Expectation value of an observable

= mean value of results of multiple measurements

$$\langle A \rangle = \sum \Big(\underbrace{\text{possible results}}_{\uparrow\ \text{eigenvalues of } \hat{A}\,!} \text{ of measurement of } A\Big)\Big(\text{prob. of getting that result}\Big)$$

$$\langle E \rangle = \sum_n E_n |c_n|^2 = \int_{-\infty}^{\infty} dx\, \Psi^* \hat{H} \Psi$$

Expectation value of position

$$\langle x \rangle = \int dx\; x\, |\Psi(x,t)|^2 = \int dx\; \Psi^*(x)\, x\, \Psi(x)$$

$$= \int dx\; \Psi^*(x)\, \hat{X}\, \Psi(x)$$

where the operator $\hat{X}$ acts on $\Psi(x)$ as multiplication by x

$$\hat{X}\Psi(x) = x\Psi(x)$$

[$\langle x \rangle$ gives mean position of particle. What about spread?]

Define uncertainty in position ΔX = root mean square deviation from mean

[construct this from end to beginning]

deviation from mean: $x - \langle x \rangle$

square " " " $(x - \langle x \rangle)^2$

mean square " " " $\int dx \, (x - \langle x \rangle)^2 \, |\psi(x)|^2$

$$\Delta X = \sqrt{\int dx \, (x - \langle x \rangle)^2 \, |\psi(x)|^2}$$

Consider $\Delta X^2 = \int dx \, \underbrace{(x - \langle x \rangle)^2}_{x^2 - 2\langle x \rangle x + \langle x \rangle^2} \, |\psi|^2$

$$= \underbrace{\int dx \, x^2 |\psi|^2}_{\langle x^2 \rangle} - 2\langle x \rangle \underbrace{\int dx \, x |\psi|^2}_{\langle x \rangle} + \langle x \rangle^2 \underbrace{\int dx \, |\psi|^2}_{1}$$

$$= \langle x^2 \rangle - 2\langle x \rangle^2 + \langle x \rangle^2$$

$$= \langle x^2 \rangle - \langle x \rangle^2$$

$$\boxed{\Delta X = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}}$$

$\langle x \rangle$ for an energy eigenstate

$$\psi(x,t) = u_n(x)\, e^{-\frac{iE_n t}{\hbar}}$$

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, x\, \psi(x,t)\, dx$$

$$= \int_{-\infty}^{\infty} \left(u_n^*(x)\, e^{\frac{iE_n t}{\hbar}}\right) x \left(u_n(x)\, e^{-\frac{iE_n t}{\hbar}}\right) dx$$

$$= \int_{-\infty}^{\infty} x\, |u_n(x)|^2\, dx$$

Particle in a box: $u_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi n x}{L}\right)$ for $0 < x < L$

$$\langle x \rangle = \frac{2}{L} \int_0^L x \sin^2\left(\frac{\pi n x}{L}\right) dx$$

Let $y = \frac{\pi n x}{L} \Rightarrow dy = \frac{n\pi}{L} dx$

$$\langle x \rangle = \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \int_0^{\pi n} y \sin^2 y\, dy$$

① Let $\sin^2 y = \frac{1}{2}(1 - \cos 2y)$

② For $\int y \cos(2y)\, dy$ use IBP $\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$

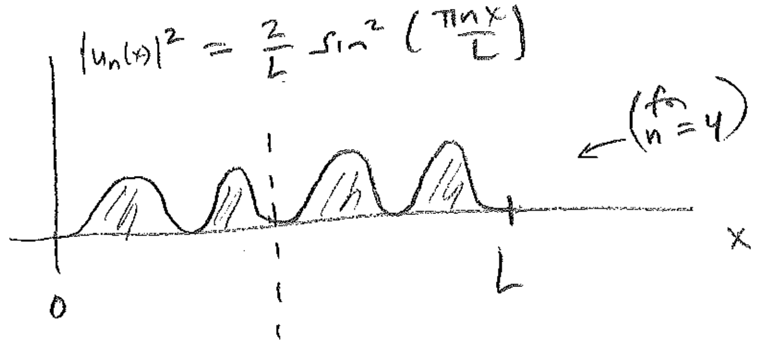
$u = y$, $dv = \cos(2y)\, dy$

$du = dy$, $v = \frac{1}{2}\sin(2y)$

③ Result: $\int_0^{\pi n} y \sin^2 y\, dy = \frac{1}{4}(\pi n)^2$

$$\Rightarrow \langle x \rangle = \frac{2}{L}\left(\frac{L}{n\pi}\right)^2 \frac{1}{4}(\pi n)^2 = \frac{L}{2}$$

[could we have anticipated this?]



[equal prob. on each side of midpt. weighted avg = $\frac{L}{2}$]

$\langle x \rangle = \frac{L}{2}$ Observe $|\psi(\frac{L}{2})|^2 = 0$.

Expectation value $\neq$ most likely position

Problem: calc Δx for $u_n(x)$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\, x^2\, \psi(x,t)\, dx$$

[Hint: integrate by parts twice!]

Review/summary

eigenfunction of $\hat{A}$ = state w/ well-defined value of observable A

eigenvalue = that well-defined value

expectation value of A = $\langle A \rangle$ = mean value of results of many measurements of A

$\langle A^2 \rangle$ = ... A^2

uncertainty $\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$ = standard deviation of measurements

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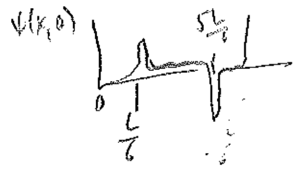
example



not an ~~eigenvalue~~ eigenfunction of $\hat{H}$

not an eigenfunction of $\hat{x}$

$$\langle x \rangle = \frac{1}{2}\left(\frac{L}{6}\right) + \frac{1}{2}\left(\frac{5L}{6}\right) = \frac{L}{2}$$

$$\langle x^2 \rangle = \frac{1}{2}\left(\frac{L}{6}\right)^2 + \frac{1}{2}\left(\frac{5L}{6}\right)^2 = \frac{26}{72} = \frac{13}{36}L^2$$

$$\Delta x = \sqrt{\frac{13}{36}L^2 + \frac{9}{36}L^2} = \sqrt{\frac{4}{36}L^2} = \frac{1}{3}L$$

17

PROBLEM ~~15~~: Consider a sequence of throws of a six-sided die.

(a) If the die is fair, what is the root-mean-squared deviation (to four significant figures) of the number thrown? [Hint: use the expression $\Delta n = \sqrt{\langle n^2 \rangle - \langle n \rangle}$ derived in class.]

(b) If a 1, 2, or 3 is twice as likely to be thrown as a 4, 5, or 6, what is the root-mean-squared deviation of the number thrown?

(c) If a 2 is twice as likely to be thrown as a 1, a 3 is three-times as likely to be thrown as a 1, etc. what is the root-mean-squared deviation of the number thrown?

Let P_n be the probability of throwing an n. The sum of probabilities must be one: $\sum_{n=1}^{6} P_n = 1$. The mean value of the number thrown is

$$\langle n \rangle = \sum_{n=1}^{6} n\, P_n.$$

In a previous exercise you computed the mean value for each of the cases above. The root-mean-squared deviation of the number thrown is

$$\Delta n = \sqrt{\langle n^2 \rangle - \langle n \rangle^2}, \qquad \text{where} \qquad \langle n^2 \rangle = \sum_{n=1}^{6} n^2\, P_n.$$

(a) If the die is fair, the probabilities are equal, so $P_n = \frac{1}{6}$. Hence

$$\langle n \rangle = \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2} = 3.5000, \qquad \langle n^2 \rangle = \frac{1}{6}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{91}{6} \approx 15.167$$

so $\Delta n = \sqrt{\frac{35}{12}} \approx 1.7078$.

(b) In this case, $P_n = \frac{2}{9}$ for $i = 1, 2, 3$ and $P_n = \frac{1}{9}$ for $i = 4, 5, 6$. Hence

$$\langle n \rangle = \frac{2}{9}(1+2+3) + \frac{1}{9}(4+5+6) = 3, \qquad \langle n^2 \rangle = \frac{2}{9}(1^2+2^2+3^2) + \frac{1}{9}(4^2+5^2+6^2) = \frac{35}{3} \approx 11.667$$

so $\Delta n = \sqrt{\frac{8}{3}} \approx 1.6330$.

(c) In this case, $P_n = \frac{n}{21}$. (Check that $\sum_{n=1}^{6} P_n = 1$.) Hence

$$\langle n \rangle = \frac{1}{21}(1^2+2^2+3^2+4^2+5^2+6^2) = \frac{13}{3} = 4.333, \qquad \langle n \rangle = \frac{1}{21}(1^3+2^3+3^3+4^3+5^3+6^3) = 21$$

so $\Delta n = \frac{\sqrt{20}}{3} = 1.4907$.

$\sum n^3 = (\sum n)^2$ ← wow!

PROBLEM ~~16~~ 18: Calculate Δx for $u_n(x)$, the nth energy eigenfunction of a particle of mass m in a one-dimensional box of width L.

As we found in class, a particle in a box of width L has probability density

$$P_n(x) = \frac{2}{L}\sin^2\left(\frac{\pi n x}{L}\right), \qquad \text{for} \qquad 0 < x < L$$

and zero outside the box, where n is the quantum number of the energy eigenfunction. The expectation value of the position is

$$\langle x\rangle = \int_{-\infty}^{\infty} dx\, x\, P(x) = \frac{2}{L}\int_0^L dx\, x\, \sin^2\left(\frac{\pi n x}{L}\right) = \frac{2}{L}\left(\frac{L}{\pi n}\right)^2 \int_0^{\pi n} dy\; y\, \sin^2 y = \frac{L}{2}$$

as we found in class. The expectation value of the square of the position is

$$\langle x^2\rangle = \int_{-\infty}^{\infty} dx\, x^2\, P(x) = \frac{2}{L}\int_0^L dx\, x^2\, \sin^2\left(\frac{\pi n x}{L}\right) = \frac{2}{L}\left(\frac{L}{\pi n}\right)^3 \int_0^{\pi n} dy\; y^2\, \sin^2 y.$$

We can do this integral by repeated integrations by parts. First we set

$$u = y^2, \qquad dv = \sin^2 y\, dy$$

which implies

$$du = 2y\, dy, \qquad v = \frac{1}{2}y - \frac{1}{4}\sin 2y$$

so that the integral becomes

$$\int_0^y dy\; y^2 \sin^2 y = \frac{y^3}{2} - \frac{y^2}{4}\sin(2y) - \int_0^y dy\left[y^2 - \frac{y}{2}\sin(2y)\right] = \frac{y^3}{6} - \frac{y^2}{4}\sin(2y) + \frac{1}{2}\int_0^y dy\, y\, \sin(2y).$$

We can do the last integral by another integration by parts, setting

$$u = y, \qquad dv = \sin(2y)\, dy$$

which implies

$$du = dy, \qquad v = -\frac{1}{2}\cos(2y)\, dy$$

so that the integral becomes

$$\int_0^y dy\, y\, \sin(2y) = -\frac{y}{2}\cos(2y) + \frac{1}{2}\int_0^y dy\, \cos(2y) = -\frac{y}{2}\cos(2y) + \frac{1}{4}\sin(2y).$$

Putting these results together we obtain

$$\int_0^y dy\; y^2 \sin^2 y = \frac{y^3}{6} - \frac{y^2}{4}\sin(2y) - \frac{y}{4}\cos(2y) + \frac{1}{8}\sin(2y).$$

Evaluating this at $y = \pi n$, and plugging into the expression for $\langle x^2\rangle$ above, we get

$$\langle x^2\rangle = \frac{2}{L}\left(\frac{L}{\pi n}\right)^3\left[\frac{\pi^3 n^3}{6} - \frac{\pi n}{4}\right] = L^2\left(\frac{1}{3} - \frac{1}{2\pi^2 n^2}\right).$$

Consequently the uncertainty in position is given by

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = L\sqrt{\frac{1}{12} - \frac{1}{2\pi^2 n^2}}.$$

Physics 2140 HW SET 5

(1) We found in class that $P_n(x) = \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)$ for $0<x<L$ and is zero outside the box.

So, $$\langle x\rangle = \frac{2}{L}\int_0^L dx\; x\sin^2\left(\frac{n\pi x}{L}\right) = \frac{2}{L}\left(\frac{L}{n\pi}\right)^2\int_0^{n\pi} dy\; y\sin^2 y$$

where we used $y = \frac{n\pi x}{L}$ and $dy = \frac{n\pi}{L}dx$.

$$\text{so } \langle x\rangle = \frac{2L}{(n\pi)^2}\left[\frac{(n\pi)^2}{4} - \frac{y\sin 2y}{4}\Big|_0^{n\pi} - \frac{\cos 2y}{8}\Big|_0^{n\pi}\right]$$ from a table of integrals

$$= \frac{2L}{(n\pi)^2}\left[\frac{n\pi^2}{4} - \frac{(0-0)}{4} - \frac{(1-1)}{8}\right] = \frac{L}{2}$$

Now,
$$\langle x^2\rangle = \frac{2}{L}\int_0^L dx\; x^2\sin^2\left(\frac{n\pi x}{L}\right) = \frac{2}{L}\left(\frac{L}{n\pi}\right)^3\int_0^{n\pi} dy\; y^2\sin^2 y$$

The first step to this is integration by parts: $\begin{cases} u = y^2 & du = 2y\,dy \\ dv = \sin^2 y\,dy & v = \frac{y}{2} - \frac{\sin 2y}{4}\end{cases}$

Then the integral becomes:
$$\int_0^{n\pi} dy\; y^2\sin^2 y = \frac{y^3}{2}\Big|_0^{n\pi} - \cancelto{0}{\frac{y^2\sin 2y}{4}\Big|_0^{n\pi}} - \int_0^{n\pi} dy\left[y^2 - \frac{y}{2}\sin 2y\right]$$

$$= \frac{(n\pi)^3}{6} + \frac{1}{2}\int_0^{n\pi} dy\; y\sin 2y$$

Here we could use integration by parts again with $\begin{cases} u = y & du = dy \\ dv = \sin(2y)\,dy & v = -\frac{1}{2}\cos(2y)\end{cases}$

but it's even faster to use the tool we learned in class: treat constants as variables (i.e. substitute a variable for a constant and use derivaties)

$$\left[\int_0^{n\pi} dy\; y\sin ay\right]_{\text{evaluated at } a=2} = \left[\int_0^{n\pi} dy\left(\frac{-\partial}{\partial a}\right)\cos ay\right]_{a=2}$$

$$= \left[-\frac{\partial}{\partial a}\int_0^{n\pi} dy\;\cos ay\right]_{a=2} = \left[-\frac{\partial}{\partial a}\left(\frac{\sin ay}{a}\right)\Big|_0^{n\pi}\right]_{a=2} = \left[-\frac{\partial}{\partial a}\left(\frac{\sin an\pi}{a}\right)\right]_{a=2}$$

$$= \frac{\sin 2n\pi}{a^2}\Big|_{a=2} - \frac{n\pi\cos 2n\pi}{a}\Big|_{a=2} = \cancelto{0}{\frac{\sin 2n\pi}{4}} - \frac{n\pi\cos 2n\pi}{2} = -\frac{n\pi}{2}$$

$$\text{So } \langle x^2\rangle = \left[\frac{(n\pi)^3}{6} - \frac{n\pi}{4}\right]\frac{2L^2}{(n\pi)^3} = L^2\left(\frac{1}{3} - \frac{1}{2\pi^2n^2}\right)$$

$$\text{Thus } \Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = L\sqrt{\frac{1}{12} - \frac{1}{2\pi^2n^2}}$$

19

PROBLEM ~~17~~: Let the normalized wavefunction describing a particle of mass m in a box of width L be given by $\psi(x,0) = Ax(L-x)$ for $0 < x < L$ and $\psi(x,0) = 0$ outside the box. (Obviously, this is not an energy eigenfunction.)
(a) Compute the value of A. Assume it to be real and positive.
(b) Carefully sketch $\psi(x,0)$, noting the position of the maximum.
(c) Compute $\langle x \rangle$. Does your answer make sense?
(d) Compute Δx.

The probability density of this wavefunction is

$$P(x,0) = |\psi(x,0)|^2 = |A|^2 x^2 (L-x)^2 2 \qquad \text{for} \qquad 0 < x < L$$

and zero outside the box.

(a) The density must be normalized to unity. Hence

$$1 = |A|^2 \int_0^L dx\, x^2(L-x)^2 = |A|^2 L^5 \int_0^1 dy\, y^2(1-y)^2 = |A|^2 L^5 \int_0^1 dy\, (y^2 - 2y^3 + y^4) = \frac{1}{30}|A|^2 L^5.$$

Since we assume A to be real, it is given by $A = \sqrt{30/L^5}$.

(b) The sketch of $\psi(x,0)$ shows it to vanish at $x = 0$ and $x = L$, with a maximum at $x = L/2$.

(c) The expectation value of position is

$$\langle x \rangle = \int_{-\infty}^{\infty} dx\; x\, P(x) = \frac{30}{L^5} \int_0^L dx\, x^3 (L-x)^2 = 30L \int_0^1 dy\, y^3(1-y)^2 = \frac{L}{2}$$

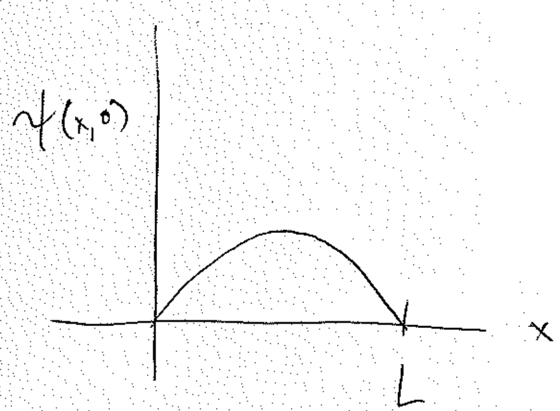
which makes perfect sense given the result from (b).

(d) The expectation value of the square of the position is

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx\; x^2\, P(x) = \frac{30}{L^5} \int_0^L dx\, x^4 (L-x)^2 = 30L^2 \int_0^1 dy\, y^4(1-y)^2 = \frac{2L^2}{7}.$$

From these two results, we may compute the uncertainty in position:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{L}{2\sqrt{7}} = 0.1890L.$$



~~after #12~~

~~18~~ 20

PROBLEM ~~19~~: Let $u_1(x)$ and $u_2(x)$ be the lowest two energy eigenfunctions for a particle of mass m in a box of length L. Let the state of the particle be described by the following *normalized* wavefunction: $\psi(x,t) = A\left[u_1(x)e^{-iE_1t/\hbar} + u_2(x)e^{-iE_2t/\hbar}\right]$.

(a) What is the value of the constant A? (Assume it to be a positive real number.)

(b) Calculate $\langle x\rangle$ for this state. Since this state is not a stationary state, the expectation value can (and does) depend on time. In this case, it executes simple harmonic motion. What is the angular frequency of oscillation? What is the amplitude? Your answers should be written in terms of m and L.

(a) The probability density for this wavefunction is

$$P(x,t) = \psi^*(x,t)\psi(x,t) = |A|^2\left[|u_1(x)|^2 + |u_2(x)|^2 + u_1^*(x)u_2(x)e^{i(E_1-E_2)t/\hbar} + u_2^*(x)u_1(x)e^{-i(E_1-E_2)t/\hbar}\right]$$

Since the eigenfunctions $u_n(x) = \sqrt{2/L}\,\sin(\pi n x/L)$ are real, one can rewrite the probability density as

$$P(x,t) = |A|^2\left[u_1^2(x) + u_2^2(x) + 2u_1(x)u_2(x)\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)\right]$$

Since the eigenfuctions $u_1(x)$ and $u_2(x)$ are orthornormal

$$\int_0^L dx\; u_1(x)^2 = \int_0^L dx\; u_2(x)^2 = 1, \qquad \int_0^L dx\; u_1(x)u_2(x) = 0$$

2008 grading

$-\frac{1}{2}$ if didn't work cross term (+ show it = 0)

the integral of the probability density is given by

$$\int_0^L dx\; P(x,t) = |A|^2\,[1+1+0] = 2|A|^2 = 1 \qquad \Longrightarrow \qquad A = \frac{1}{\sqrt{2}}$$

since we assume A to be real and positive.

(b) The expectation value of position for a particle with this wavefunction is

$$\langle x\rangle = \int dx\; xP(x,t) = \frac{1}{2}\left[\int_0^L dx\; xu_1^2(x) + \int_0^L dx\; xu_2^2(x) + 2\cos\left(\frac{(E_1-E_2)t}{\hbar}\right)\int_0^L dx\; xu_1(x)u_2(x)\right]$$

The expectation value for the eigenfunctions for a particle in a box were computed in class to be $L/2$. The last integral is

$$\int_0^L dx\; xu_1(x)u_2(x) = \frac{2}{L}\int_0^L dx\; x\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right) = \frac{2}{L}\left(\frac{L}{\pi}\right)^2\int_0^\pi dy\; y\sin y\sin(2y)$$

The integral can be done in a variety of ways, giving $-8/9$, hence we have

$-\frac{1}{2}$ if did not write ω in terms of m & L

$$\langle x\rangle = \frac{L}{2} - \frac{16L}{9\pi^2}\cos\left(\frac{3\hbar\pi^2 t}{2mL^2}\right)$$

where we have used the eigenenergies $E_n = \hbar^2\pi^2n^2/2mL^2$ for a particle in a box. Hence the expectation value oscillates about the center of the box with angular frequency $\omega = 3\hbar\pi^2/2mL^2$ and amplitude $16L/9\pi^2 \approx 0.18L$.

-3 if set up but did not do the integrals or did not calc ω

or maybe not → (-1 if write $\int_0^\pi y \sin y \sin 2y = -\frac{8}{9}$ w/ no explanation)

Here's one way to do the integral

$$I = \int_0^\pi 2y \sin^2 y \cos y \, dy$$

$u = 2y \qquad dv = \sin^2 y \cos y \, dy$

$du = 2\,dy \qquad v = \frac{1}{3}\sin^3 y$

$$I = \underbrace{\frac{2}{3} y \sin^3 y \Big|_0^\pi}_{0} - \frac{2}{3}\int_0^\pi \sin^3 y \, dy$$

$$= -\frac{2}{3}\int_0^\pi (1-\cos^2 y)\sin y \, dy \qquad u = \cos y$$

$$= -\frac{2}{3}\int_{-1}^{1} (1-u^2)\, du$$

$$= -\frac{2}{3}u + \frac{2}{9}u^3 \Big|_{-1}^{1} = -\frac{8}{9}$$

Another way: $\sin y \sin 2y = \frac{1}{2}(\cos y - \cos 3y)$

so $\int y \, \frac{1}{2}(\cos y - \cos 3y) = -\frac{8}{9}$

Griffiths (2e) 2.9

\#21

PROBLEM ~~20~~: Let the wavefunction describing a particle of mass m in a box of width L be given at time $t = 0$ by $\psi(x, 0) = Ax(L - x)$ for $0 < x < L$ and $\psi(x, 0) = 0$ outside the box. (This is the same wavefunction considered in an earlier problem.)

(a) Compute $\langle E \rangle$, the expectation value of energy for this state. Note that it is not one of the energy eigenvalues of the particle in the box (though it is close).

(b) If one were to measure the energy of a particle described by this wavefunction, one would obtain one of the energy eigenvalues E_n with a probability given by $|c_n|^2$, where $\psi(x, 0)$ is written as the linear combination of eigenfunctions $\psi(x, 0) = \sum_n c_n u_n(x)$. Assume that $\psi(x, 0)$ can be approximately written as $c_1 u_1(x) + c_3 u_3(x)$, with other terms in the infinite series negligible. (Given the form of $\psi(x, 0)$, this is a very good approximation. Why is there no contribution from $u_2(x)$?) Use your result from (a) to estimate the values of $|c_1|^2$ and $|c_3|^2$, the probabilities that the particle will be found to have energies E_1 and E_3 respectively. (Three significant figures please.)

In an earlier problem, you computed the normalization constant A for this wavefunction to be $A = \sqrt{30/L^5}$.

(a) The expectation value of energy for this state is given by

$$\langle E \rangle = \int_{-\infty}^{\infty} dx\, \psi^*(x, 0)\, \hat{H}\, \psi(x, 0) = \frac{30}{L^5} \int_0^L dx\, x(L - x) \left(\frac{-\hbar^2}{2m} \frac{d^2}{dx^2} \right) x(L - x) = \frac{30\hbar^2}{mL^5} \int_0^L dx\, x(L - x) = \frac{5\hbar^2}{mL^2}$$

Don't compute $\langle H^2 \rangle$ because of δ-fcn at the boundary.

This is not equal to any of the eigenenergies $E_n = \hbar^2 \pi^2 n^2 / 2L^2$, though it is very close to the ground state energy $E_1 \approx 4.9348\hbar^2/L^2$.

(b) We can think of $\psi(x, 0) = Ax(L - x)$ as a linear combination of energy eigenfunctions, consisting primarily of $u_1(x)$ but with an admixture of even excited states $u_{2m+1}(x)$ as well. (There are no contributions from $u_{2m}(x)$ because they are odd functions, whereas $Ax(L - x)$ is even.) Let us approximate $\psi(x, 0)$ as a linear combination of the two lowest even eigenfunctions $c_1 u_1(x) + c_3 u_3(x)$. Normalization of the wavefunction implies

$$|c_1|^2 + |c_3|^2 = 1$$

The expectation value of the energy is

$$|c_1|^2 \frac{\hbar^2 \pi^2}{2L} + |c_3|^2 \frac{9\hbar^2 \pi^2}{2L} = \frac{5\hbar^2}{mL^2}$$

pre

One may solve these two equations to find

$$|c_1|^2 = \frac{9}{8} - \frac{5}{4\pi^2} \approx 0.99835, \qquad |c_3|^2 = \frac{5}{4\pi^2} - \frac{1}{8} \approx 0.00165$$

exact: $c_n = \left(\frac{8\sqrt{15}}{\pi^3} \right) \frac{1}{n^3}$

$c_1 = 0.9993$ $\quad |c_1|^2 = 0.99855$

$c_3 = 0.03701$ $\quad |c_3|^2 \approx 0.00137$

$$1 = \int_0^L |\Psi(x,0)|^2 dx = A^2 \int_0^L x^2(L-x)^2 dx = A^2 L^5 \left(\int_0^1 y^2(1-y)^2 dy = \frac{1}{5} - \frac{2}{4} + \frac{1}{3} = \frac{1}{30} \right)$$

$$A = \sqrt{\frac{30}{L^5}}$$

$$\langle E \rangle = \int_0^L \psi^*(x,0)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\psi = A^2 \int_0^L dx\, x(L-x)\left(+\frac{\hbar^2}{m}\right) = \frac{\hbar^2}{m}\frac{30 L^3}{L^5} \underbrace{\int_0^1 y(1-y)\,dy}_{\frac{1}{2}-\frac{1}{3}}$$

$$= \frac{5\hbar^2}{mL^2}$$

$$E_1 = \frac{\pi^2}{2}\frac{\hbar^2}{mL^2} = 4.935\frac{\hbar^2}{mL^2}$$

$$E_3 = \frac{9\pi^2}{2}\frac{\hbar^2}{mL^2}$$

Assume $\psi = c_1 u_1 + c_3 u_3$

$$\Rightarrow \langle E \rangle = \underbrace{|c_1|^2}_{1-|c_3|^2} \frac{\pi^2}{2} + |c_3|^2 \frac{9\pi^2}{2} = 5$$

$$\frac{\pi^2}{2} + 4\pi^2 |c_3|^2 = 5$$

$$|c_3|^2 = \frac{5 - \frac{\pi^2}{2}}{4\pi^2} = \frac{5}{4\pi^2} - \frac{1}{8} = 0.00165$$

$$|c_1|^2 = 0.9983$$

(exact answer $c_n = \frac{8\sqrt{15}}{\pi^3}\frac{1}{n^3}$ so $|c_3|^2 = \frac{960}{\pi^6 (3)^6} = 0.00137$.)