Recall $p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z \right)$

p^μ is a 4-vector $\Rightarrow$ $p'^\mu = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu p^\nu$

$$\begin{pmatrix} \frac{E'}{c} \\ p_x' \\ p_y' \\ p_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

$E' = \gamma(E - \beta c p_x)$

$p_x' = \gamma(p_x - \beta \frac{E}{c})$

$p_y' = p_y$

$p_z' = p_z$

Consider a particle <u>at rest</u> in S'

$E' = mc^2, \quad p_x' = p_y' = p_z' = 0$

Let S be moving at speed v in $+x$ direction w.r.t. S'

In S, particle is moving at speed v in $+x$ direct

$E = \gamma(E' + \beta c p_x') = \gamma(mc^2 + 0) = \gamma mc^2$ ✓

$p_x = \gamma(p_x' + \beta \frac{E'}{c}) = \gamma(0 + \beta \frac{mc^2}{c}) = \gamma m v$ ✓

$p_y = p_y' = 0$

$p_z = p_z' = 0$

Define the total 4-momentum of a system of particles

$$p^\mu_{sys} = \sum_i p_i^\mu \qquad i \text{ labels the different particles}$$

$$p^\mu_{sys} = (p^0_{sys}, \vec{p}_{sys}) \xrightarrow{} \text{total 3-momentum of the system}$$

$E_{sys} = c\,p^0_{sys}$ = total energy of the system

In a isolated system, both energy + 3-momentum are conserved

$$\therefore \quad E^{sys}_{init} = E^{sys}_{final}$$

$$\vec{p}^{\,sys}_{init} = \vec{p}^{\,sys}_{final}$$

Therefore 4-momentum of the system is conserved

$$p^\mu_{sys,init} = p^\mu_{sys,final}$$

This is valid in any IRF since p^μ is a 4-vector

$$\sum_{i=\text{init}} E_i = \sum_{j=\text{final}} E_j$$

$$\sum_{i=\text{init}} (m_i c^2 + K_i) = \sum_{j=\text{final}} (m_j c^2 + K_j)$$

Neither mass (rest energy) nor kinetic energy are <u>separately</u> conserved, only their total

Rest energy can be converted to kinetic energy
(e.g. radioactivity $U \rightarrow Th + \alpha$
fission $U \rightarrow Ba + Kr$)

Kinetic energy can be converted into rest energy

(cosmic rays: π + μ created
particle accelerators: Higgs)

Particle decay, or nuclear fission

$X_0 \rightarrow X_1 + X_2$

example

$^{236}_{92}U \longrightarrow {}^{147}_{56}Ba + {}^{89}_{36}Kr$

$m_0 = 236.046u$ $m_1 = 146.934u$ $m_2 = 88.918u$

Observe that $m_0 > m_1 + m_2$ so mass energy → kinetic

[$1u = \frac{1}{12}(^{12}C)$ atom]

Define $Q \equiv (\underbrace{m_0 - m_1 - m_2}_{0.194})c^2 = 181$ meV

where $(1u)c^2 = 931.50$

Energy conservation

$E_0 = E_1 + E_2$

If X_0 at rest $\Rightarrow$ $m_0c^2 = (m_1c^2 + K_1) + (m_2c^2 + K_2)$

$\underbrace{(m_0 - m_1 - m_2)c^2}_{Q} = K_1 + K_2 = 181$ meV

[chemical reactions release 1-10 eV]

$\Rightarrow Q =$ total kinetic energy of the fragments

How much energy does each fragment get?

To answer this, need use 3-momentum conservation ts

Momentum conservation

$$\vec{p}_0 = \vec{p}_1 + \vec{p}_2$$

X_0 at rest $\Rightarrow \vec{p}_0 = 0$

$\Rightarrow \vec{p}_1 = -\vec{p}_2$ (equal + opposite)

$p_1 = p_2$ (magnitudes equal)

$$E_0 = E_1 + E_2$$

$$m_0 c^2 = \sqrt{(cp_1)^2 + (m_1 c^2)^2} + \sqrt{\underbrace{(cp_2)^2}_{(cp_1)^2} + (m_2 c^2)^2}$$

Hint: don't try to solve for p_1.
Instead use $(cp_1)^2 = E_1^2 - (m_1 c^2)^2$

$$m_0 c^2 = E_1 + \sqrt{E_1^2 - (m_1 c^2)^2 + (m_2 c^2)^2}$$

$$(m_0c^2 - E_1)^2 = E_1^2 - (m_1c^2)^2 - (m_2c^2)^2$$

$$(m_0c^2)^2 - 2E_1(m_0c^2) = -(m_1c^2)^2 - (m_2c^2)^2$$

$$E_1 = \frac{(m_0^2 + m_1^2 - m_2^2)\,c^2}{2m_0}$$

By symmetry $\quad E_2 = \frac{(m_0^2 + m_2^2 - m_1^2)}{2m_0}\,c^2$

Kinetic energy $\quad K_1 = E_1 - m_1c^2$

$$= \frac{(m_0^2 - 2m_0m_1 + m_1^2) - m_2^2}{2m_0}\,c^2$$

$$= \frac{(m_0 - m_1)^2 - m_2^2}{2m_0}\,c^2$$

$$= \underbrace{\left(\frac{m_0 - m_1 + m_2}{2m_0}\right)}_{f_1 \text{ fraction that particle 1 gets}}\underbrace{(m_0 - m_1 - m_2)c^2}_{Q\text{, total energy released } 181\,\text{MeV}}$$

For example, Barium: $f_1 = \frac{236 - 147 + 89}{2(236)} = 0.377 \Rightarrow K_1 = 68\,\text{MeV}$

Krypton $\Rightarrow K_2 = 113\,\text{MeV}$

If $m_1 \approx m_2$ the fraction $= \frac{1}{2}$

Consider a system of particles w/ 4-momentum $p^\mu_{sys} = \left(\frac{E_{sys}}{c}, \vec{p}_{sys} \right)$

Define the <u>invariant mass of the system</u> as

$$M_{sys} c^2 = \sqrt{E_{sys}^2 - (c\vec{p}_{sys})^2}$$

which is both <u>conserved</u> (doesn't change over time)
and <u>invariant</u> (same in every frame)

It is invariant because we can express it as a Lorentz scalar

$$\sum p^\mu_{sys} \eta_{\mu\nu} p^\nu_{sys} = -\left(\frac{E_{sys}}{c}\right)^2 + (\vec{p}_{sys})^2 = -M_{sys}^2 c^2$$

Let's evaluate it in the "center-of-mass" or "zero momentum" frame.
In this frame $E_{sys} = E_{cm}$ and $\vec{p}_{tot} = 0$ so

$$M_{sys} c^2 = E_{cm} = \sum_i E_i^{cm} = \sum (m_i c^2 + K_i^{cm})$$

$$M_{sys} = \sum \left(m_i + \frac{K_i^{cm}}{c^2} \right)$$

The invariant mass is the sum of masses of the constituents
<u>plus</u> the sum of kinetic energies in the cm frame

Antiparticles

have same mass + opposite charge of the corresponding particle

electron (e^-) $\rightarrow$ positron (e^+)
proton (p) $\rightarrow$ antiproton ($\bar{p}$)

e^+ predicted by Dirac ~1930
discovered in cosmic rays by Anderson ~1932

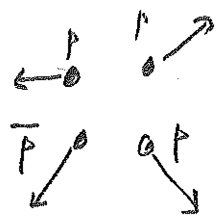
$\bar{p}$ produced in Berkeley Bevatron ~1955
by conversion of kinetic energy $\rightarrow$ rest energy (mass)

Colliding beam experiment

Two oncoming protons in center-of-mass or zero-momentum frame

Each proton has the same magnitude of momentum and ∴ the same energy E_p

After the collision



[need new proton to balance charge, baryon #]

$$E_{final} = 4 m_p c^2 + K_{final}^{cm}$$

Energy conservation

$$E_{init} = E_{final}$$

$$2 E_p = 4 m_p c^2 + K_{final}^{cm}$$

$$E_p = 2 m_p c^2 + \tfrac{1}{2} K_f^{cm}$$

What is the minimum kinetic energy of each proton necessary for this rxn?

all final state particles at rest

$$K_p = E_p - m_p c^2 = m_p c^2 + \tfrac{1}{2} K_f^{cm} \geq m_p c^2$$

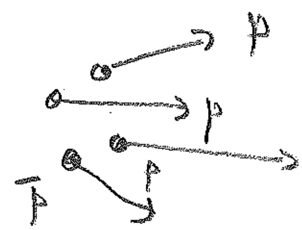
$$\Rightarrow K_p^{min} = m_p c^2 = 938 \text{ MeV}$$

[problem: $\gamma = 2 \Rightarrow \frac{v}{c} = \frac{\sqrt{3}}{2} = 0.866.$]

Fixed target experiment [actually done at Berkeley]

Let incoming proton have momentum $\vec{P}_p$ and energy E_p

After the collision



[particles moving to right]

The final state invariant mass is $Mc^2 = \sum (m_i c^2 + K_i^{cm})$

(evaluate in cm frame)

$$= 4m_p c^2 + K_i^{cm} \geq 4m_p c^2$$

The initial state invariant mass is

$$(Mc^2)^2 = E_{sys}^2 - (c\vec{P}_{sys})^2$$

$$= (m_p c^2 + E_p)^2 - (c\vec{p}_p)^2$$

$$= (m_p c^2)^2 + 2(m_p c^2) E_p + \underbrace{E_p^2 - c^2 p_p^2}_{(m_p c^2)^2}$$

if all final state particles moving together

Solve for $E_p = \dfrac{(Mc^2)^2 - 2(m_p c^2)^2}{2(m_p c^2)} \geq \dfrac{(4m_p c^2)^2 - 2(m_p c^2)^2}{2(m_p c^2)} = 7 m_p c^2$

What is the minimum kinetic energy of the incoming proton?

$$K_p = E_p - m_p c^2 \geq 6 m_p c^2$$

$\Rightarrow K_p^{min} = 6 m_p c^2 = 5630 \text{ MeV}$ [6x as much as colliding beam]

[prob: calc speeds of initial + final particles]

Alternative approach

$$\sqrt{m_p^2 + p^2} + m_p = 4 \cdot \sqrt{m_p^2 + \left(\frac{p}{4}\right)^2}$$

$$p = 4\sqrt{3}\, m$$

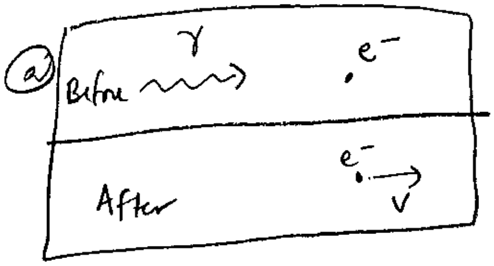
$$E = \sqrt{m^2 + 48m^2} = 7m \checkmark$$

Relativity

A.P. French, problem 6-14

There are probably many valid ways to prove these reactions are impossible. Here are some of them:

(a)



En cons: $E_\gamma + m = E_e$

Mom cons: $p_\gamma = p_e$

$$p_\gamma^2 = p_e^2$$

use $E_\gamma = p_\gamma$, $E_e^2 = p_e^2 + m^2$

$$E_\gamma^2 = E_e^2 - m^2$$

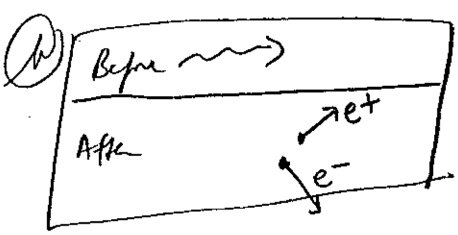
$$E_\gamma^2 = (E_\gamma + m)^2 - m^2$$

$$= E_\gamma^2 + 2mE_\gamma$$

$$0 = 2mE_\gamma$$

$\Rightarrow E_\gamma = 0$ i.e. only possible if photon has no energy!

(b)



En cons $E_\gamma = E_{e+} + E_{e-}$

p_x cons $p_\gamma = p_{e+x} + p_{e-x}$

Since $E_\gamma = p_\gamma$ we must have

$$E_{e+} + E_{e-} = p_{e+x} + p_{e-x}$$

$$\sqrt{p_{e+}^2 + m^2} + \sqrt{p_{e-}^2 + m^2} = p_{e+x} + p_{e-x}$$

But the left hand side is $> |p_{e+}| + |p_{e-}|$

and the r.h. side is $\leq |p_{e+}| + |p_{e-}|$ since $p_x \leq |p|$

So this is <u>not</u> possible.

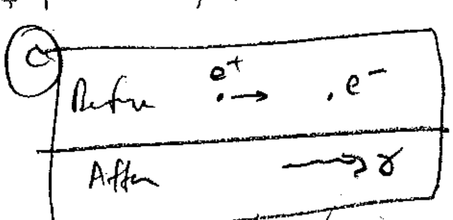
$$\sqrt{p_+^2 + m^2} + \sqrt{p_-^2 + m^2} = |\vec{p}_+ + \vec{p}_-|$$

$$p_+^2 + p_-^2 + 2m^2 + 2\sqrt{(p_+^2 + m^2)(p_-^2 + m^2)} = p_+^2 + p_-^2 + 2\vec{p}_+ \cdot \vec{p}_-$$

$$p_+^2 p_-^2 + (p_+^2 + p_-^2)m^2 + m^4 = (\vec{p}_+ \cdot \vec{p}_- - m^2)^2$$

$$p_+^2 p_-^2 + (p_+^2 + p_-^2)m^2 = (\vec{p}_+ \cdot \vec{p}_-)^2 - 2m(\vec{p}_+ \cdot \vec{p}_-)$$

(c)



En cons. $E_{e+} + m = E_\gamma$

mom cons: $p_{e+} = p_\gamma$

$$p_{e+}^2 = p_\gamma^2$$

$$E_{e+}^2 - m^2 = E_\gamma^2 = (E_{e+} + m)^2$$

$$-m^2 = 2E_{e+}m + m^2$$

$\Rightarrow E_e = -m \Rightarrow$ Not possible!

A good time to talk abt e^+e^- annihilation into 2γ.

$$m + \sqrt{m^2 + p^2} = p$$

$$m^2 + p^2 = p^2 - 2mp + m^2$$

$2mp = 0$ but $p = 0$ is not a solution!