In QM, every observable A (something that can be measured) is associated with a linear operator $\hat{A}$ which acts on wavefunctions ψ to give another function

e.g. energy E $\rightarrow$ $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$

position X $\rightarrow$ $\hat{X} = x$ [multiply by x]

momentum P $\rightarrow$ $\hat{P} = ?$ [T.B.D.]

- An <u>eigenfunction</u> of $\hat{A}$ is a wavefunction ψ_a that satisfies the eigenvalue equation

$$\hat{A}\psi_a = a\psi_a$$

- ψ_a represents a state w/ well-defined value of observable A. i.e. if we measure A of state ψ_a, result will always be the eigenvalue a

- State ψ_a has <u>zero</u> <u>uncertainty</u> for the observable A

i.e. $\Delta A = 0$

proof: $\langle A\rangle = \int dx\, \psi_a^* \hat{A}\psi_a = a\int dx\, \psi_a^*\psi_a = a$

$$\langle A^2\rangle = \int dx\, \psi_a^* \underbrace{\hat{A}^2\psi_a}_{\underbrace{\hat{A}\,a\psi_a}_{a^2\psi_a}} = a^2$$

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} = \sqrt{a^2 - a^2} = 0.$$

[can talk about compatibility, incompatibility, & non commuting operators X, P]

$\hat{p} = ?$

Eigenfunctions of $\hat{p}$ are states of well-defined momentum

$$\hat{p}\psi_p = p\psi_p$$

For a free particle, a state of well defined momentum & energy is

$$\psi = e^{\frac{ipx}{\hbar}} e^{-\frac{iEt}{\hbar}}$$

$$\underbrace{\underbrace{i\hbar \frac{\partial \psi}{\partial t}}_{-\frac{iE}{\hbar}\psi}}_{E\psi} = \hat{H}\psi$$

$$\hat{p}\, e^{\frac{ipx}{\hbar}} = p\, e^{\frac{ipx}{\hbar}}$$

$$\hat{p} = B\frac{d}{dx} \Rightarrow \hat{p}\, e^{\frac{ipx}{\hbar}} = B\frac{ip}{\hbar} e^{\frac{ipx}{\hbar}} = p\, e^{\frac{ipx}{\hbar}}, \quad B = \frac{\hbar}{i}$$

$$\boxed{\hat{p} = \frac{\hbar}{i}\frac{d}{dx}}$$

Even though we "derived" $\hat{p}$ using a free particle, the result is valid for any particle

$$\langle p \rangle = \int dx\, \psi^* \hat{p}\psi = \frac{\hbar}{i}\int dx\, \psi^* \frac{d\psi}{dx}$$

[exercise: calc $\langle p \rangle$, Δp for particle in a box]

$$\langle p \rangle = 0, \qquad \Delta p = \frac{\hbar\pi n}{L}$$

[After exercise presented]

Energy eigenfunctions for particle in a box

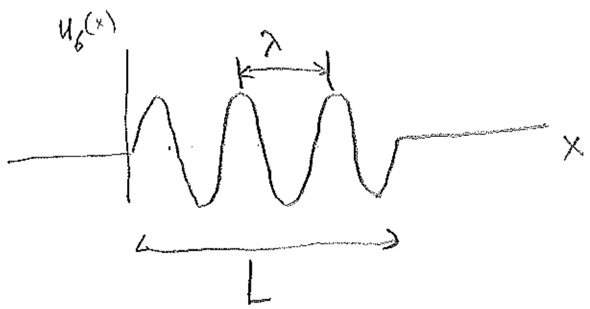
$$u_n(x) = \sqrt{\frac{2}{L}}\ \sin\left(\frac{\pi n x}{L}\right)$$

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \qquad \Delta E_n = 0$$

$$\langle x \rangle = \frac{L}{2} \qquad \Delta x = L\sqrt{\frac{1}{12} - \frac{1}{2\pi^2 n^2}}$$ [from problem]

$$\langle p \rangle = 0 \qquad \Delta p = \frac{\hbar \pi n}{L}$$ [from exercise]

[neither position nor momentum eigenstate]



$$\lambda = \frac{2L}{n}$$ from inspection

$$p = \frac{h}{\lambda} = \frac{hn}{2L} = \frac{\pi \hbar n}{L}$$

Would "expect" momentum measurement to give $\sim \frac{\pi \hbar n}{L}$

Why $\langle p \rangle = 0$? $\qquad p = \pm \frac{\pi \hbar n}{L}$

[particle can move either direction]

A measurement of p gives [see plot]

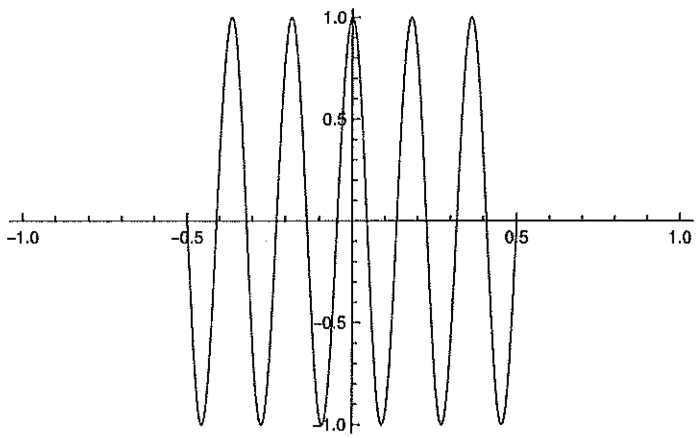
```
f[x_] := If[Abs[x] < 1 / 2, Cos[n Pi x], 0]

nn = 11;

FourierTransform[f[x], x, p]
```

$$\frac{2\,p\,\mathrm{Cos}\left[\frac{n\pi}{2}\right]\,\mathrm{Sin}\left[\frac{p}{2}\right] - 2\,n\,\pi\,\mathrm{Cos}\left[\frac{p}{2}\right]\,\mathrm{Sin}\left[\frac{n\pi}{2}\right]}{\sqrt{2\pi}\,\left(p^2 - n^2\,\pi^2\right)}$$

```
Plot[f[x] /. n → nn, {x, -1, 1}]
```



$n = 11$

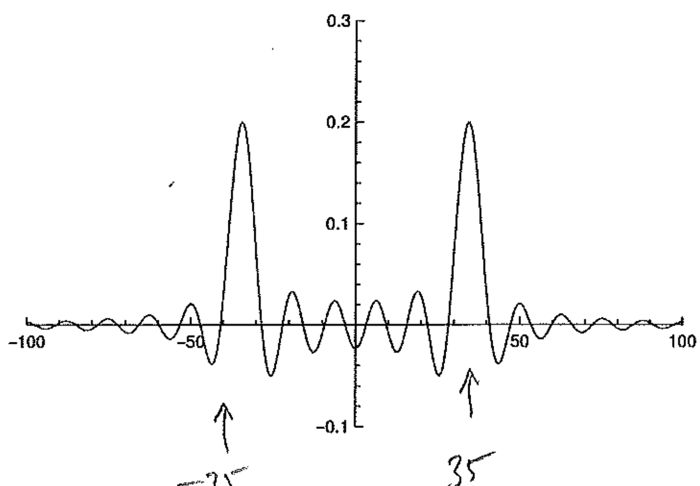
$\lambda = \frac{2}{11} L$

$p = \frac{h}{\lambda} = \frac{11\pi\hbar}{L}$

```
FourierTransform[f[x] /. n → nn, x, p]
```

$$\frac{11\sqrt{2\pi}\,\mathrm{Cos}\left[\frac{p}{2}\right]}{p^2 - 121\,\pi^2}$$

```
Plot[%, {p, -100, 100}, PlotRange → {{-100, 100}, {-.1, .3}}]
```



let $\hbar = L = 1$

$p = 11\pi \approx 35$

Rest of these plots
just for my reference

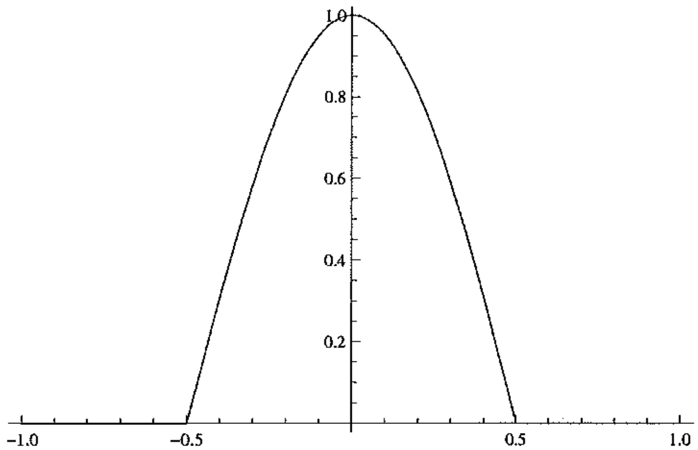
In[34]:= `f[x_] := If[Abs[x] < 1 / 2, Cos[n Pi x], 0]`

In[35]:= `FourierTransform[f[x], x, p]`

Out[35]=
$$\frac{2\,p\,\mathrm{Cos}\left[\frac{n\pi}{2}\right]\,\mathrm{Sin}\left[\frac{p}{2}\right] - 2\,n\,\pi\,\mathrm{Cos}\left[\frac{p}{2}\right]\,\mathrm{Sin}\left[\frac{n\pi}{2}\right]}{\sqrt{2\pi}\,\left(p^2 - n^2\pi^2\right)}$$

In[36]:= `Plot[f[x]  /. n -> 1, {x, -1, 1}]`

Out[36]=

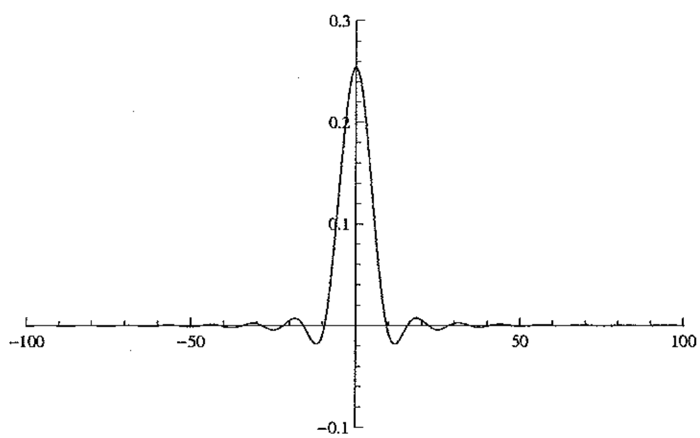


In[37]:= `FourierTransform[f[x] /. n → 1, x, p]`

Out[37]=
$$\frac{\sqrt{2\pi}\,\mathrm{Cos}\left[\frac{p}{2}\right]}{-p^2 + \pi^2}$$

In[38]:= `Plot[%, {p, -100, 100}, PlotRange → {{-100, 100}, {-.1, .3}}]`

Out[38]=



In[41]:= `f[x_] := If[Abs[x] < 1 / 2, Cos[n Pi x], 0]`

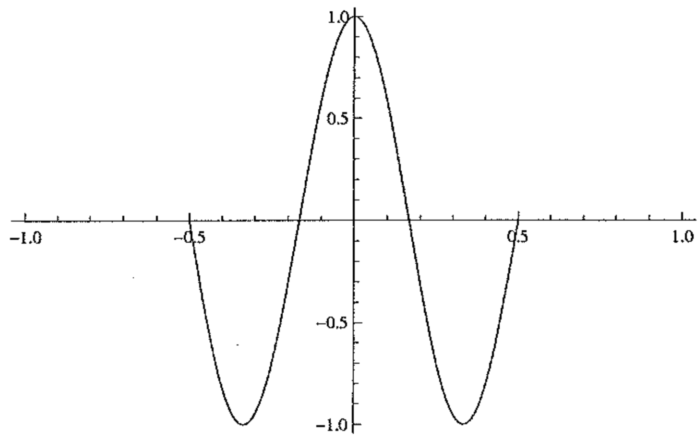
In[42]:= `nn = 3`

Out[42]= 3

In[43]:= `FourierTransform[f[x], x, p]`

Out[43]= $$\frac{2\,p\,\mathrm{Cos}\left[\frac{n\pi}{2}\right]\,\mathrm{Sin}\left[\frac{p}{2}\right] - 2\,n\,\pi\,\mathrm{Cos}\left[\frac{p}{2}\right]\,\mathrm{Sin}\left[\frac{n\pi}{2}\right]}{\sqrt{2\pi}\,\left(p^2 - n^2\pi^2\right)}$$

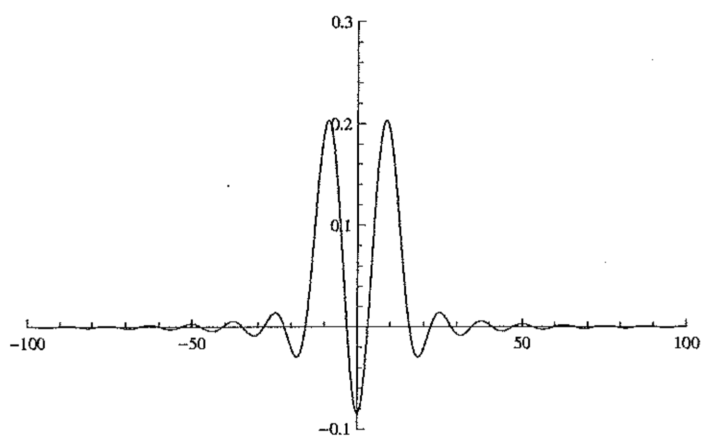
In[44]:= `Plot[f[x] /. n → nn, {x, -1, 1}]`

Out[44]=


In[45]:= `FourierTransform[f[x] /. n → nn, x, p]`

Out[45]= $$\frac{3\sqrt{2\pi}\,\mathrm{Cos}\left[\frac{p}{2}\right]}{p^2 - 9\pi^2}$$

In[46]:= `Plot[%, {p, -100, 100}, PlotRange → {{-100, 100}, {-.1, .3}}]`

Out[46]=


In[47]:= `f[x_] := If[Abs[x] < 1 / 2, Cos[n Pi x], 0]`

In[48]:= `nn = 5`

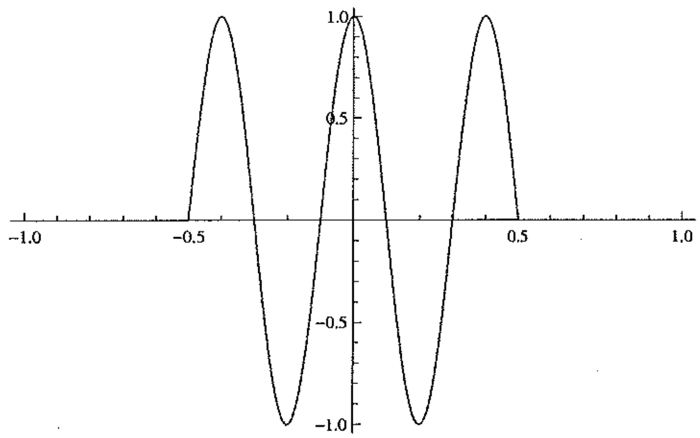
Out[48]= 5

In[49]:= `FourierTransform[f[x], x, p]`

Out[49]=
$$\frac{2\,p\,\mathrm{Cos}\left[\frac{n\pi}{2}\right]\,\mathrm{Sin}\left[\frac{p}{2}\right]-2\,n\,\pi\,\mathrm{Cos}\left[\frac{p}{2}\right]\,\mathrm{Sin}\left[\frac{n\pi}{2}\right]}{\sqrt{2\pi}\,\left(p^2-n^2\pi^2\right)}$$

In[50]:= `Plot[f[x] /. n → nn, {x, -1, 1}]`

Out[50]=

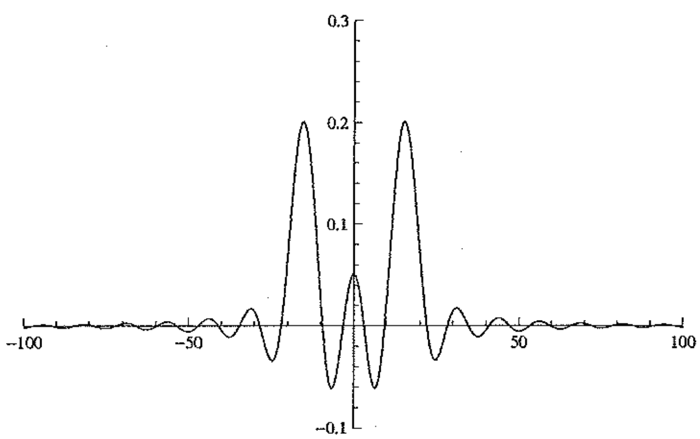


In[51]:= `FourierTransform[f[x] /. n → nn, x, p]`

Out[51]=
$$-\frac{5\sqrt{2\pi}\,\mathrm{Cos}\left[\frac{p}{2}\right]}{p^2-25\,\pi^2}$$

In[52]:= `Plot[%, {p, -100, 100}, PlotRange → {{-100, 100}, {-.1, .3}}]`

Out[52]=



In[1]:= `f[x_] := If[Abs[x] < 1 / 2, Cos[n Pi x], 0]`

In[2]:= `nn = 51`

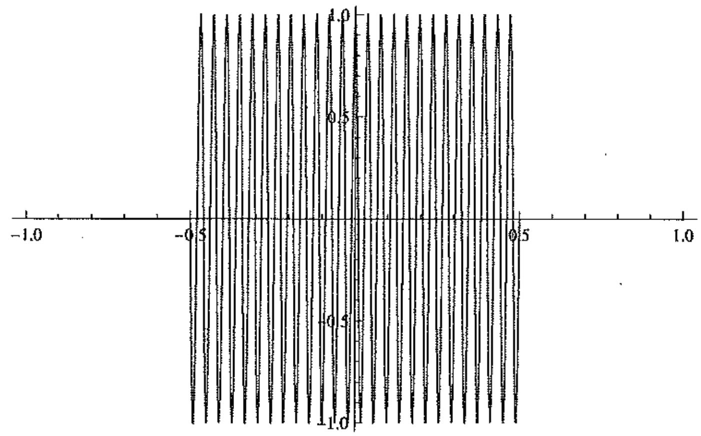
Out[2]= 51

In[3]:= `FourierTransform[f[x], x, p]`

Out[3]= $$\frac{2\,p\,\mathrm{Cos}\left[\frac{n\pi}{2}\right]\,\mathrm{Sin}\left[\frac{p}{2}\right]-2\,n\,\pi\,\mathrm{Cos}\left[\frac{p}{2}\right]\,\mathrm{Sin}\left[\frac{n\pi}{2}\right]}{\sqrt{2\pi}\,\left(p^2-n^2\pi^2\right)}$$

In[4]:= `Plot[f[x] /. n → nn, {x, -1, 1}]`

Out[4]=

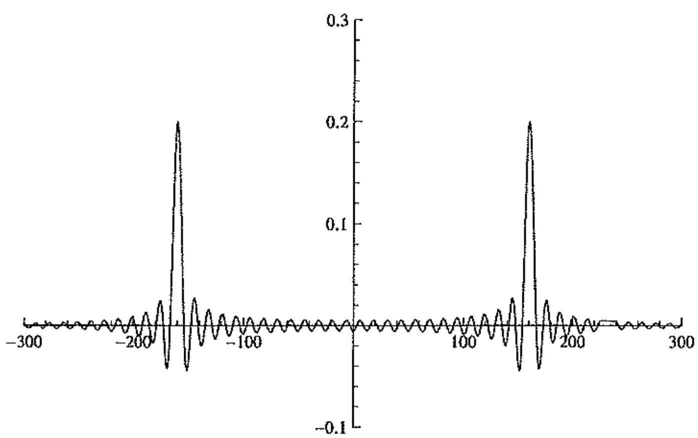


In[5]:= `FourierTransform[f[x] /. n → nn, x, p]`

Out[5]= $$\frac{51\sqrt{2\pi}\,\mathrm{Cos}\left[\frac{p}{2}\right]}{p^2-2601\,\pi^2}$$

In[6]:= `Plot[%, {p, -300, 300}, PlotRange → {{-300, 300}, {-.1, .3}}]`

Out[6]=



Note $\Delta x \sim L \qquad \Delta p \sim \frac{1}{L}$

$$\Delta x \, \Delta p = \hbar \pi n \sqrt{\frac{1}{12} - \frac{1}{2\pi^2 n^2}}$$

$$= \hbar \sqrt{\frac{\pi^2 n^2}{12} - \frac{1}{2}} \qquad n = 1, 2, \ldots$$

minimum value for $n = 1$

$$(\Delta x \, \Delta p)_{min} = \hbar \sqrt{\frac{\pi^2}{12} - \frac{1}{2}} \approx 0.57 \, \hbar$$

In general one can prove

$$\Delta x \, \Delta p \geq \frac{\hbar}{2} \qquad \text{(Heisenberg)}$$

If $\Delta x \, \Delta p = \frac{\hbar}{2}$ we say inequality is <u>saturated</u>

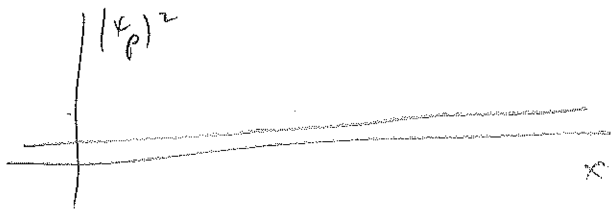
Consider a free particle ($V(x) = 0$)
described by a momentum & energy eigenstate

$$\psi_p(x,t) = A(p)\, e^{\frac{ipx}{\hbar}}\, e^{-\frac{iE_p t}{\hbar}} \qquad E_p = \frac{p^2}{2m}$$

If measure its momentum or energy, the result will be p or E_p respectively w/ 100% probability.

What if measure its position?

Probability density $|\psi_p|^2 = |A(p)|^2$ (indep of x and t)



Problems w/ this description

① stationary state ⇒ does not move

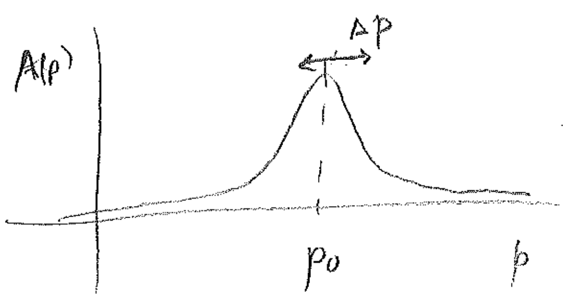
② not localized ⇒ equally likely to be found anywhere

③ not normalizable $\int_{-\infty}^{\infty} |A(p)|^2 dx = \infty$

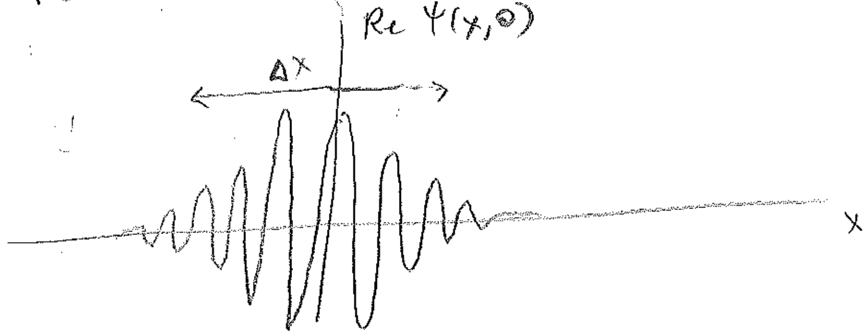
All 3 problems can be solved by forming a wavepacket (linear combination of eigenstates)

$$\psi(x,t) = \int dp\ \psi_p(x,t)$$
$$= \int dp\ A(p)\, e^{\frac{ipx}{\hbar}}\, e^{-\frac{ip^2 t}{2m\hbar}}$$

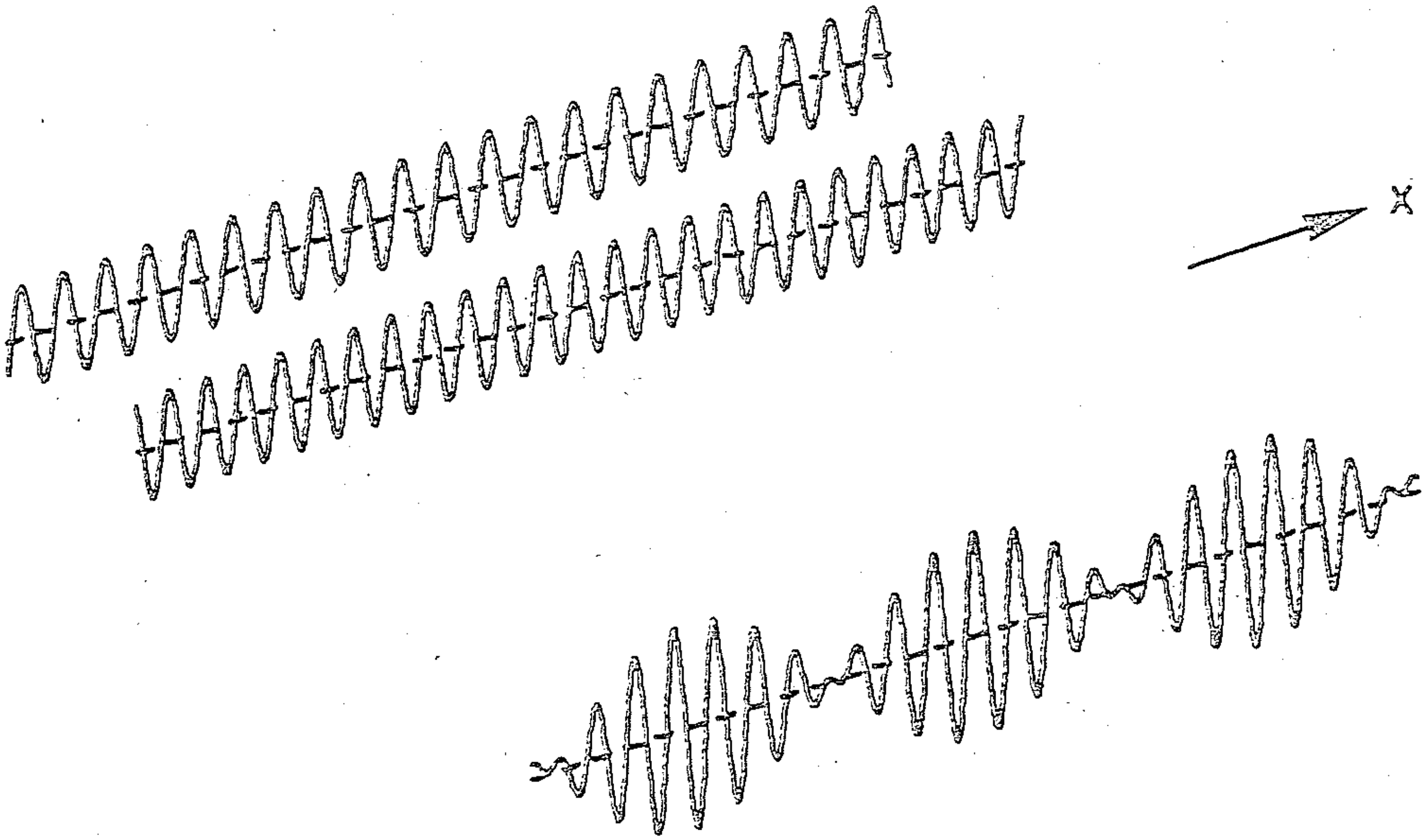
where $A(p)$ is a function peaked near p_0 w/ width $\sim \Delta p$

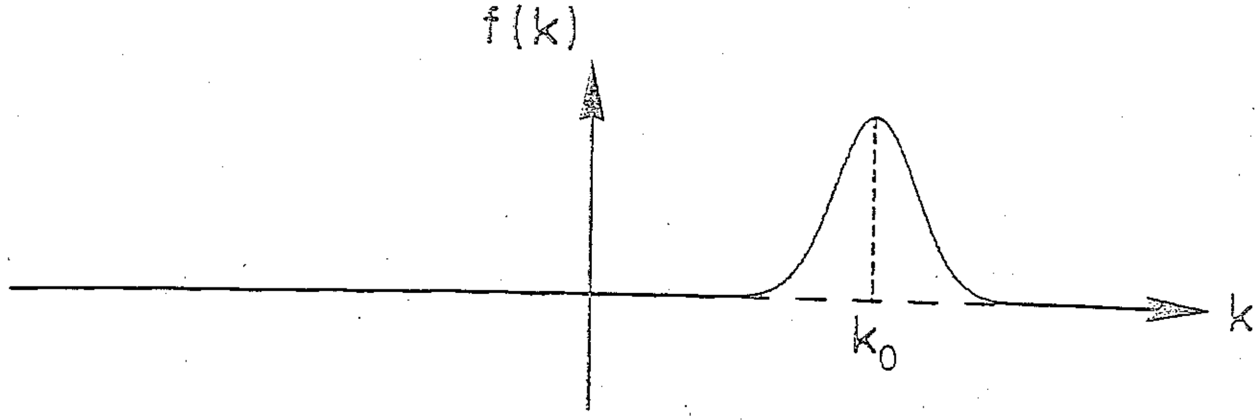


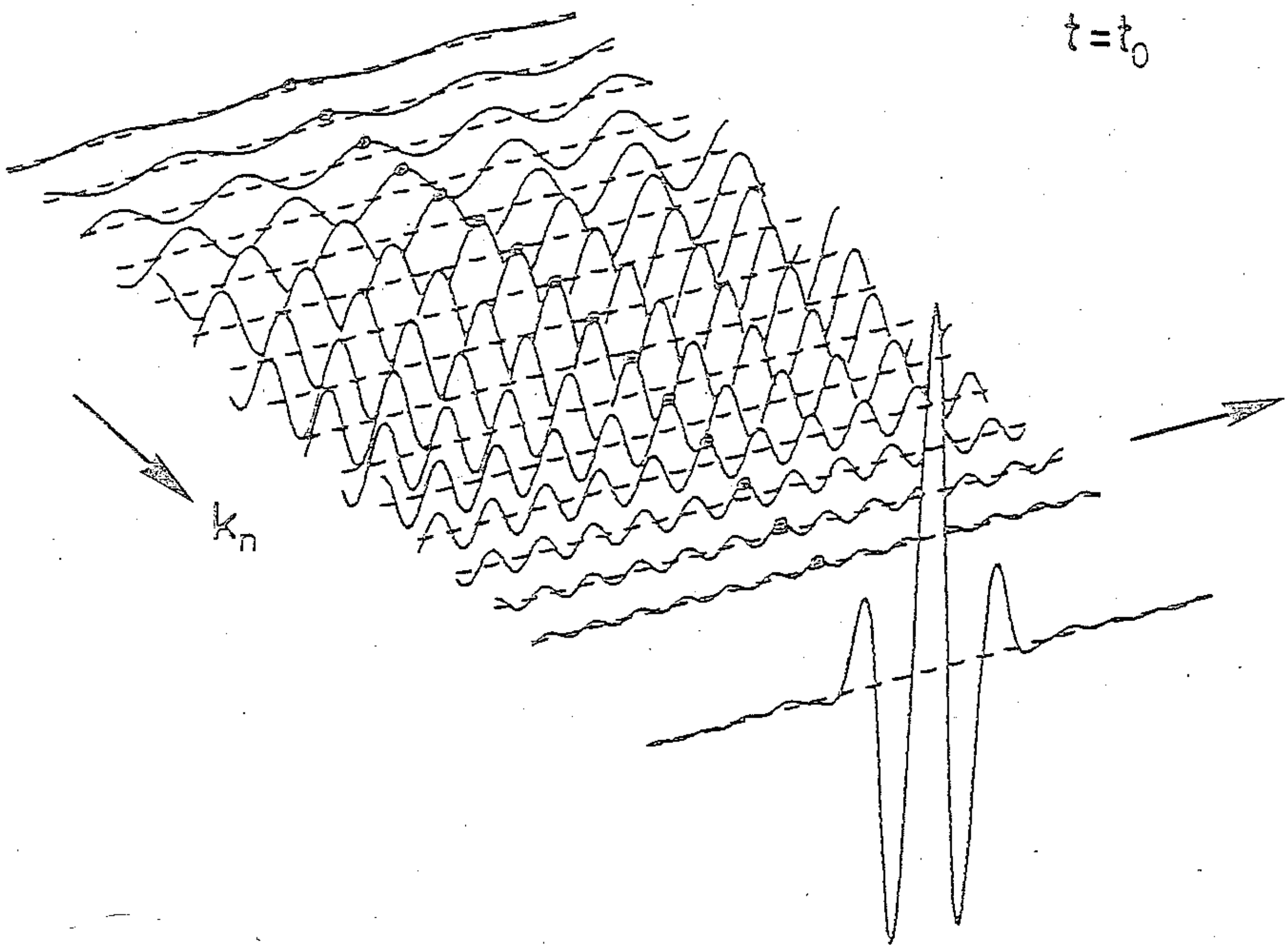
Constructive interference between different components results in

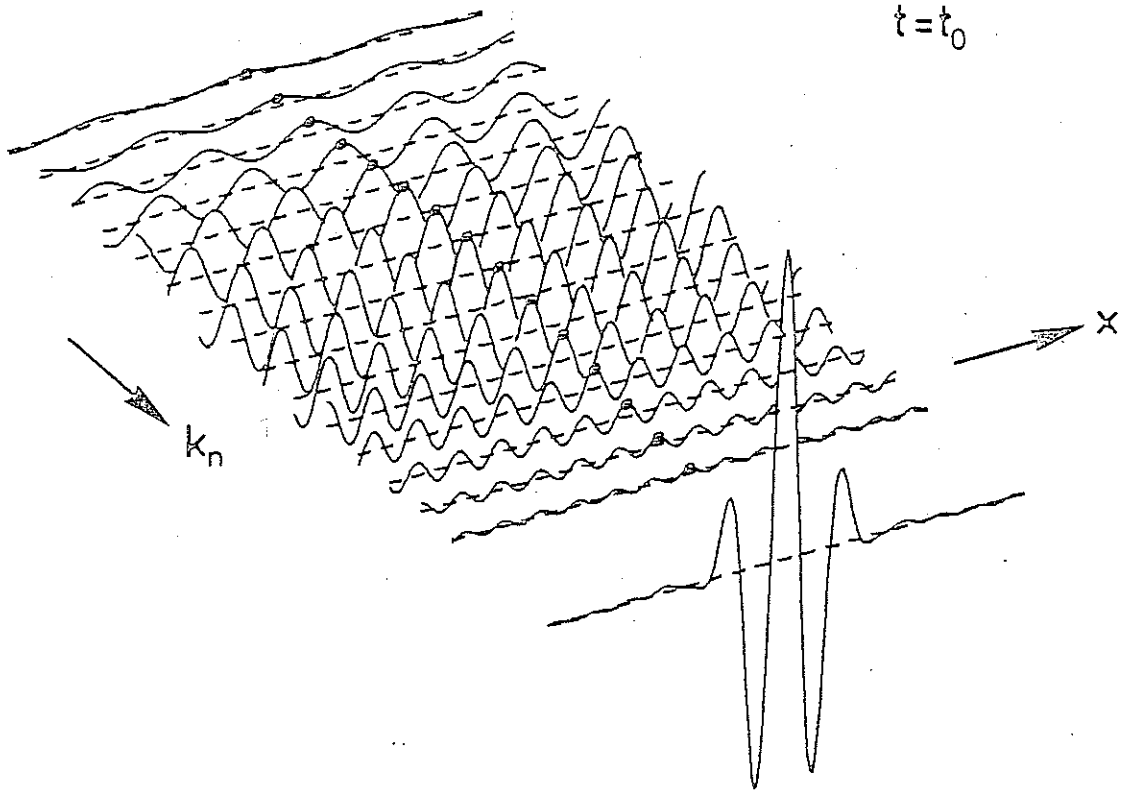


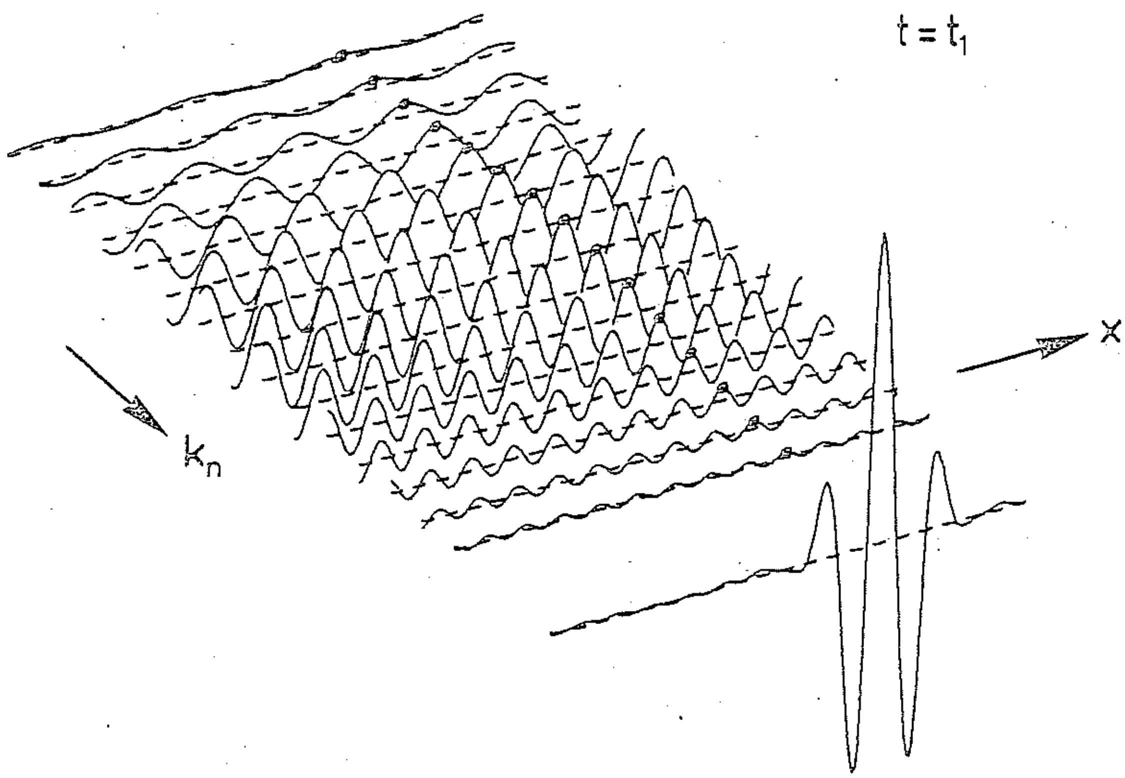
The wavepacket
1. moves (w/ speed $\frac{p_0}{m}$)
2. is localized to within $\Delta x \sim \frac{\hbar}{\Delta p}$
3. can be normalized.

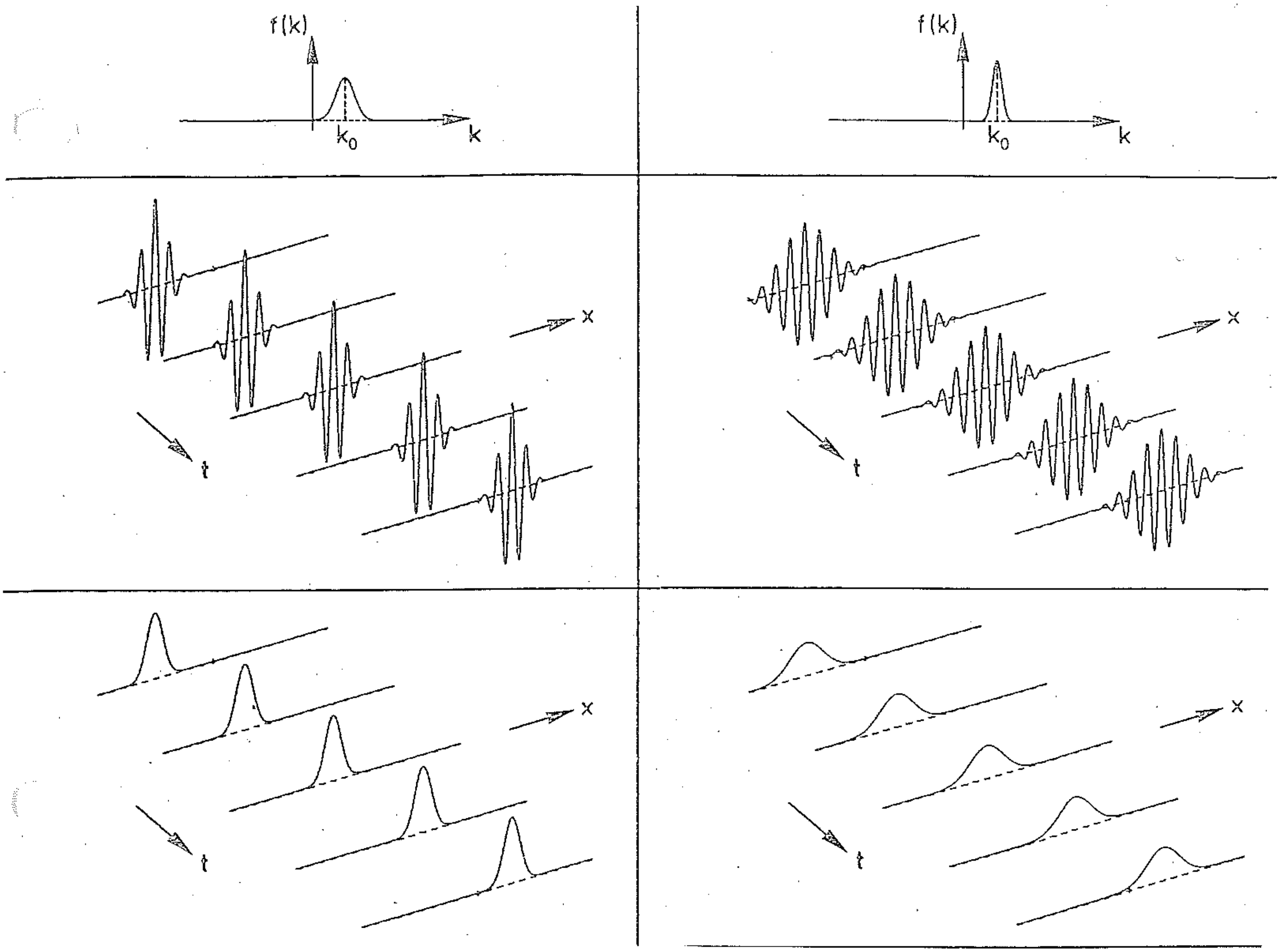

x


f(k)
k
k_0


$t = t_0$
x
k_n


t = t0
x
kn


t = t1
x
kn



f(k)
k_0
k
f(k)
k_0
k
x
t
x
t
x
t
x
t