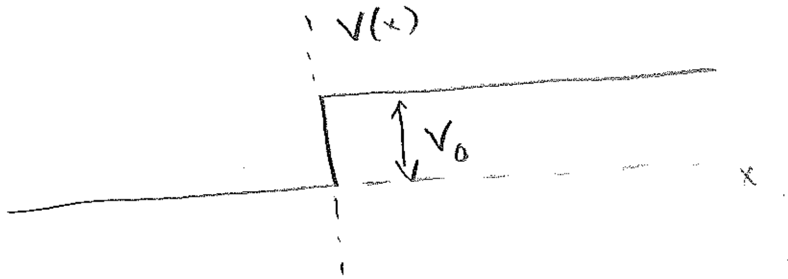
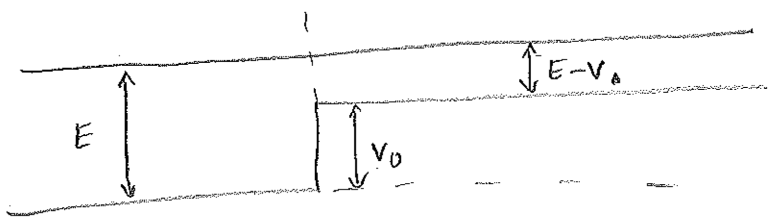
Step function potential: classical analysis

$$V(x) = \begin{cases} 0 , & x < 0 \\ V_0 , & x > 0 \end{cases}$$



$$F = -\frac{dV}{dx}$$

Particle is free in regions where V is const but experiences a large leftward force when it crosses $x = 0$



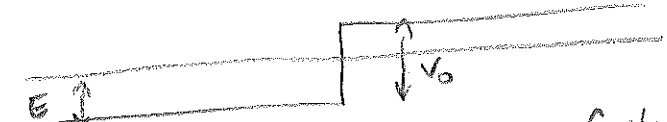
$$E = K + V$$
$$K = \frac{p^2}{2m} = E - V$$

If $E > V_0$, $x < 0$ and $x > 0$ are both allowed regions

In $x < 0$ region, $p_1 = \sqrt{2mE}$

In $x > 0$ region, $p_2 = \sqrt{2m(E - V_0)}$

Particle slows down when it moves from $x < 0$ to $x > 0$



If $E < V_0$, $x > 0$ is a forbidden region

In $x < 0$ region, $p_1 = \pm\sqrt{2mE}$

Particle coming from left bounces back

Step function potential: quantum analysis

① solve t.i.s.e : $-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + V(x)u = Eu$

② examine initial conditions

③ apply boundary conditions

$\boxed{x<0 \text{ region}}$

$V(x) = 0 \Rightarrow \frac{d^2u}{dx^2} = -\frac{2mE}{\hbar^2}u$

Try free particle solution $u = \underbrace{Ae^{\frac{ip_1x}{\hbar}}}_{\text{right-moving wave}} + \underbrace{Be^{-\frac{ip_1x}{\hbar}}}_{\text{left moving wave}}$

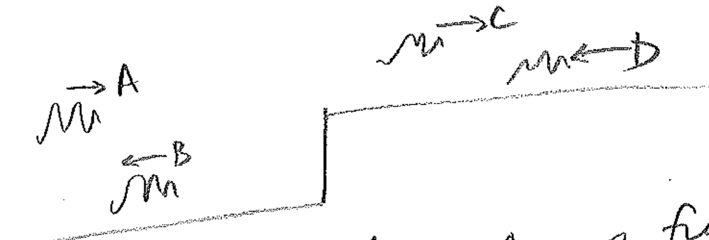
$-\frac{p_1^2}{\hbar^2}u = -\frac{2mE}{\hbar^2}u \Rightarrow p_1 = \sqrt{2mE}$ (as before)

$\boxed{x>0 \text{ region}}$

$V(x) = V_0 \Rightarrow \frac{d^2u}{dx^2} = -\frac{2m(E-V_0)}{\hbar^2}u$

Try $u = \underset{\text{right}}{Ce^{\frac{ip_2x}{\hbar}}} + \underset{\text{left}}{De^{-\frac{ip_2x}{\hbar}}}$

$-\frac{p_2^2}{\hbar^2}u = -\frac{2m(E-V_0)}{\hbar^2}u \Rightarrow p_2 = \sqrt{2m(E-V_0)}$



② If initially the particle is coming from the left, the

A = amplitude of incident (incoming) wave

B = " " reflected wave

C = " " transmitted wave

D = unphysical $\Rightarrow$ Set D = 0. on physical grounds

(3) Recall boundary conditions
if $V(x)$ is finite, both $u(x)$ and $\frac{du}{dx}$ must be continuous

$$u(x \to 0^-) = u(x \to 0^+)$$

$$A + B = C$$

$$\frac{du}{dx}(x \to 0^-) = \frac{du}{dx}(x \to 0^+)$$

$$\left(\frac{ip_1}{\hbar} A e^{\frac{ip_1 x}{\hbar}} - \frac{ip_1}{\hbar} B e^{-\frac{ip_1 x}{\hbar}}\right)\Big|_{x=0} = \frac{ip_2}{\hbar} C e^{\frac{ip_2 x}{\hbar}}\Big|_{x=0}$$

$$\frac{ip_1}{\hbar}(A - B) = \frac{ip_2}{\hbar} C$$

$$A - B = \frac{p_2}{p_1} C$$

Add. $$2A = \left(1 + \frac{p_2}{p_1}\right) C$$

$$C = \frac{2p_1}{p_1 + p_2} A$$

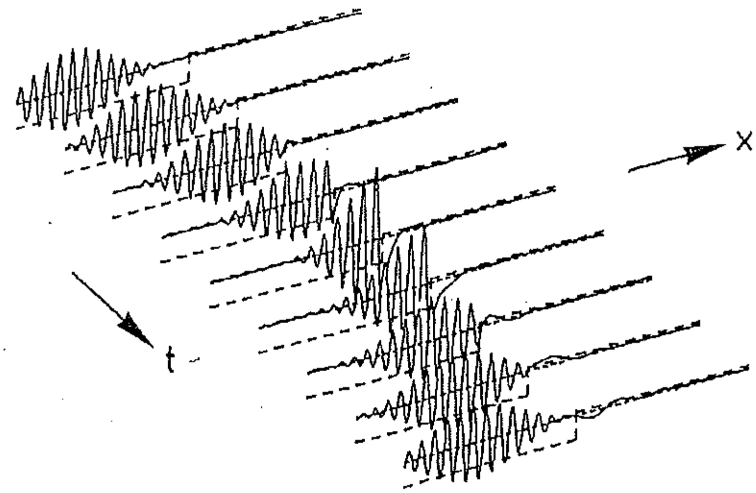
then $$B = C - A = \left(\frac{p_1 - p_2}{p_1 + p_2}\right) A$$

Energy eigenfunctions

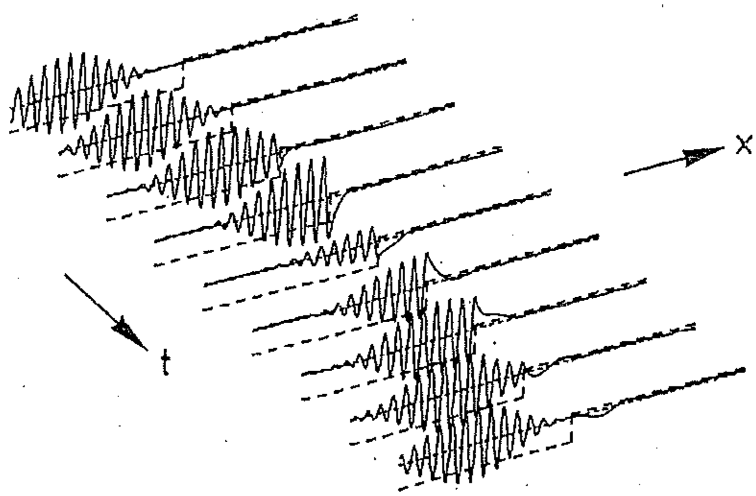
$$u(x) = \begin{cases} A\left(e^{\frac{ip_1 x}{\hbar}} + \left[\frac{p_1 - p_2}{p_1 + p_2}\right] e^{-\frac{ip_1 x}{\hbar}}\right) & x < 0 \\ A\left(\left[\frac{2p_1}{p_1 + p_2}\right] e^{\frac{ip_2 x}{\hbar}}\right) & x > 0 \end{cases}$$

overall normalization (A) not determined
by t.i.S.e nor by boundary conditions

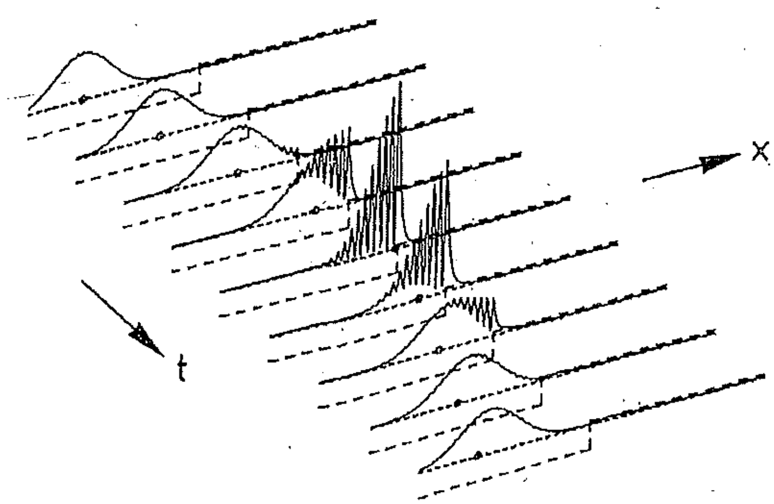
$\mathrm{Re}\,\psi(x,t)$



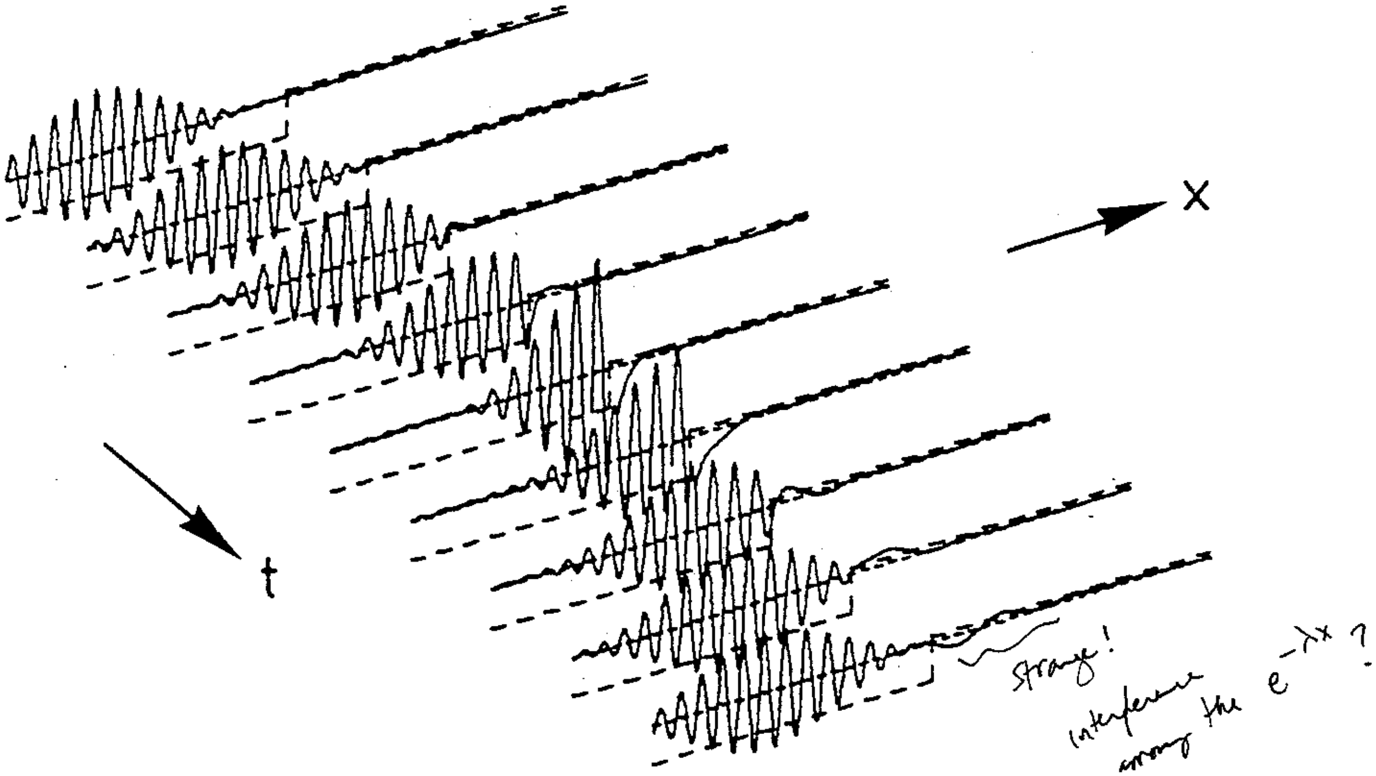
$\mathrm{Im}\,\psi(x,t)$

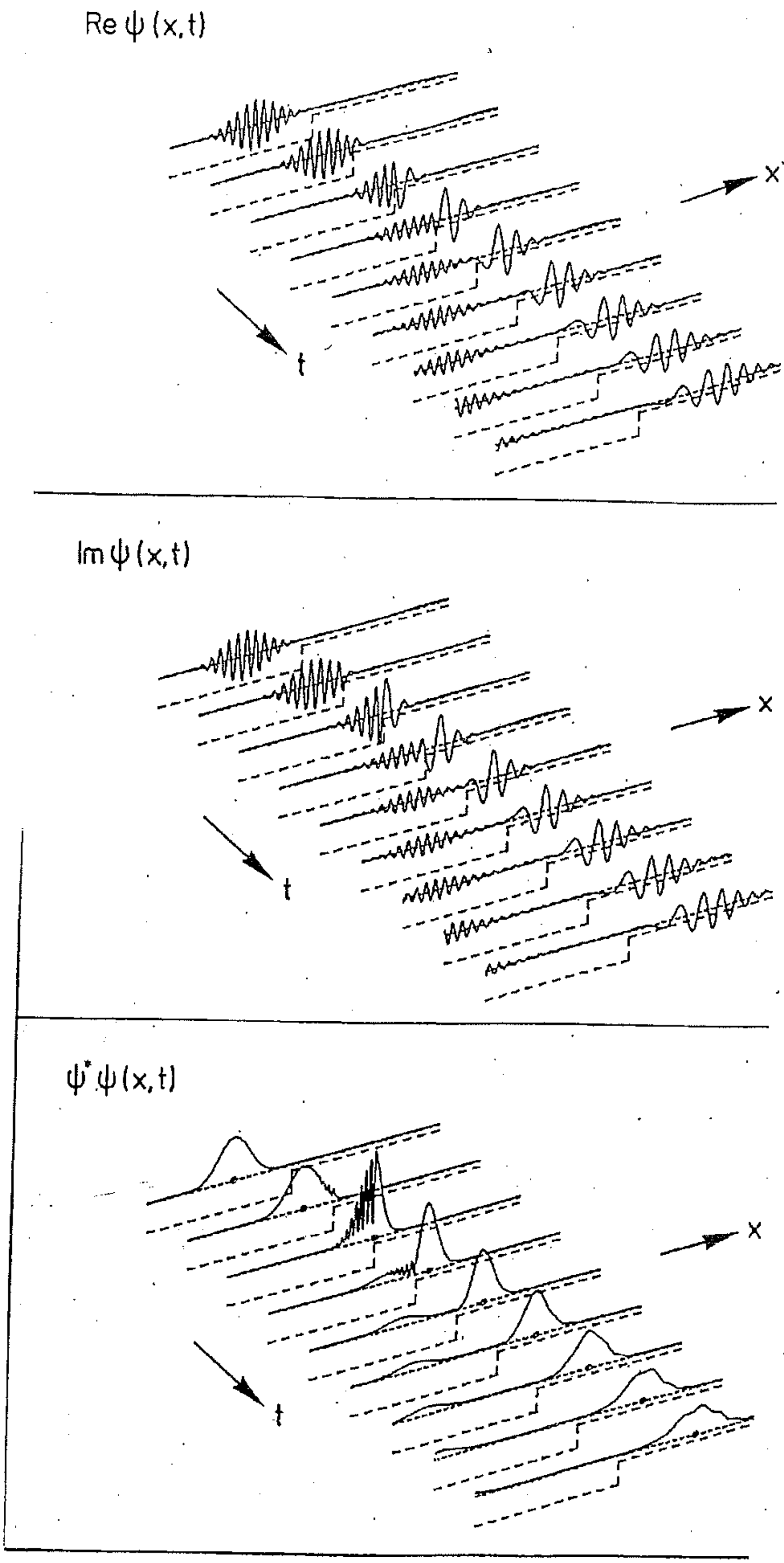


$\psi^* \psi(x,t)$



$\mathrm{Re}\,\psi(x,t)$




Re ψ (x,t)
x
t
Im ψ (x,t)
x
t
ψ*ψ (x,t)
x
t

Probability density $= \psi^* \psi$

Probability current = amt of probability flowing through a point per unit time

$$= \frac{\hbar}{2mi}\left(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx}\right)$$

Eg. $\psi = A e^{\frac{ipx}{\hbar}}$ (right moving)

$$J = \frac{\hbar}{2m}\left(A^* e^{-\frac{ipx}{\hbar}} \frac{ip}{\hbar} A e^{\frac{ipx}{\hbar}} - A e^{\frac{ipx}{\hbar}}\left(-\frac{ip}{\hbar}\right) A^* e^{-\frac{ipx}{\hbar}}\right)$$

$$= \frac{p}{m}|A|^2 = v|A|^2$$

$\psi = B e^{-\frac{ipx}{\hbar}}$ (left moving)

$$= -\frac{p}{m}|B|^2$$

Reflection coefficient R = prob of a particle reflecting off step potential

$$\equiv -\frac{J_{ref}}{J_{inc}} = -\frac{\left(-\frac{p_1}{m}\right)|B|^2}{\left(\frac{p_1}{m}|A|^2\right)} = \left|\frac{B}{A}\right|^2$$

$$R = \left(\frac{p_1 - p_2}{p_1 + p_2}\right)^2$$

- particle does not split
- probability amplitude splits

Assign ex 36 & 37

$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$

$E > V_0$ calc T vs E

31

~~PROBLEM 24~~: Probability current represents probability flowing out of or into a region. Conservation of probability then requires that the probability density $P(x,t) = \psi^*\psi$ increase or decrease correspondingly. This is expressed mathematically by the *continuity equation*:

$$\frac{\partial J}{\partial x} + \frac{\partial P}{\partial t} = 0$$

Prove this equation by using the fact that ψ obeys the time-dependent Schrödinger equation. (Hint: take the complex conjugate of the time-dependent Schrödinger equation, and use the fact that the potential energy is a real function.)

Using

$$J = \frac{\hbar}{2mi}\left[\psi^*\left(\frac{\partial\psi}{\partial x}\right) - \left(\frac{\partial\psi^*}{\partial x}\right)\psi\right]$$

we find that

$$\begin{aligned}\frac{\partial J}{\partial x} &= \frac{\hbar}{2mi}\left[\psi^*\left(\frac{\partial^2\psi}{\partial x^2}\right) + \left(\frac{\partial\psi^*}{\partial x}\right)\left(\frac{\partial\psi}{\partial x}\right) - \left(\frac{\partial\psi^*}{\partial x}\right)\left(\frac{\partial\psi}{\partial x}\right) - \left(\frac{\partial^2\psi^*}{\partial x^2}\right)\psi\right] \\ &= \frac{\hbar}{2mi}\left[\psi^*\left(\frac{\partial^2\psi}{\partial x^2}\right) - \left(\frac{\partial^2\psi^*}{\partial x^2}\right)\psi\right]\end{aligned}$$

Next recall that the time-dependent Schrödinger equation takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi$$

The complex conjugate of this equation is

$$-i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + V(x)\psi^*$$

where we use the fact that the potential energy is a real function: $V(x)^* = V(x)$.

Then

$$\begin{aligned}\frac{\partial P}{\partial t} &= \psi^*\frac{\partial\psi}{\partial t} + \frac{\partial\psi^*}{\partial t}\psi \\ &= \frac{1}{i\hbar}\psi^*\left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi\right] + \frac{1}{-i\hbar}\left[-\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} + V(x)\psi^*\right]\psi \\ &= -\frac{\hbar}{2mi}\psi^*\frac{\partial^2\psi}{\partial x^2} + \frac{\hbar}{2mi}\frac{\partial^2\psi^*}{\partial x^2}\psi\end{aligned}$$

Combining these two calculations we see that

$$\frac{\partial J}{\partial x} + \frac{\partial P}{\partial t} = 0$$

Exercise

In the handout
The transmission coefficient T is defined

as $\frac{J_{trans}}{J_{inc}}$ where J_{trans} is the probability

current associated with the transmitted wave.

(a) Calculate the transmission coefficient T for an energy eigenfunction for a step function potential w/ $E > V_0$

$$J_{trans} = \frac{p_2}{m}\left(\frac{2p_1}{p_1+p_2}\right)^2 |A|^2$$

$$T = \frac{p_2}{p_1}\frac{4p_1^2}{(p_1+p_2)^2} = \frac{4p_1p_2}{(p_1+p_2)^2}$$

(b) Since an incident particle either has to reflect or to transmit, the probabilities must add up to 100%

Show that $R + T = 1$

$$\frac{4p_1p_2}{(p_1+p_2)^2} + \frac{(p_1-p_2)^2}{(p_1+p_2)^2} = 1 \checkmark$$

Exercise

(a) Compute $\frac{B}{A}$ for an eigenfunction of the step function potential w/ $E < V_0$

$$A + B = D$$

$$\frac{ip_1}{\hbar}(A - B) = -\alpha D$$

$$A - B = \frac{i\hbar\alpha}{p_1} D$$

$$2A = \left(1 + \frac{i\hbar\alpha}{p_1}\right) D$$

$$D = \frac{2p_1}{p_1 + i\hbar\alpha} A$$

$$B = D - A = \left(\frac{p_1 - i\hbar\alpha}{p_1 + i\hbar\alpha}\right) A$$

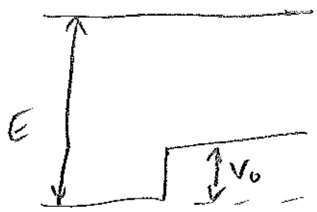
$$\frac{B}{A} = \frac{p_1 - i\hbar\alpha}{p_1 - i\hbar\alpha}$$

(b) Compute R, the reflection coefficient

$$R = \left|\frac{B}{A}\right|^2 = \frac{p_1 - i\hbar\alpha}{p_1 + i\hbar\alpha}\,\frac{p_1 + i\hbar\alpha}{p_1 - i\hbar\alpha} = 1$$

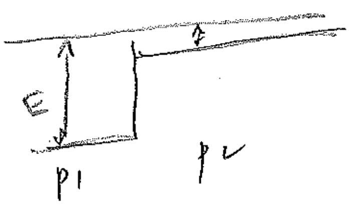
(c) Compute T, transmission coefficient.

Suppose $E \gg V_0$ then $p_1 \approx p_2$ and $R \approx 0$



particle unlikely to reflect

Suppose E is just slightly greater than V_0
classically particle can still escape

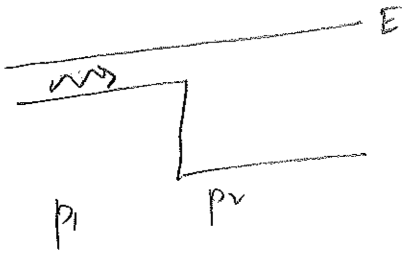


$p_2 \ll p_1$ so $R = \left(\frac{p_1 - p_2}{p_1 + p_2}\right)^2 \approx \left(\frac{p_1}{p_1}\right)^2 = 1$

quantum mechanically, there is a large probability of reflection

(conduction electrons unable to escape
→ reduces photocell efficiency if λ just below λ_{min})

(Eisberg Resnick p. 193)



Classically particle always goes into $x > 0$ region

but $p_1 \ll p_2$ so $R \approx \left(\frac{-p_2}{p_2}\right)^2 = 1$

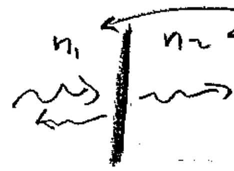
so quantum mechanical reflection

(E R. p. 198
Gr (2e), p. 2.35)

K6

Two comments: ① reflection caused by mismatch in de Broglie wavelength $\lambda = \frac{h}{p}$ (as in case of all wave reflections) ie $\lambda_1 \neq \lambda_2 \Rightarrow p_1 - p_2 \neq 0$ $\Rightarrow R \neq 0$

e.g. light at an interface



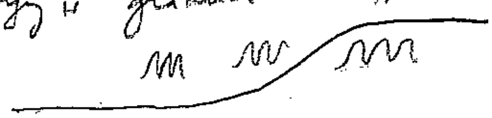
← index of refraction

$\boxed{\nu = \frac{v}{\lambda} \quad \text{so } \lambda \sim v \sim \frac{c}{n}}$

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 \qquad \text{Recall } \lambda = \frac{\lambda_{\text{in vacuo}}}{n}$$

② reflection occurs only if change in potential occurs over an interval small relative to de Broglie wavelength.

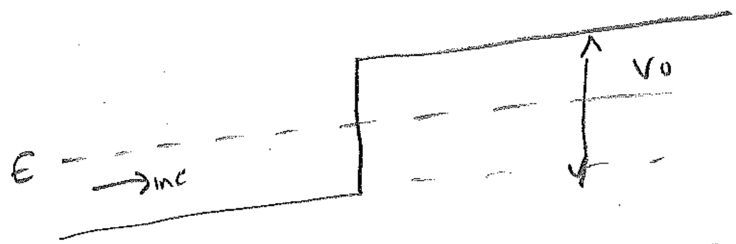
For macroscopic objects, de Broglie wavelength is much less than atomic dimensions, so any change in potential energy is gradual & there is no sudden change in λ



$\Rightarrow$ no reflection

K7

Step function potential w/ $E < V_0$



classically reflection always occurs.
$x > 0$ is a forbidden region

quantum analysis

$\boxed{x<0 \text{ region}}$ $$u = \underset{\uparrow \atop \text{inc.}}{A e^{\frac{ip_1 x}{\hbar}}} + \underset{\uparrow \atop \text{refl.}}{B e^{-\frac{ip_1 x}{\hbar}}}$$ $p_1 = \sqrt{2mE}$

$\boxed{x>0 \text{ region}}$ $$\frac{d^2u}{dx^2} = \frac{2m}{\hbar^2}(V_0 - E)\, u$$

so $e^{\frac{ip_2 x}{\hbar}}$ is <u>not</u> a solution

particle is not allowed in $x > 0$

Instead try $u = e^{\alpha x}$

$$\frac{d^2u}{dx^2} = \alpha^2 e^{\alpha x} = \frac{2m}{\hbar^2}(V_0 - E) e^{\alpha x} \Rightarrow \alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

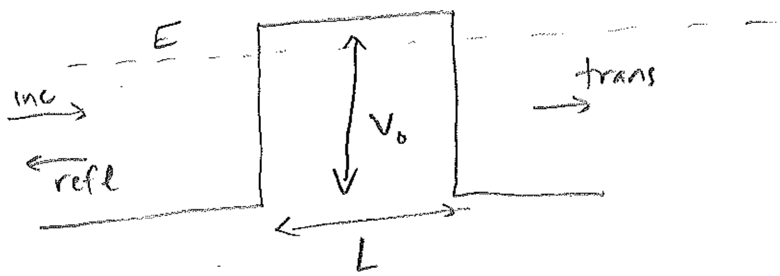
General solution:
$$u = Ce^{\alpha x} + De^{-\alpha x}$$

However, the solution must be valid for all $x > 0$, and $Ce^{\alpha x} \xrightarrow[x \to \infty]{} \infty$
Therefore we set $C = 0$ on physical grounds.
[Finish this for exercise]

K8

Barrier tunnelling

Let $V = \begin{cases} 0, & x < 0 \\ V_0, & 0 < x < L \\ 0, & x > 0 \end{cases}$



Let a particle have energy $E < V_0$ so that the barrier represents a forbidden region.

Classically the particle can only reflect off the barrier.

Quantum mechanically, the wavefunction does not vanish in the forbidden region & there is a (generally small) probability that the particle can "tunnel" through the barrier

$\boxed{x<0 \text{ Region}}$ $\quad u = Ae^{ikx} + Be^{-ikx}$ $\qquad k = \frac{p}{\hbar} = \frac{\sqrt{2mE}}{\hbar}$

$\boxed{0<x<L \text{ Region}}$ $\quad u = Ce^{\alpha x} + De^{-\alpha x}$ $\qquad \alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$

$\boxed{x>L \text{ Region}}$ $\quad u = Fe^{ikx}$

Impose boundary conditions at $x = 0$ and $x = L$ to obtain [Problem]

$$A = \frac{e^{ikL}}{4ik\alpha}\left[(\alpha + ik)^2 e^{-\alpha L} - (\alpha - ik)^2 e^{\alpha L}\right] F$$

~~K12~~ K9

-- Consider $\alpha L \gg 1$

$$A \approx \frac{i}{4k\alpha} e^{ikL} (\alpha - ik)^2 e^{\alpha L} F$$

$$A^* A \approx \frac{1}{16 k^2 \alpha^2} (\alpha^2 + k^2)^2 e^{2\alpha L} F^* F$$

Now $\alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$ $\qquad k = \frac{\sqrt{2mE}}{\hbar}$

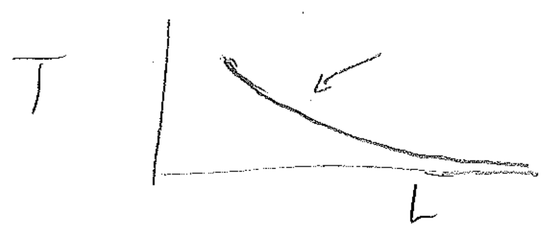
$$\Rightarrow \alpha^2 + k^2 = \frac{2mV_0}{\hbar^2}$$

$$|A|^2 = \frac{\hbar^4 \left(\frac{2mV_0}{\hbar}\right)^2}{16(2mE)(2m[V_0 - E])} e^{2\alpha L} |F|^2$$

$$= \frac{V_0^2}{16 E(V_0 - E)} e^{2\alpha L} |F|^2$$

$$T = \left|\frac{F}{A}\right|^2 = \frac{16 E (V_0 - E)}{V_0^2} e^{-\frac{2L}{\hbar}\sqrt{2m(V_0 - E)}}$$

exponential dependence on barrier width

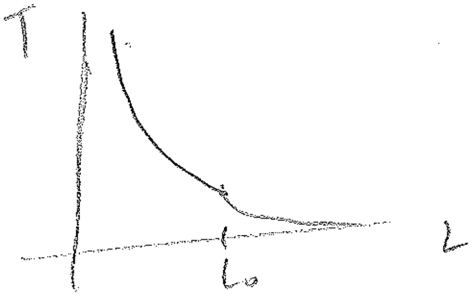


K10

Define $L_0 = \frac{\hbar}{2\sqrt{2m(V_0-E)}}$

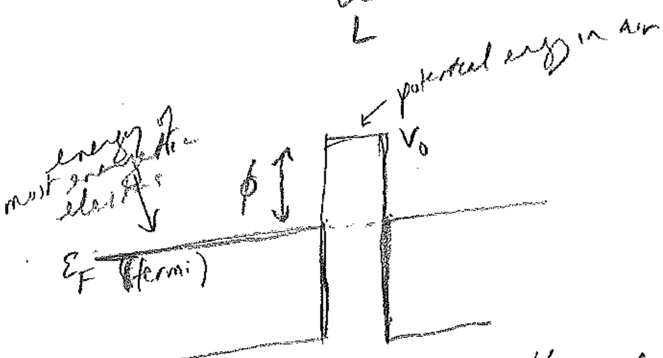
then

$$T \approx e^{-\frac{L}{L_0}}$$



Tunnelling is appreciable only if L not much bigger than L_0.

Electron tunnelling across air gap between metal plates



ϕ = work function of the metal

$\sim$ few eV

(cf Photoelectric effect)

$\Rightarrow V_0 - E_F = \phi$

$\Rightarrow L_0 = \frac{\hbar}{2\sqrt{2m\phi}}$

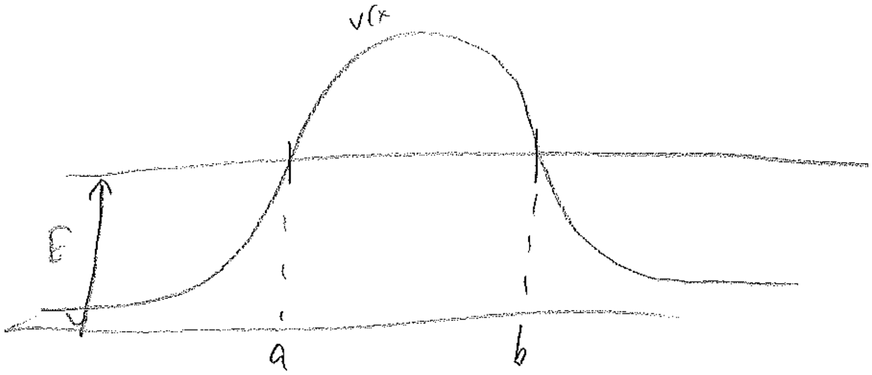
$\sim \frac{1E-34 \text{ Js}}{3\sqrt{(9E-31\text{ kg})(1.6E-19 \text{ J})}}$

$\sim$ E-10 m $\sim$ Å

[I often skip this]

K11

If potential barrier has variable height



$V(a) = V(b) = E$ $\quad$ a, b = turning points (classical)

then the tunnelling (transmission) probability is

<u>approximately</u>

$$T \approx e^{-2\gamma}$$

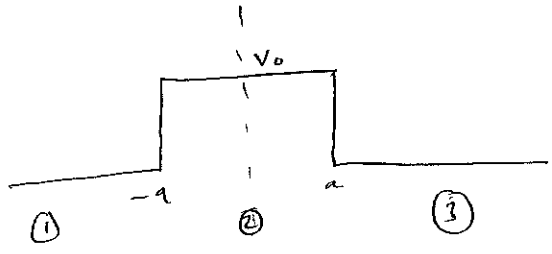
where

$$\gamma = \frac{1}{\hbar}\int_a^b dx\,\sqrt{2m(V(x)-E)}$$

is the "tunnelling factor"

OLD SOLUTION

Tunneling through a square barrier



$$u_1 = A e^{ikx} + B e^{-ikx} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$u_2 = C e^{\lambda x} + D e^{-\lambda x} \qquad \lambda = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)}$$

$$u_3 = F e^{ikx}$$

$$k^2 + \lambda^2 = \frac{2mV_0}{\hbar^2}$$

Continuity at $x = -a$:

$$A e^{-ika} + B e^{+ika} = C e^{-\lambda a} + D e^{\lambda a}$$
$$ikA e^{-ika} - ikB e^{ika} = \lambda C e^{-\lambda a} - \lambda D e^{\lambda a}$$
$$2ikA e^{-ika} = (ik+\lambda) C e^{-\lambda a} + (ik-\lambda) D e^{\lambda a}$$

continuity at $x = a$:

$$C e^{\lambda a} + D e^{-\lambda a} = F e^{ika}$$
$$\lambda C e^{\lambda a} - \lambda D e^{-\lambda a} = ik F e^{ika}$$
$$2\lambda C e^{\lambda a} = (\lambda + ik) F e^{ika}$$
$$-2\lambda D e^{-\lambda a} = (-\lambda + ik) F e^{ika}$$

(6 pts if get to here)

$$\therefore \quad 2ikA e^{-ika} = \frac{(ik+\lambda)^2}{2\lambda} e^{-2\lambda a + ika} F + \frac{(ik-\lambda)^2}{-2\lambda} e^{2\lambda a + ika} F$$

$$A = F e^{2ika} \frac{1}{4ik\lambda}\Big[\underbrace{+(\lambda^2 - k^2 + 2ik\lambda) e^{-2\lambda a} - (\lambda^2 - k^2 - 2ik\lambda) e^{2\lambda a}}_{-(\lambda^2-k^2)\,2\sinh(2\lambda a) + 4ik\lambda\cosh(2\lambda a)}\Big]$$

$$A = F e^{2ika}\Big[\cosh(2\lambda a) + i\frac{(\lambda^2 - k^2)}{2k\lambda}\sinh(2\lambda a)\Big]$$

intermediate check

$$\boxed{A = \frac{F e^{2ika}}{4ik\lambda}\left[(\lambda+ik)^2 e^{-2\lambda a} - (\lambda - ik)^2 e^{2\lambda a}\right]}$$

$$\left|\frac{A}{F}\right|^2 = \frac{(\lambda^2+k^2)^2(e^{-4\lambda a} + e^{4\lambda a}) - (\lambda+ik)^4 - (\lambda-ik)^4}{16k^2\lambda^2}$$

$$(\lambda+ik)^4 + (\lambda-ik)^4 = 2(\lambda^4 - 6\lambda^2k^2 + k^4) = 2(\lambda^2+k^2)^2 - 16\lambda^2k^2$$

$$\left|\frac{A}{F}\right|^2 = 1 + \frac{(\lambda^2+k^2)^2}{16\lambda^2k^2}\big(\underbrace{e^{-4\lambda a} + e^{4\lambda a} - 2}_{\substack{2\cosh(4\lambda a) - 2 \\ 2\cosh^2(2\lambda a) + 2\sinh^2(2\lambda a) - 2 \\ 4\sinh^2(2\lambda a)}}\big)$$

$$= 1 + \left(\frac{\lambda^2+k^2}{2k\lambda}\right)^2 \sinh^2(2\lambda a)$$

$$\left|\frac{A}{F}\right|^2 = \underbrace{\cosh^2(2\lambda a)}_{1+\sinh^2} + \left(\frac{\lambda^2 - k^2}{2k\lambda}\right)^2 \sinh^2(2\lambda a)$$

$$= 1 + \underbrace{\left(\frac{\lambda^2 + k^2}{2k\lambda}\right)^2}\sinh^2(2\lambda a)$$

$$\frac{\frac{2m}{\hbar^2}(V_0 - E) + E}{2\left(\frac{2m}{\hbar^2}\right)\sqrt{E(V_0-E)}} = \frac{V_0}{2\sqrt{E(V_0-E)}}$$

$$\boxed{T = \left|\frac{F}{A}\right|^2 = \left[1 + \frac{V_0^2}{4E(V_0-E)}\sinh^2\left(\frac{2a}{\hbar}\sqrt{m(V_0-E)}\right)\right]^{-1}}$$

in forbidden region (if asked about J in forbidden region)

$$(C^* e^{\alpha x} + D^* e^{-\alpha x})(\alpha C e^{\alpha x} - \alpha D e^{-\alpha x})$$
$$-(\alpha C^* e^{\alpha x} - \alpha D^* e^{-\alpha x})(C e^{\alpha x} + D e^{-\alpha x})$$

$$= -2\alpha C^* D + 2\alpha D^* C$$

$$J \quad \frac{\hbar}{2mi}(2\alpha)\underbrace{(D^* C - C^* D)}_{\text{imaginary}}$$

so non-zero

in allowed

$$\frac{\hbar}{2mi}\Big[(A^* e^{-ikx} + B^* e^{ikx})(ikA e^{ikx} - ikB e^{-ikx})$$
$$-(-ikA^* e^{-ikx} + ikB^* e^{ikx})(A e^{ikx} + B e^{-ikx})\Big]$$

$$= \frac{\hbar k}{m}\left(|A|^2 - |B|^2\right)$$ no cross term

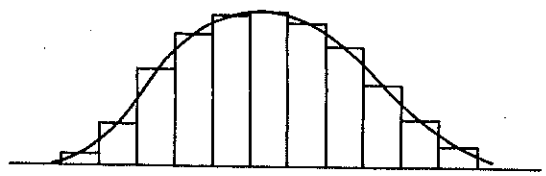


Figure 5-4. Approximation of smooth barrier by a juxtaposition of square potential barriers.

multiplicative[2] when they are small (in effect, with most of the flux reflected, the transmission through each slice is an independent, improbable event), we may write, approximately

$$\begin{aligned}\ln |T|^2 &\approx \sum_{\substack{\text{partial}\\ \text{barriers}}} \ln |T_{\substack{\text{partial}\\ \text{barrier}}}|^2 \\ &\approx -2 \sum \Delta x \, \langle \kappa \rangle \\ &\approx -2 \int_{\text{barrier}} dx \, \sqrt{(2m/\hbar^2)[V(x) - E]}\end{aligned} \tag{5-42}$$

In the partial barriers, Δx is the width and $\langle \kappa \rangle$ the average value of κ for that barrier. In the last step a limit of infinitely narrow barriers was taken. It is clear from the expression that the approximation is least accurate near the "turning points" where the energy and potential are nearly equal, since there (5-40) is not a good approximation to (5-35). It is also important that $V(x)$ be a slowly varying function of x, since otherwise the approximation of a curved barrier by a stack of square ones is only possible if the latter are narrow, and there again (5-40) is a poor approximation. A proper treatment, using the WKB approximation includes a discussion of the behavior near the turning points. For most purposes, it is still a fair approximation to write

$$|T|^2 \approx e^{-2\int dx \sqrt{(2m/\hbar^2)[V(x) - E]}} \tag{5-43}$$

with the integration over the region in which the square root is real.

TUNNELING PHENOMENA

The phenomenon of particle tunneling is quite common in atomic and nuclear physics, and we discuss several examples at this point.

Cold Emission

Consider electrons in a metal. As noted in our discussion of the photoelectric effect in Chapter 1, these electrons are held in a metal by a potential, which, to first approximation, may be described by a box of finite depth, as shown in Fig. 5-5*a*.

[2]This statement is only correct for the most important exponential part, as can be seen from the fact that doubling the width will only approximately square the transmission coefficient $|T|^2$.

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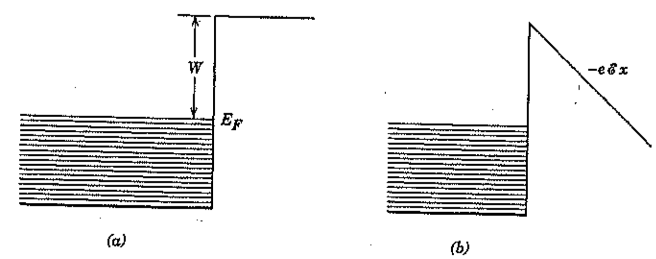


Figure 5-5. (*a*) Electronic energy levels in metal. E_F is the Fermi energy and W is the work function. (*b*) Potential altered by an external electric field.

The electrons are actually stacked up in energy levels that are very dense, since the box is very wide. It is a property of electrons[3] that no more than two of them can occupy any given energy level; thus for the lowest energy state of the metal, all the levels up to a certain energy, called the *Fermi energy* (which depends on the density of free electrons), are filled. When the temperature is above absolute zero, a few electrons are thermally excited to higher levels, but even at room temperature, the number is small. The difference between the Fermi energy and the top of the well is what is required to bring an electron out; it is the *work function* discussed in connection with the photoelectric effect. Electrons can be removed by transferring energy to them, either by photons, or by heating them. They can also be removed by the application of an external electric field $\mathscr{E}$. *Cold emission* occurs because the external field changes the potential seen by an electron from W to $(W - e\mathscr{E}x)$ (Fig. 5-5*b*), if the electron is at the top of the "sea" of levels. The transmission coefficients are

$$|T|^2 = e^{-2\int_0^a dx[2m(W - e\mathscr{E}x)/(\hbar^2)]^{1/2}} \tag{5-44}$$

Since

$$\int dx(A + Bx)^{1/2} = \frac{(A + Bx)^{3/2}}{3B/2}$$

this leads to

$$\frac{\sqrt{2}}{3}$$

$$|T|^2 = e^{-4\sqrt{2/3}\sqrt{mW/\hbar^2}(W/e\mathscr{E})} \tag{5-45}$$

The Fowler-Nordheim formula, as (5-45) is called, describes the emission only qualitatively. One effect, which is easily included, is the additional attraction of the electron back to the plate, caused by the image charge. The other effect, much harder to handle, is that there are surface imperfections in the metal surface, which change the electric field locally, and since $\mathscr{E}$ appears in the exponent, this can make a large difference. Incidentally, we see that the exponent may be written in

[3]This property of electrons is described by the Pauli exclusion principle, which will be discussed in Chapter 8.

PROBLEM 25: Consider an electron in a metal which has work function ϕ. The work function is defined as the minimum energy required for an electron to escape from the surface of the metal. If a strong electric field is applied, however, the electron may be able to escape through a process called "cold emission". This is a quantum mechanical process whereby the electron tunnels through the potential barrier.

Let the potential energy of the electron be given by $V(x) = \epsilon_F + \left(1 - \frac{x}{L}\right)\phi$, where $x = 0$ denotes the surface of the metal, and ϵ_F is the Fermi energy (the energy of the highest energy electron).

(a) What is the magnitude of the electric field to which this potential energy corresponds?

(b) Compute the tunneling factor γ for an electron with energy ϵ_F to tunnel from $x = 0$ to $x = L$.

(c) Compare your result with the tunneling factor for the same electron through a square barrier of width L and constant height $V_0 = \epsilon_F + \phi$ (and therefore no electric field).

(a) The potential energy of the electron is $V(x) = \epsilon_F + \left(1 - \frac{x}{L}\right)\phi$, so the force on the electron is

$$F_x = -\frac{dV}{dx} = \frac{\phi}{L}$$

and therefore the electric field is

$$E_x = \frac{F_x}{(-e)} = -\frac{\phi}{eL}$$

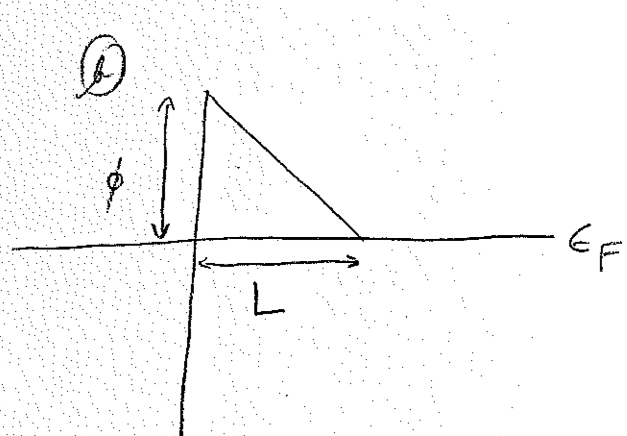
where e is the (magnitude of the) charge on an electron. The magnitude of the electric fields is $\phi/(eL)$.

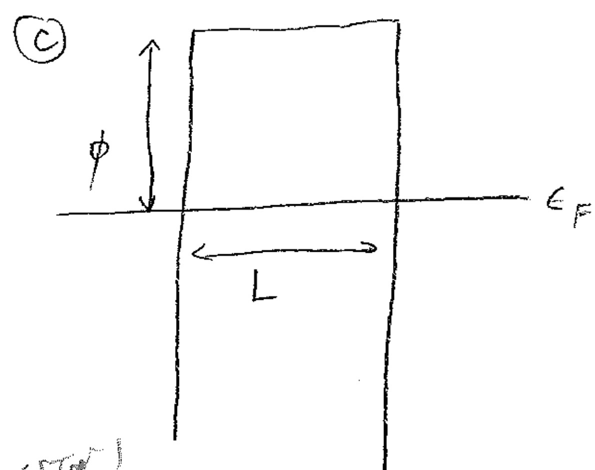
(b) The tunneling factor is

$$\gamma = \frac{1}{\hbar}\int_a^b dx\sqrt{2m(V(x)-E)} = \frac{\sqrt{2m\phi}}{\hbar}\int_0^L dx\sqrt{1-\frac{x}{L}} = \frac{-2L\sqrt{2m\phi}}{3\hbar}\left(1-\frac{x}{L}\right)^{3/2}\Bigg|_0^L = \frac{2}{3}\frac{\sqrt{2m\phi}}{\hbar}L.$$

(c) The tunneling factor through a square barrier of width L and constant height $V_0 = \epsilon_F + \phi$ is

$$\gamma = \frac{1}{\hbar}\int_0^L dx\sqrt{2m\phi} = \frac{\sqrt{2m\phi}}{\hbar}L.$$





Scanning tunneling microscope (STM)
of Gasiorowicz