A Lorentz boost in the x-direction

$$ct' = \gamma(ct - \beta x) \qquad \beta = \frac{v}{c}$$

$$x' = \gamma(x - \beta \cdot ct) \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$y' = y$$

$$z' = z$$

can be expressed in matrix form

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

This suggests we define a <u>four-vector</u>

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \quad \text{where} \begin{cases} x^0 = ct \\ x^1 = x \\ x^2 = y \\ x^3 = z \end{cases}$$

[Greek index $\mu = 0, 1, 2, 3$]

which transforms under a <u>general</u> Lorentz transformation as

$$\begin{pmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \underbrace{\begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix}}_{\Lambda = \text{Lorentz transformation matrix}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

This can be written more compactly in index notation

$$x^{\mu\prime} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad \text{where } \Lambda^{\mu}{}_{\nu} = \text{matrix element of } \Lambda$$

(row) (column)

$(\mu = 0, 1, 2, 3)$

Def: A fourvector is <u>any</u> object $A^{\mu} = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix}$

that transforms as $A^{\mu\prime} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} A^{\nu}$

under a Lorentz transformation.

<u>examples</u>

$$x^{\mu} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$ ← all components have same units

wave four-vector

$$k^{\mu} = \begin{pmatrix} k^0 \\ k^1 \\ k^2 \\ k^3 \end{pmatrix} = \begin{pmatrix} \frac{\omega}{c} \\ k_x \\ k_y \\ k_z \end{pmatrix}$$ ← all components have same units

A Lorentz transformation matrix Λ must obey

$$\Lambda^T \eta \Lambda = \eta$$

where

$$\eta \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{Minkowski metric}$$

Lorentz transformations include <u>boosts</u> and <u>rotations</u>

e.g.

Boost in x-direction $\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Rotation in xy-plane $\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}$

[ex. show these obey $\Lambda^T \eta \Lambda = \eta$]

"The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.

H. Minkowski, "Space and Time," 1908

Def: <u>Lorentz scalar</u> = a quantity invariant under Lorentz transformation

e.g. scalar product of two 4-vector

$$\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} A^\mu \eta_{\mu\nu} B^\nu$$

$$= A^0 \underbrace{\eta_{00}}_{-1} B^0 + A^0 \underbrace{\eta_{01}}_{0} B^1 + \dots + A^1 \underbrace{\eta_{11}}_{1} B^1 + \dots$$

$$= -A^0B^0 + \underbrace{A^1B^1 + A^2B^2 + A^3B^3}_{\vec{A}\cdot\vec{B}}$$

examples

$$\sum\sum k^\mu \eta_{\mu\nu} k^\nu = -\left(\frac{\omega}{c}\right)\left(\frac{\omega}{c}\right) + \vec{k}\cdot\vec{k} = -\frac{1}{c^2}\underbrace{(\omega^2 - c^2k^2)}_{=0}$$

$$\sum\sum k^\mu \eta_{\mu\nu} x^\nu = -\left(\frac{\omega}{c}\right)(ct) + \vec{k}\cdot\vec{x} = \underbrace{\vec{k}\vec{x} + \omega t}_{\text{phase of travelling wave}}$$

invariant = nodes are nodes in any frame

Proof that scalar product is invariant (similar to earlier proof)

$$\sum\sum A^{\mu'}\eta_{\mu\nu}B^{\nu'} = \underbrace{(A^{0'}\ A^{1'}\ A^{2'}\ A^{3'})}_{(A^0\ A^1\ A^2\ A^3)\Lambda^T}\ \eta\ \underbrace{\begin{pmatrix} B^{0'} \\ B^{1'} \\ B^{2'} \\ B^{3'} \end{pmatrix}}_{\Lambda\begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix}}$$

$$= (A^0\ A^1\ A^2\ A^3)\ \underbrace{\Lambda^T\eta\Lambda}_{\eta}\ \begin{pmatrix} B^0 \\ B^1 \\ B^2 \\ B^3 \end{pmatrix}$$

$$= \sum\sum A^{\mu}\eta_{\mu\nu}B^{\nu}$$

n6

Einstein's
Principle of relativity

All the laws of physics are invariant under (Lorentz) boosts

ie

- Eqns of physics have the same form in reference frames related by Lorentz transformations.

They must be expressed in terms of quantities covariant under Lorentz transformations

Lorentz-scalar = Lorentz scalar

4-vector = 4-vector

tensor = tensor

Just talk about this

e.g. wave equation

$$-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} + \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} = 0$$ (invariant under rotations)

Although not invariant under Galilean boosts, it is invariant under Lorentz boosts → (see 3rd class)

probably skip

Proof: Define 4-gradient $\partial^\mu = \begin{cases} -\frac{1}{c}\frac{\partial}{\partial t} & \mu = 0 \\ \frac{\partial}{\partial x^i} & \mu = 1, 2, 3 \end{cases}$

[minus sign necessary so that ∂^μ txs as 4-vector]

This transforms as a 4-vector

wave eqn $\Rightarrow$ $(-\partial^0\partial^0 + \partial^1\partial^1 + \partial^2\partial^2 + \partial^3\partial^3)A = 0$

$$\Big(\underbrace{\sum_{\mu,\nu=0}^{3} \partial^\mu \eta_{\mu\nu} \partial^\nu}_{\text{Lorentz scalar [called d'Alembertian]}}\Big) A = 0$$

↳ invariant under Lz txs

NOT DONE IN CLASS

Why is $\partial^0 = -\partial_0$?

$$x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu$$

$$\Lambda^\mu{}_\nu \eta^{\nu\lambda} \frac{\partial A}{\partial x^\lambda} = \Lambda^\mu{}_\nu \eta^{\nu\lambda} \frac{\partial A}{\partial x'^\kappa} \frac{\partial x'^\kappa}{\partial x^\lambda}$$

$$= \underbrace{\Lambda^\mu{}_\nu \eta^{\nu\lambda} \Lambda^\kappa{}_\lambda}_{\eta^{\mu\kappa}} \frac{\partial A}{\partial x'^\kappa}$$

ie $\Lambda^\mu{}_\nu (\eta^{\nu\lambda} \partial_\lambda A) = \eta^{\mu\kappa} (\partial_\kappa A)'$

ie $\Lambda^\mu{}_\nu \partial^\nu A = \partial^\kappa A'$

ie ∂^μ is a 4-vect

NOT DONE

$\not{V2}$

4-gradient $\frac{\partial \psi}{\partial x^\mu} = \begin{pmatrix} \frac{\partial \psi}{\partial x^0} \\ \frac{\partial \psi}{\partial x^1} \\ \frac{\partial \psi}{\partial x^2} \\ \frac{\partial \psi}{\partial x^3} \end{pmatrix} = \begin{pmatrix} \frac{1}{c}\frac{\partial \psi}{\partial t} \\ \\ \vec{\nabla}\psi \\ \\ \end{pmatrix}$

Exercise 11

Show that

$\frac{\partial \psi}{\partial x^\mu}$ does not transform as a 4-vector

but $\partial^\mu \psi \equiv \eta^{\mu\nu} \frac{\partial \psi}{\partial x^\nu}$ does.

$\eta^{\mu\nu}$ = inverse of $\eta_{\mu\nu}$ $\quad \eta^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$$\partial^\mu \psi = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} -\frac{1}{c}\frac{\partial \psi}{\partial t} \\ \vec{\nabla}\psi \end{pmatrix} = \begin{pmatrix} -\frac{1}{c}\frac{\partial \psi}{\partial t} \\ \vec{\nabla}\psi \end{pmatrix}$$

invariant $\eta_{\mu\nu} \partial^\mu \partial^\nu \psi = 0$

$$= -\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} + \nabla^2 \psi = 0, \qquad \frac{\partial^2 \psi}{\partial t^2} - c^2 \nabla^2 \psi = 0$$

wave eqn

Exercise

NOT DONE

$$t' = \gamma t - \gamma\beta x$$
$$x' = -\gamma\beta t + \gamma x$$

$$t = \gamma t' + \gamma\beta x'$$
$$x = \gamma\beta t' + \gamma x'$$

$$\frac{\partial\psi}{\partial t'} = \gamma \frac{\partial\psi}{\partial t} + \gamma\beta \frac{\partial\psi}{\partial x}$$

$$\frac{\partial\psi}{\partial x'} = \gamma\beta \frac{\partial\psi}{\partial t} + \gamma \frac{\partial\psi}{\partial x}$$

$$\left(-\frac{\partial\psi}{\partial t'}\right) = \gamma\left(-\frac{\partial\psi}{\partial t}\right) - \gamma\beta\left(\frac{\partial\psi}{\partial x}\right)$$

$$\left(+\frac{\partial\psi}{\partial x'}\right) = -\gamma\beta\left(-\frac{\partial\psi}{\partial t}\right) + \gamma\left(\frac{\partial\psi}{\partial x}\right)$$

so $\begin{pmatrix} -\frac{\partial\psi}{\partial t'} \\ \frac{\partial\psi}{\partial x'} \end{pmatrix}$ transforms as $\begin{pmatrix} t' \\ x' \end{pmatrix}$

Let two events A + B have spacetime coordinates x^μ_A + x^μ_B

Let $\Delta x^\mu = x_B^\mu - x_A^\mu = \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$ be the spacetime separation between events

Δx^μ is a 4-vector

Define the <u>spacetime</u> <u>interval</u> Δs^2 between A and B as

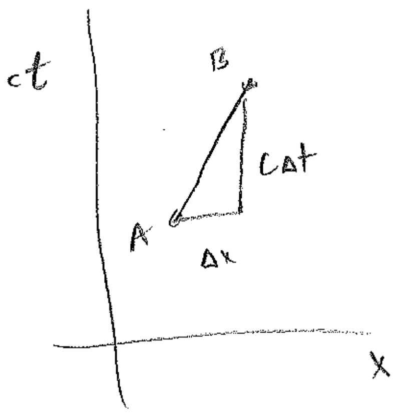
$$\Delta s^2 = \sum \sum \Delta x^\mu \eta_{\mu\nu} \Delta x^\nu = -(c\,\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Since Δs^2 is a Lorentz scalar (same in all IRFs)

we may evaluate it in any convenient frame

Given two events A & B, suppose it is possible to travel from A to B at constant velocity $\vec{v}$ ($< c$).

Then $\Delta\vec{x} = \vec{v}\,\Delta t$.



Spacetime interval between A & B is

$$\Delta s^2 = -(c\,\Delta t)^2 + (\Delta\vec{x})^2$$
$$= -(c\,\Delta t)^2 + (\vec{v}\,\Delta t)^2$$
$$= -c^2(\Delta t)^2\left(1 - \frac{v^2}{c^2}\right)$$
$$= -\frac{c^2(\Delta t)^2}{\gamma^2}$$

The <u>proper</u> time between events $\Delta\tau$ is the time between them in a frame (if one exists) in which they occur at the same place.

Evaluate Δs^2 in the frame

$$\Delta s^2 = -(c\,\Delta\tau)^2 + 0$$

Since Δs^2 is invariant $\Rightarrow -c^2(\Delta\tau)^2 = -\frac{c^2}{\gamma^2}(\Delta t)^2$

$$\Rightarrow \Delta t = \gamma\,\Delta\tau \quad \text{(time dilation)}$$

Proper time between events is less than the time between them in any other frame (since $\gamma \geq 1$)