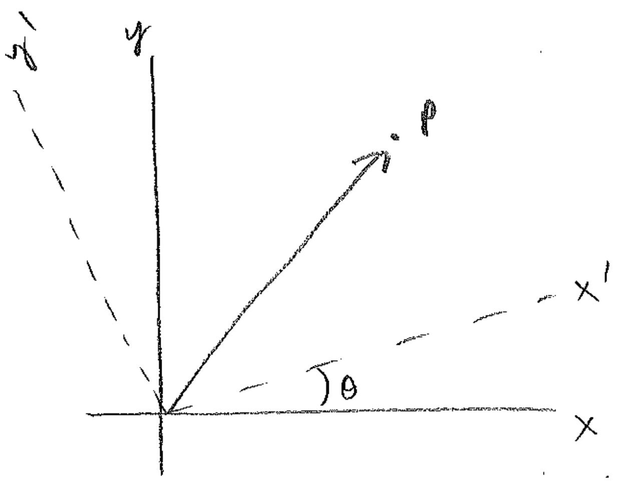
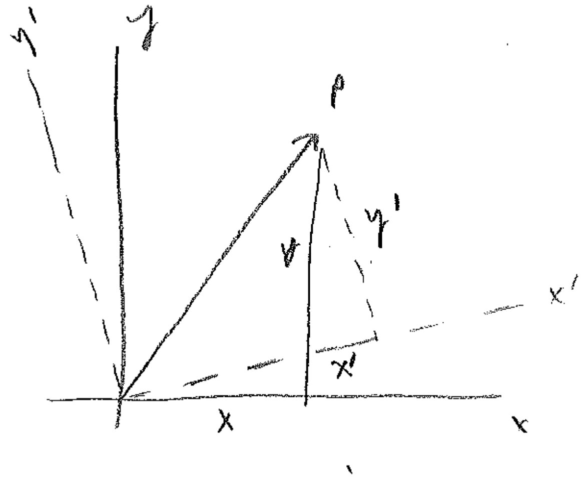
Rotations in two dimensions



P = physical point

$\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \end{pmatrix}$

are coordinates of P in two different coordinate systems.



Use diagram to show that

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

[exercise]

Can rewrite this as a matrix equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

[explain matrix multiplication]

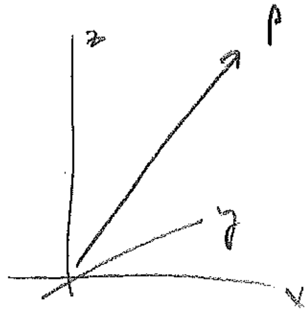
Distance from P to origin $= x^2 + y^2 = x'^2 + y'^2$

$\therefore$ $x^2 + y^2$ is invariant under rotations

[exercise]

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 \\ -a_1 \end{bmatrix}$$

Rotations in three dimensions



$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$ [superscripts]

A rotation about an arbitrary axis through the origin can be described by a rotation matrix R

$$\vec{x}\,' = R\,\vec{x}$$

$$\begin{pmatrix} x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{pmatrix} = \begin{pmatrix} R^1{}_1 & R^1{}_2 & R^1{}_3 \\ R^2{}_1 & R^2{}_2 & R^2{}_3 \\ R^3{}_1 & R^3{}_2 & R^3{}_3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Matrix elements are denoted $R^i{}_j$ ← row index (i), ← column index (j)

For example

$$x^{1\prime} = R^1{}_1 x^1 + R^1{}_2 x^2 + R^1{}_3 x^3 = \sum_{j=1}^{3} R^1{}_j x^j$$

An index that is repeated & summed over is called a "dummy index"

$$x^{i\prime} = \sum_{j=1}^{3} R^i{}_j x^j \qquad i = 1, 2, 3$$

An index that appears on both sides of an equation & is not summed over is a "free index"

A rotation matrix must be "orthogonal" which means

$$R^T R = \mathbb{1} \qquad [\Sigma R^i{}_k \delta_{ij} R^j{}_l = \delta_{kl}]$$

where

$$R^T = \text{transpose of } R \qquad [\text{exchange rows \& columns}]$$

$$= \begin{pmatrix} R^1{}_1 & R^2{}_1 & R^3{}_1 \\ R^1{}_2 & R^2{}_2 & R^3{}_2 \\ R^1{}_3 & R^2{}_3 & R^3{}_3 \end{pmatrix} \qquad [\text{reflect across diagonal}]$$

$$\mathbb{1} = \text{unit matrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

one can show that $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is orthogonal [exer]

rotation through θ in xy plane

one can show that the product of rotation matrices

$R_\theta R_\phi$ is the same as $R_{\theta+\phi}$ [exer]

Rotation matrices form a "group", called $O(3)$

orthogonal — 3×3 matrices

[show non commutativity using a book or eraser]

Position vector transforms as $x^{i\prime} = \sum_{j=1}^{3} R^i{}_j x^j$.

What about velocity?

$$\vec{v} = \frac{d\vec{x}}{dt} \Rightarrow v^i = \frac{dx^i}{dt}$$

Then $v^{i\prime} = \frac{dx^{i\prime}}{dt} = \frac{d}{dt}\left(\sum R^i{}_j x^j\right) = \sum R^i{}_j \frac{dx^j}{dt} = \sum R^i{}_j v^j$

Similarly $\vec{a} = \frac{d\vec{v}}{dt} \Longrightarrow a^{i\prime} = \sum R^i{}_j a^j$

We say $\vec{v}$ and $\vec{a}$ are "vectors under rotation" because their components transform like those of $\vec{x}$

(other examples: $\vec{L} = \vec{x} \times m\vec{v}$, $\vec{\nabla} f = \left(\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \frac{\partial f}{\partial x^3}\right)$)

[proof intentionally omitted →]

Def: <u>3-vector (under rotations)</u>: any quantity $\vec{A}$ whose components $\begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix}$ transform under rotations as $\vec{A}' = R\vec{A}$

as $A^{i\prime} = \sum_{j=1}^{3} R^i{}_j A^j$, just like $\vec{x}$

[A 2nd rank tensor (under rotations) is any object $\mathbb{B}$ whose components B^{ij} transform as $B^{ij\prime} = \sum_{k=1}^{3}\sum_{l=1}^{3} R^i{}_k R^j{}_l B^{kl}$]

Aside to instructor

Proof that $\vec{\nabla} f$ is a vector is more involved. If we wish to avoid vector calculus in this course, we should omit the proof

$$x'^i = R^i{}_j x^j$$

Consider

$$R^i{}_j \frac{\partial f}{\partial x^j} = R^i{}_l \delta^{lj} \frac{\partial f}{\partial x^j}$$

$$= R^i{}_l \delta^{lj} \frac{\partial f}{\partial x^{k'}} \frac{\partial x^{k'}}{\partial x^j}$$

$$= \underbrace{R^i{}_l \delta^{lj} R^k{}_j}_{\delta^{ik}} \frac{\partial f}{\partial x^{k'}}$$

RR^T

$$= \frac{\partial f}{\partial x^{i'}}$$

(one could try proving it for a concrete 2d rotation)

Why is it important that all 3-vectors transform under rotations in the same way: $\vec{A}' = RA$?

<u>Principle of Isotropy</u>: there is no preferred direction in space so the laws of physics must be invariant under rotations; that is, the eqns of physics have the same form when expressed in coordinates related by rotations

Isotropy is guaranteed if the eqns of physics are expressed in terms of 3-vectors.

Newton 2nd law:
$$\vec{F} = m\vec{a}$$
$$R\vec{F} = Rm\vec{a}$$

Because $\vec{F}$ and $\vec{a}$ are both 3-vectors
$$\vec{F}' = R\vec{F} \quad \text{and} \quad \vec{a}' = R\vec{a}$$
$$\Rightarrow \vec{F}' = m\vec{a}' \qquad \text{(same form)}$$

Similarly, momentum conservation
$$\sum_{\text{init}} \vec{p} = \sum_{\text{final}} \vec{p}$$

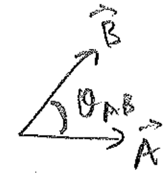
$$\Rightarrow \sum_{\text{init}} \vec{p}' = \sum_{\text{final}} \vec{p}'$$

Def: <u>scalar (under rotations)</u>

a quantity that is invariant under rotations

e.g. time
~~energy~~
<u>magnitude</u> of a vector [e.g. exercise $|x|^2$]
any dot product of vectors

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}| \cos \theta_{AB}$$



(Under a rotation of $\vec{A}$ and $\vec{B}$, θ_{AB} does not change)

Another proof in terms of components

$$\vec{A} \cdot \vec{B} = A^1B^1 + A^2B^2 + A^3B^3 = (A^1, A^2, A^3)\begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}$$

$$\begin{pmatrix} A^{1\prime} \\ A^{2\prime} \\ A^{3\prime} \end{pmatrix} = R \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix} \Rightarrow (A^{1\prime}\ A^{2\prime}\ A^{3\prime}) = (A^1\ A^2\ A^3)\, R^T$$

$$\begin{pmatrix} B^{1\prime} \\ B^{2\prime} \\ B^{3\prime} \end{pmatrix} = R \begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}$$

$$\vec{A}' \cdot \vec{B}' = (A^{1\prime}\ A^{2\prime}\ A^{3\prime}) \begin{pmatrix} B^{1\prime} \\ B^{2\prime} \\ B^{3\prime} \end{pmatrix} = (A^1\ A^2\ A^3)\underbrace{R^T R}_{\mathbb{1} \text{ because } R \text{ is orthogonal}}\begin{pmatrix} B^1 \\ B^2 \\ B^3 \end{pmatrix}$$

$$= \vec{A} \cdot \vec{B} \quad \checkmark$$

An eqn of physics expressed in terms of scalars obeys the principle of isotropy.

energy conservation $\quad E_i = E_f$

energy is a scalar so $E_i' = E_i$ and $E_f' = E_f$

$\therefore \; E_i' = E_f'$ (same form)

classical wave equation

$$\frac{\partial^2 A}{\partial t^2} = c^2 \left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2} \right)$$

$$= c^2 \; \underbrace{\vec{\nabla} \cdot \vec{\nabla}} \; A$$

dot product of gradient operator $\vec{\nabla}$

[called Laplacian]

Both $\frac{\partial^2}{\partial t^2}$ and $\vec{\nabla} \cdot \vec{\nabla}$ are scalars under rotation

$\Rightarrow$ wave eqn obeys isotropy