t} \\ \vec{\nabla}\psi \end{pmatrix}$$

NOT DONE

$\nabla 2$

Exercise! Show that $\frac{\partial\psi}{\partial x^\mu}$ does not transform as a 4-vector

but $\partial^\mu\psi \equiv \eta^{\mu\nu}\frac{\partial\psi}{\partial x^\nu}$ does.

$\eta^{\mu\nu}$ = inverse of $\eta_{\mu\nu}$ $\qquad \eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$$\partial^\mu\psi = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \frac{1}{c}\frac{\partial\psi}{\partial t} \\ \vec{\nabla}\psi \end{pmatrix} = \begin{pmatrix} -\frac{1}{c}\frac{\partial\psi}{\partial t} \\ \vec{\nabla}\psi \end{pmatrix}$$

invariant $\eta_{\mu\nu}\partial^\mu\partial^\nu\psi = 0$

$$= -\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi = 0, \qquad \frac{\partial^2\psi}{\partial t^2} - c^2\nabla^2\psi = 0$$

wave eqn

~~Exercise~~

NOT DONE

$$t' = \gamma t - \gamma\beta x$$
$$x' = -\gamma\beta t + \gamma x$$

$$t = \gamma t' + \gamma\beta x'$$
$$x = \gamma\beta t' + \gamma x'$$

$$\frac{\partial\psi}{\partial t'} = \gamma \frac{\partial\psi}{\partial t} + \gamma\beta \frac{\partial\psi}{\partial x}$$

$$\frac{\partial\psi}{\partial x'} = \gamma\beta \frac{\partial\psi}{\partial t} + \gamma \frac{\partial\psi}{\partial x}$$

$$\left(-\frac{\partial\psi}{\partial t'}\right) = \gamma\left(-\frac{\partial\psi}{\partial t}\right) - \gamma\beta\left(\frac{\partial\psi}{\partial x}\right)$$

$$\left(+\frac{\partial\psi}{\partial x'}\right) = -\gamma\beta\left(-\frac{\partial\psi}{\partial t}\right) + \gamma\left(\frac{\partial\psi}{\partial x}\right)$$

so $\begin{pmatrix} -\frac{\partial\psi}{\partial t'} \\ \frac{\partial\psi}{\partial x'} \end{pmatrix}$ transforms as $\begin{pmatrix} t' \\ x' \end{pmatrix}$

Exercise

$$-(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2$$

$$= -(\gamma c\Delta t - \gamma\beta\Delta x)^2 + (-\gamma\beta c\Delta t + \gamma\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

$$= (-\gamma^2 + \gamma^2\beta^2)(c\Delta t)^2 + \underbrace{2\gamma^2\beta c\Delta t\Delta x - 2\gamma^2\beta c\Delta t\Delta x}_{0}$$

$$+ (\gamma^2 - \gamma^2\beta^2)(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

but $\gamma^2 - \gamma^2\beta^2 = \dfrac{1-\beta^2}{1-\beta^2} = 1$ so

$$= -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$