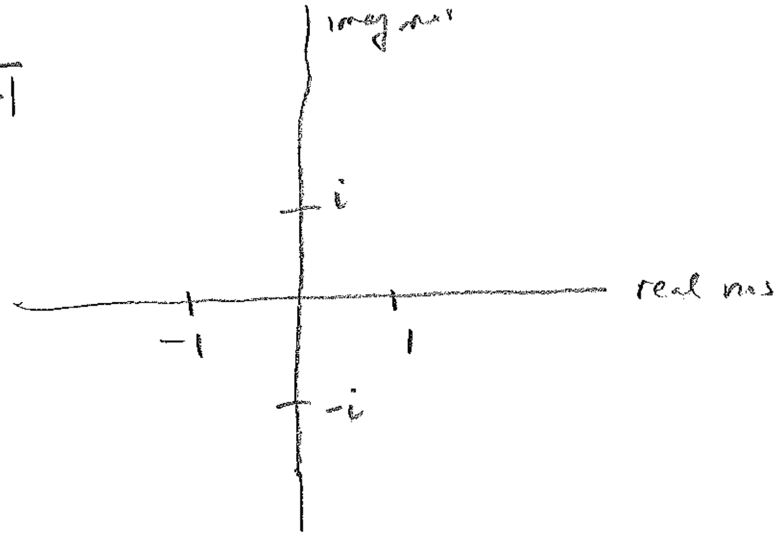
Complex numbers

$i = \sqrt{-1}$

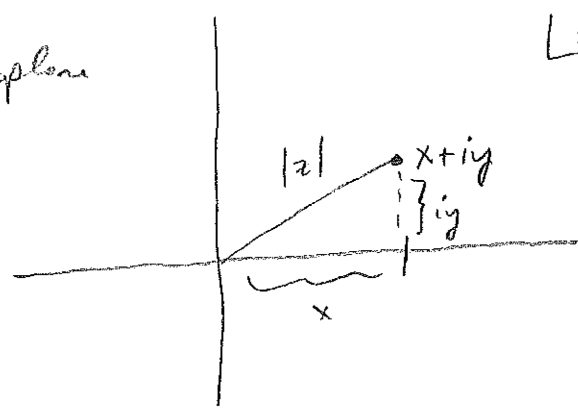


[complex plane] Let x, y be real numbers
Then a complex number z
has the form $x + iy$

$x = \mathrm{Re}\, z$
$y = \mathrm{Im}\, z$

Complex plane

$\lfloor z$



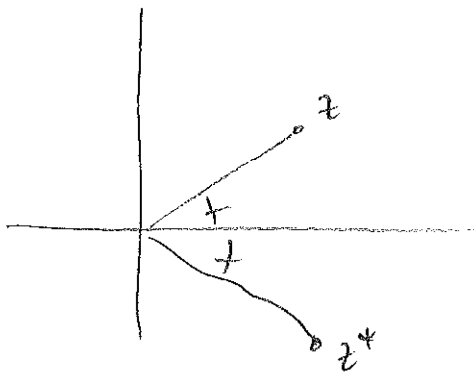
$|z| \equiv$ (magnitude) of complex number $= \sqrt{x^2 + y^2}$ [distance from origin]
modulus

(same as magnitude of vector)

If $z = x + iy$

then complex conjugate $z^* = x - iy$

[reflect across real axis]



If $z = z^*$, z is real

If $z = -z^*$, z is imag.

$z + z^* = 2x$ so $x = \mathrm{Re}\, z = \frac{1}{2}(z + z^*)$

$z - z^* = 2iy$ so $y = \mathrm{Im}\, z = \frac{1}{2i}(z - z^*)$

$zz^* = (x+iy)(x-iy) = x^2 - i^2 y^2 = x^2 + y^2 = |z|^2$

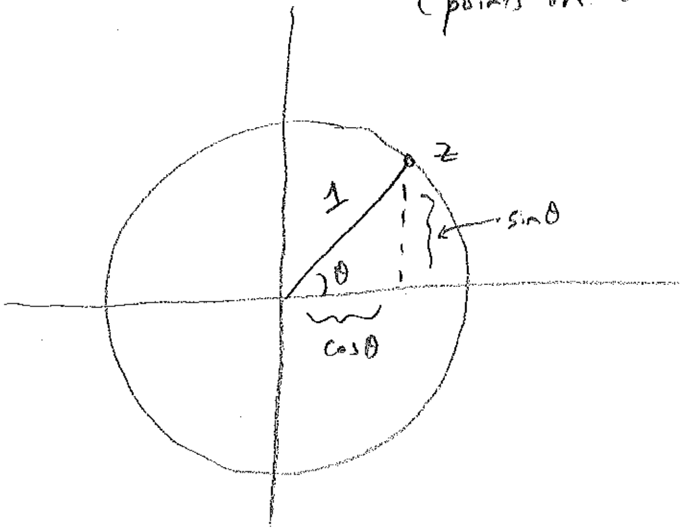
$\Rightarrow zz^*$ is always <u>real</u> & <u>non-negative</u>

Note $|z|^2 \neq z^2$

$z^2 = (x+iy)(x+iy) = (x^2 - y^2) + 2ixy$

"complex phase" = complex number w/ modulus one, $|z| = 1$

(points on unit circle)



$$z = \cos\theta + i\sin\theta$$

let $z_1 = \cos\theta_1 + i\sin\theta_1$

$z_2 = \cos\theta_2 + i\sin\theta_2$

Then $z_1 z_2 = (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)$

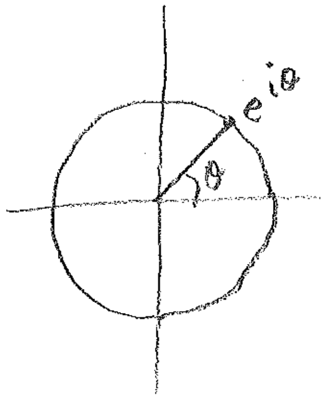
$= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

What function also behaves like this? exponential

This suggests

Euler formula: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

This implies $e^{i\theta_1} e^{i\theta_2} = e^{i\theta_1 + i\theta_2} = e^{i(\theta_1 + \theta_2)}$



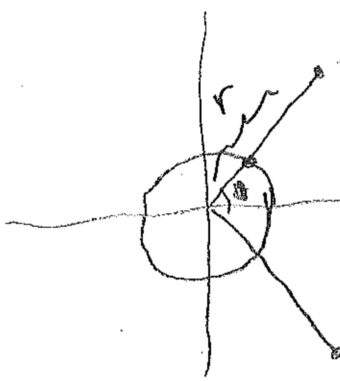
$\theta = 0$: $\quad e^{i\theta} = 1 \quad \checkmark$

$\theta = \frac{\pi}{2}$: $\quad e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i \quad \checkmark$

$\theta = \pi$: $\quad e^{i\pi} = \cos\pi + i\sin\pi = -1$

$$e^{\frac{3\pi i}{2}} = -i$$

$$e^{2\pi i} = 1$$



$z = re^{i\theta}$ (polar form of complex #)

$z = x+iy$ (Cartesian form)

$z^* = re^{i(-\theta)} = re^{-i\theta}$

Then $|z|^2 = z^*z = (re^{i\theta})^*(re^{i\theta})$

$$= re^{-i\theta}\, re^{i\theta}$$

$$= r^2 e^0$$

$$= r^2$$

Given a complex number z, its inverse $1/z$ is also a complex number, which can be written in the form $a + ib$ (where a and b are real numbers). If $z = 1 + 2i$, what are a and b? [Hint: first multiply $1/z$ by z^*/z^*.]

$$\frac{1}{(1+2i)} \frac{(1-2i)}{(1-2i)} = \frac{1-2i}{1+4} = \frac{1}{5} + \left(-\frac{2}{5}\right)i$$

Exercise: Prove the Euler formula using Taylor series for exponential & trigonometric functions

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Exercise: Use Euler formula to show that

$e^{i\theta}$ behaves "as expected"

under differentiation w.r.t. θ

[one can instead use this to motivate the i in exponent]

ie let $e^{a\theta} = \cos\theta = \cos\theta + i\sin\theta$

then $\frac{d}{d\theta} \Rightarrow \quad a e^{a\theta} = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = ie^{i\theta}$

$\Rightarrow a = i$