[4-vecs]

$$x^\mu = (ct, \vec{x})$$

$$k^\mu = (\tfrac{\omega}{c}, \vec{k})$$

$$\partial^\mu = (-\tfrac{1}{c}\tfrac{\partial}{\partial t}, \vec{\nabla})$$

$$\vec{v} \overset{?}{\longrightarrow} v^\mu$$

$$\frac{d\vec{x}}{dt} \overset{?}{\longrightarrow} \frac{dx^\mu}{dt}$$

dx^μ is a 4-vector
but dt is not a scalar

Try instead

$\frac{dx^\mu}{d\tau}$ where $d\tau$ is time as measured in the frame of the moving object (proper time)
invariant

Define 4-velocity (proper velocity) $\boxed{u^\mu = \frac{dx^\mu}{d\tau}}$

$$\Rightarrow \quad u^0 = \frac{dx^0}{d\tau} = \frac{d(ct)}{d\tau} = \frac{dt}{d\tau}c = \gamma c$$

$$\vec{u} = \frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{x}}{dt} = \gamma\vec{v}$$

For small speeds, $\gamma \approx 1$ and $\vec{u} \approx \vec{v}$
but at high speeds $\vec{u}$ is not the velocity

(As $v \to c$, $u \to \infty$)

Recall $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \dots$

Momentum: $\vec{p} = m\vec{v}$

Conservation of momentum is a law of physics & ∴ must be invariant under boosts
$\Rightarrow$ momentum must transform in a well-defined way, e.g. as a 4-vector p^μ

But v^μ is not a 4-vector, so def of momentum must be modified

Define $\boxed{p^\mu = m u^\mu}$

$$\Rightarrow \begin{cases} p^0 = m u^0 = \gamma m c \\ \vec{p} = m\vec{u} = \gamma m \vec{v} = \underbrace{m\vec{v}}_{\text{nonrel. approx}} + \underbrace{\frac{1}{2}\frac{m}{c^2}v^2\vec{v} + \frac{3}{8}\frac{m}{c^4}v^4\vec{v}}_{\text{relativistic corrections}} + \ldots \end{cases}$$

High speed Experiments show that $\sum\vec{p} = \sum\gamma m\vec{v}$, not $\sum m\vec{v}$, is conserved in collisions, etc.

If $\sum p^\mu$ is conserved, then both $\sum\vec{p}$ and $\sum p^0$ are conserved.

What is this quantity p^0 that is conserved?

$$p^0 = \gamma m c = mc + \frac{1}{2}\frac{mv^2}{c} + \frac{3}{8}\frac{mv^4}{c^3} + \ldots$$

Multiply by c

$$cp^0 = \gamma m c^2 = \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\tfrac{1}{2}mv^2}_{\text{nonrel Kinetic energy}} + \underbrace{\frac{3}{8}m\frac{v^4}{c^2} + \ldots}_{\text{relativistic correction to Kin}}$$

cp^0 must be the total energy (which is conserved)

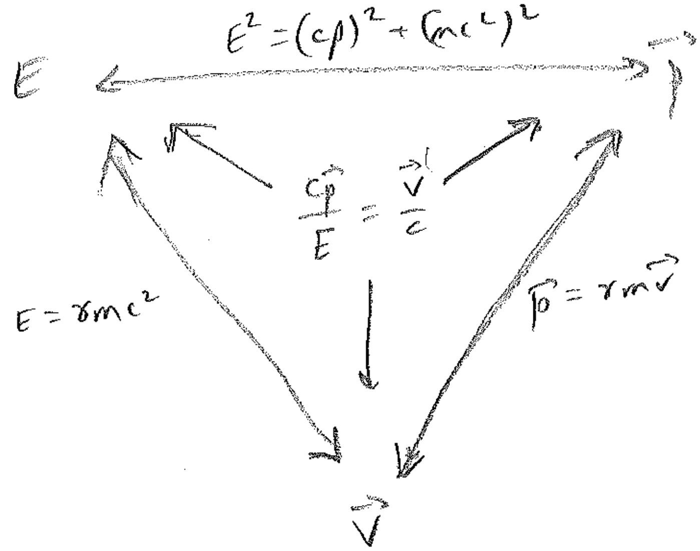
$$\boxed{\begin{array}{l} \text{relativistic } E = \gamma m c^2 = mc^2 + K \\ \text{relativistic } K = \cancel{\gamma m c^2} \Rightarrow (\gamma - 1)\, mc^2 \\ \text{relativistic } \vec{p} = \gamma m \vec{v} \end{array}}$$

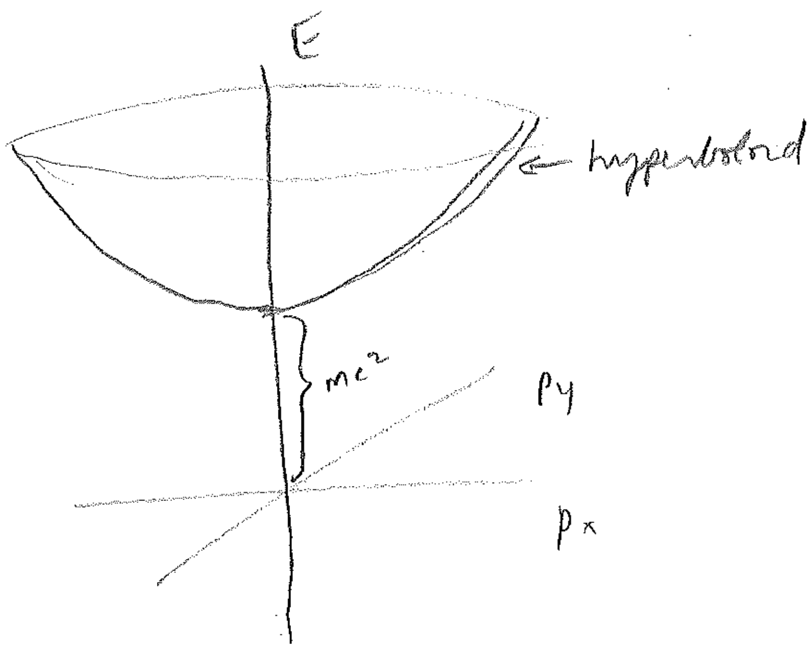
observe ratio $\frac{\vec{p}}{E} = \frac{\vec{v}}{c^2}$ so $\boxed{\vec{v} = \frac{c^2\vec{p}}{E}}$

[Exercise: use $E = \gamma m c^2$ and $\vec{p} = \gamma m \vec{v}$ to show]

$$E^2 = (c\vec{p})^2 + (mc^2)^2$$

If $\vec{p} = 0$, then $E = mc^2$





In nature there are also massless particles (ie rest mass is zero) energy not zero $E = \cancel{m}c^2 + K$, $m \to 0$

Then $E^2 = (cp)^2$ ie $E = c|\vec{p}|$

Therefore $\frac{v}{c} = \frac{cp}{E} = 1$ ie $v = c$

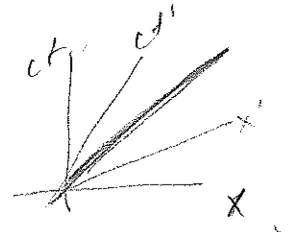
massless particles must move at <u>light speed</u>. (photons) (gravitons)

$E_{rest} = mc^2 \Rightarrow$ massless particle has no rest energy only kinetic energy ~~($E = K = pc$)~~ ← could be confusing

(Note: $E = \gamma mc^2$ + $\vec{p} = \gamma m\vec{v}$ break down since $m \to 0$, $\gamma \to \infty$)

What if you boost into a frame in which it is at rest? not possible!

$\gamma = \infty$



$v \to c$ (axes collapse)

Massless particles travel at $v = c$ in all IRF's