Galilean boost by v in +x direction

$t = t'$

$x = x' + vt'$

$y = y'$

$z = z'$

These are accurate for low speed boosts ($v \ll c$) but break down for high speeds...

The eqns correct for all speeds are

Lorentz boost by v in +x direction

$t = \gamma(t' + \frac{v}{c^2}x')$

time dilation — relativity of simultaneity

$x = \gamma(x' + vt')$

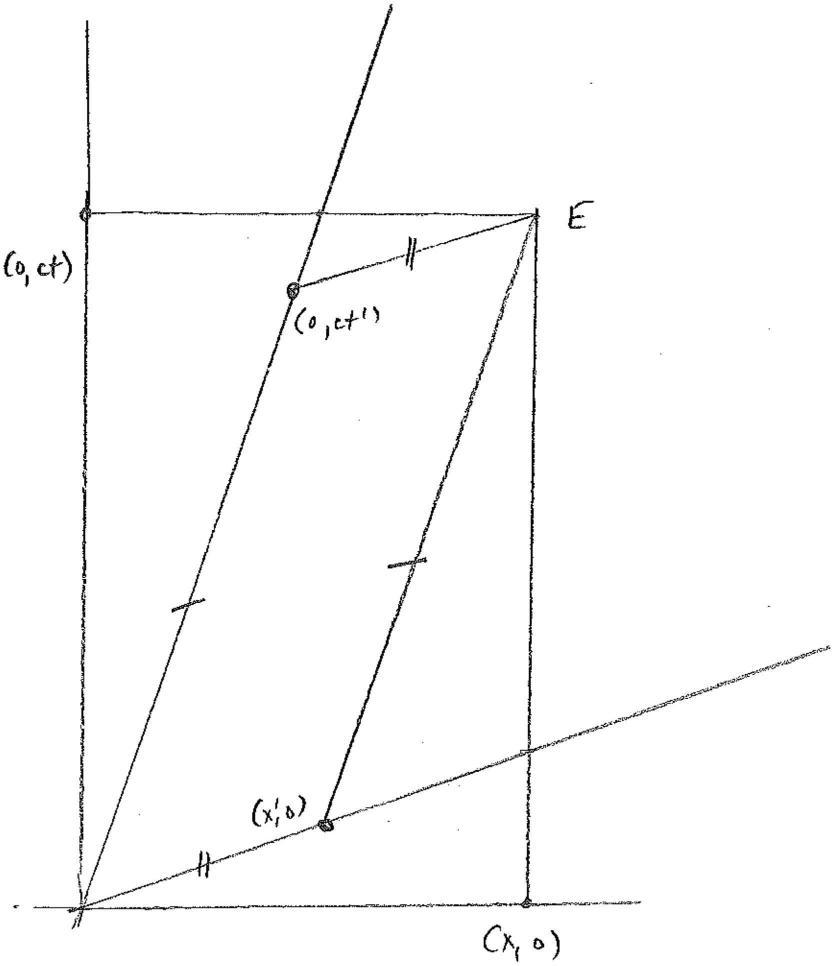
length contraction along direction of motion

$y = y'$

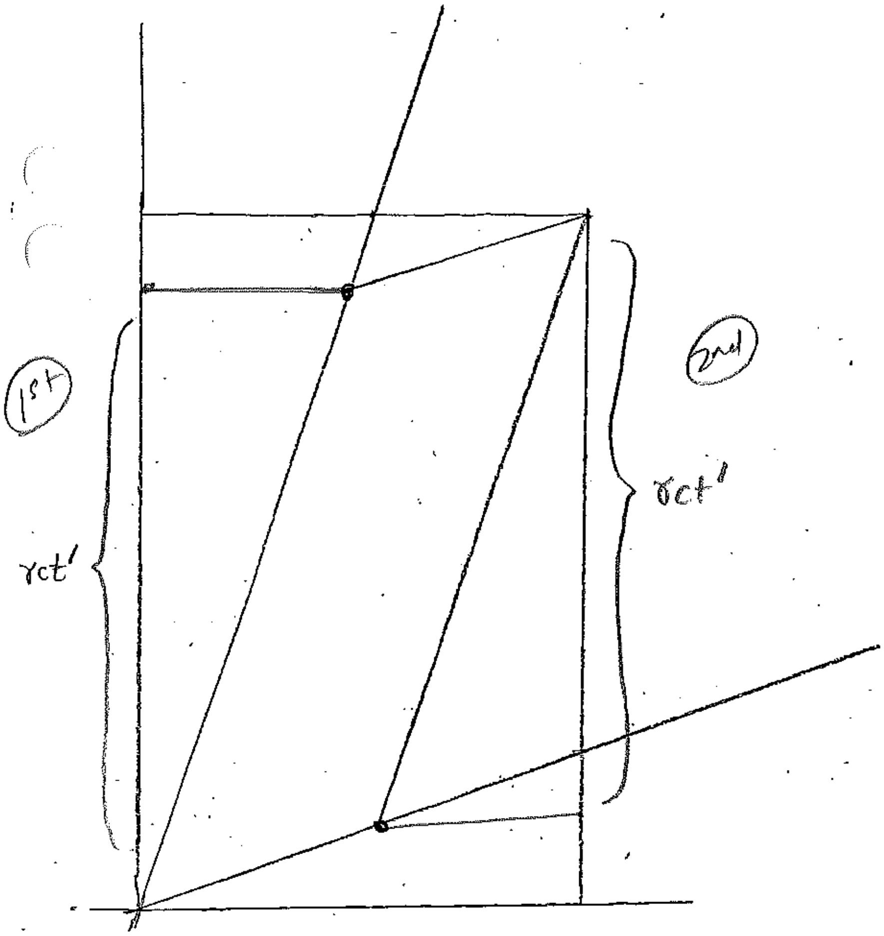
$z = z'$

no length contraction in transverse directions

(Lorentz $\rightarrow$ Galilean if $\frac{v}{c} \ll 1$, $\gamma \approx 1$

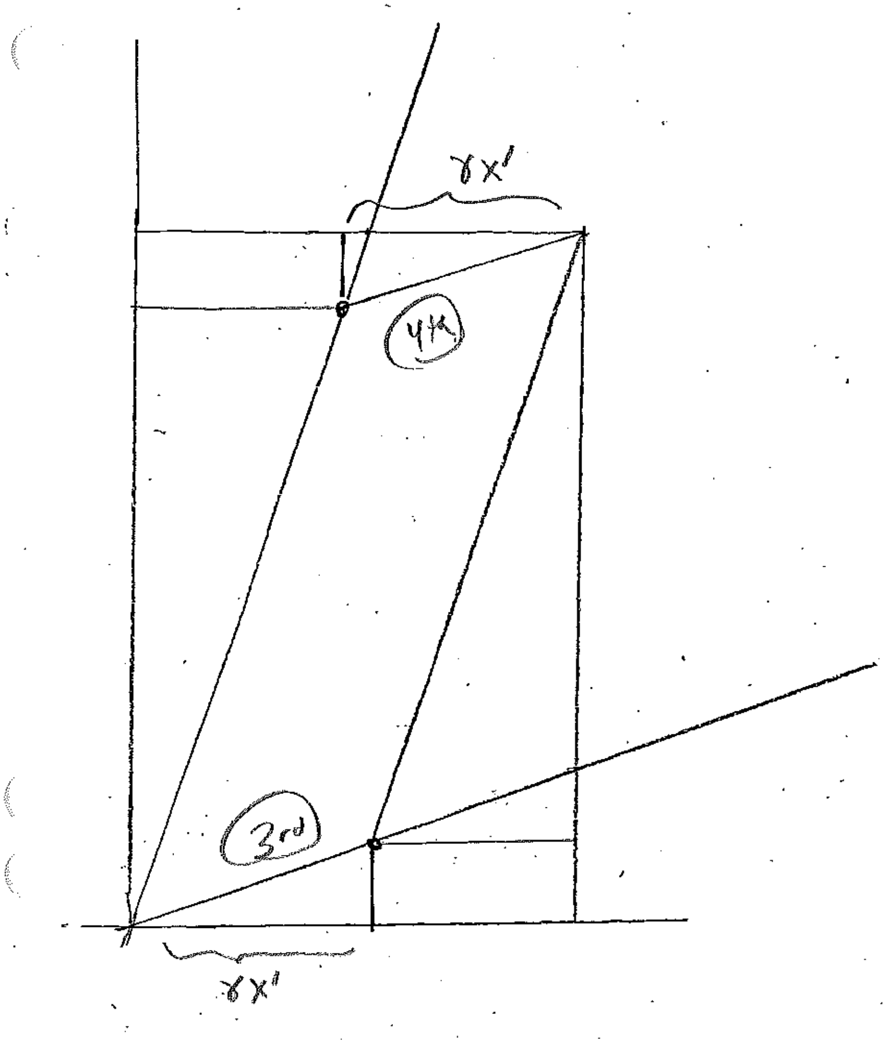


[To derive this, draw a big picture in your notes]

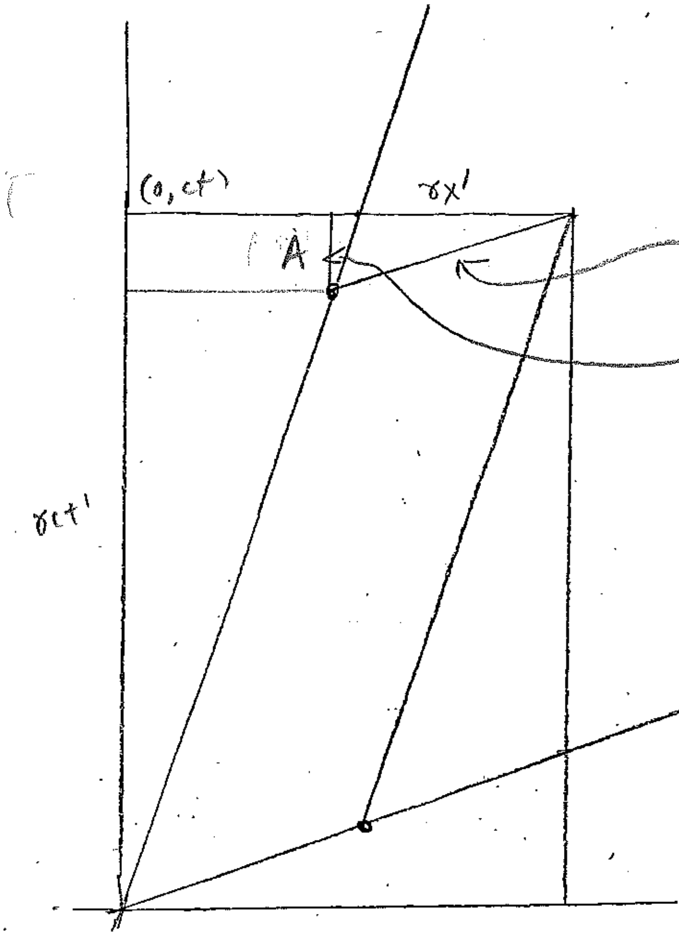


[time dilation]

[hypotenuse of similar triangle are equal due to parallelogram
⇒ congruent triangles]



[length contraction]



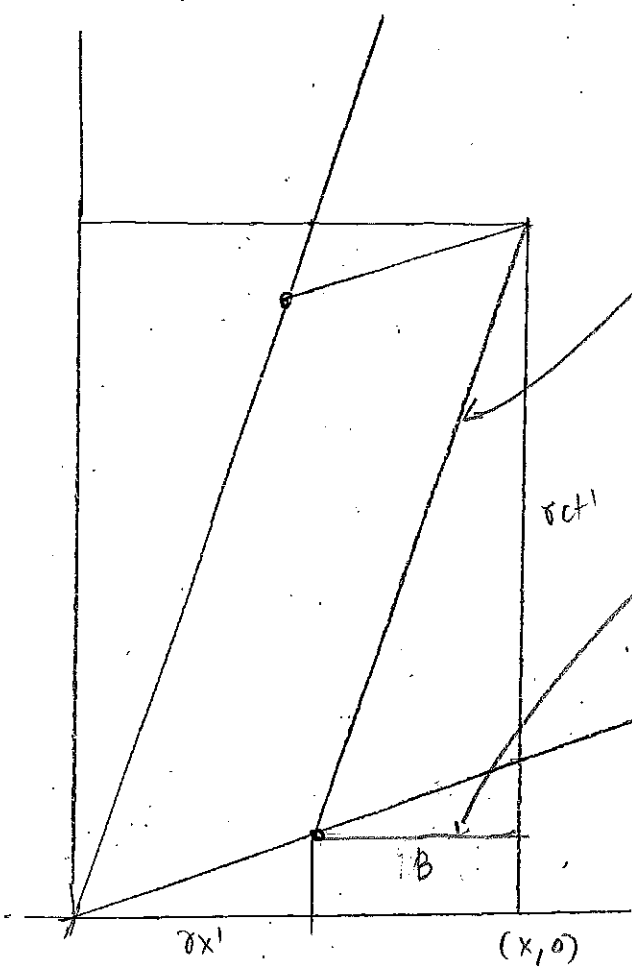
Slope $= \frac{v}{c} = \beta$

$\beta = \frac{A}{\gamma x'} \Rightarrow A = \beta\gamma x'$

$\Rightarrow ct = \gamma ct' + \beta\gamma x'$

$t = \gamma(t' + \frac{\beta}{c} x')$

$$\boxed{t = \gamma\left(t' + \frac{v}{c^2} x'\right)}$$



Slope $= \frac{c}{v} = \frac{1}{\beta}$

$\frac{1}{\beta} = \frac{\gamma ct'}{B}$

$B = \beta\gamma ct'$

$x = \gamma x' + \beta\gamma ct'$

$$\boxed{x = \gamma(x' + vt')}$$

Exercise

Given $t = \gamma(t' + \frac{v}{c^2}x')$

$x = \gamma(x' + vt')$

Solve for x' and t' to obtain

$t' = \gamma(t - \frac{v}{c^2}x)$

$x' = \gamma(x - vt)$

An easier way to see this is to observe that S is moving w/ velocity $-v$ w.r.t. S'

so replace $v \to -v$ and $\gamma \to \gamma$

Mnemonic: origin of S' $(x'=0)$ moves w/ speed v to right w.r.t. S

ie $x = vt$ or $x - vt = 0$

[exercise: redo exercise w/ $ct_{\pm 1}$ of train clock using Lorentz tx]

Given two events E_1 & E_2,
the differences in positions & times Δx & Δt
obey

$\Delta t' = \gamma(\Delta t - \frac{v}{c^2}\Delta x)$

$\Delta x' = \gamma(\Delta x - v\Delta t)$

Exercise

Given $x' = \gamma(x - vt)$ where $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

$t' = \gamma\left(t - \frac{vx}{c^2}\right)$

Eliminate t:

$$x' + vt' = \gamma(x - vt) + v\gamma\left(t - \frac{vx}{c^2}\right)$$

$$= \gamma\left(1 - \frac{v^2}{c^2}\right)x = \sqrt{1 - \frac{v^2}{c^2}}\, x$$

$$\therefore \quad x = \gamma(x' + vt')$$

Eliminate x:

$$\frac{v}{c^2}x' + t' = \frac{v\gamma}{c^2}(x - vt) + \gamma\left(t - \frac{vx}{c^2}\right)$$

$$= \gamma\left(-\frac{v^2}{c^2} + 1\right)t$$

$$= \sqrt{1 - \frac{v^2}{c^2}}\, t$$

$$\therefore \quad t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

Note: these are the same as the original eqns but with $x \leftrightarrow x'$
$t \leftrightarrow t'$
$v \rightarrow -v$

which makes sense physically.

Exercise

At what time does synchronizing pulse arrive at ~~rear~~ head clock

In S' : event occurs at $\begin{aligned} x' &= +l_0 \\ t' &= +\frac{l_0}{c} \end{aligned}$

then $$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

$$= \gamma\left(\frac{l_0}{c} + \frac{v l_0}{c^2}\right)$$

$$= \gamma\left(\frac{l_0}{c} + \frac{v}{c^2}(+l_0)\right)$$

$$= \frac{l_0}{c^2}\frac{(c+v)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{l_0}{c}\frac{c+v}{\sqrt{c^2-v^2}}$$

$$= \frac{l_0}{c}\sqrt{\frac{c+v}{c-v}}$$ as calc'd earlier!

position $$x = \gamma(x' + vt')$$

$$= \gamma\left(+l_0 + \frac{v l_0}{c}\right)$$

$$= +\frac{\gamma l_0}{c}(c+v) = + l_0\sqrt{\frac{c+v}{c-v}}$$

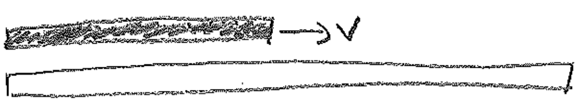
(makes sense since pulse travels at speed c in S too!)

Length contraction paradox

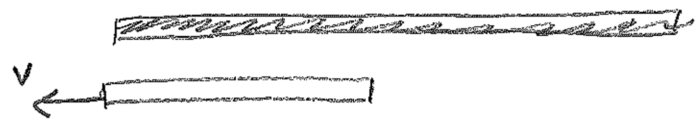
Each observer (correctly) claims the other's length contracts. How can both be right?

Consider two meter sticks moving w/ relative speed $v=\frac{\sqrt{3}}{2}c \Rightarrow \gamma=2$.

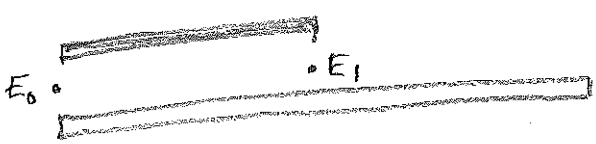
Snapshot of events occurring simultaneously in S $(t=0)$

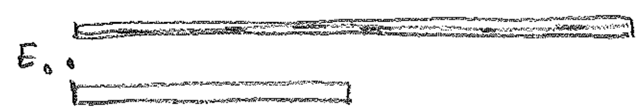


Snapshot of events occurring simultaneously in S' $(t'=0)$



[Now add events]



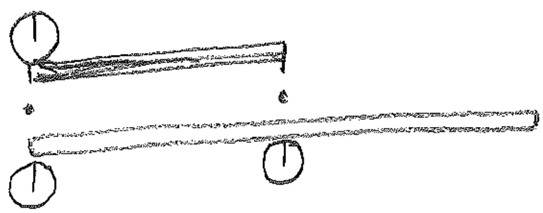


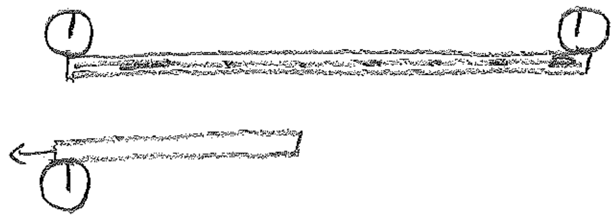
event E_0 = left ends of sticks coincide

event E_1 = right end of top stick coincides w/ 50 cm mark on bottom stick

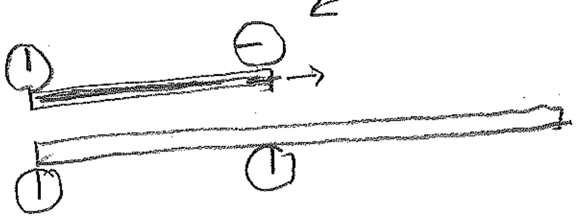
E_0 & E_1 are simultaneous in S but not in S'

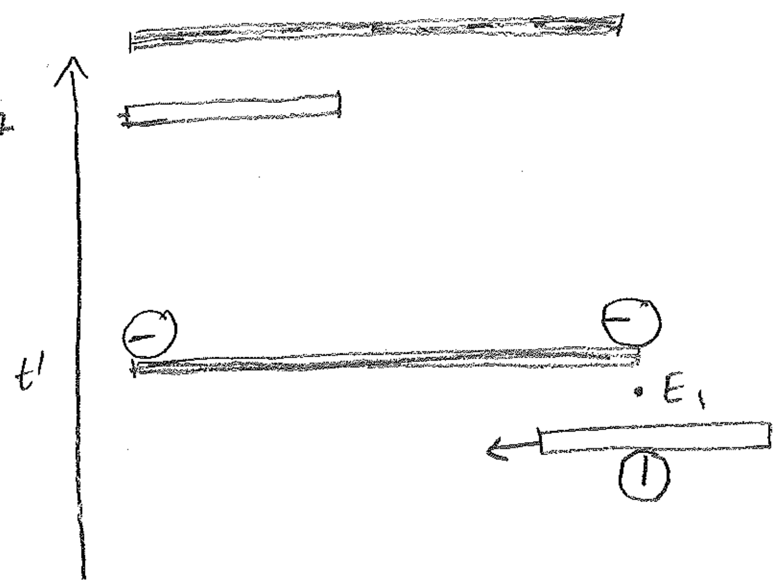
[Now add clocks, first on non moving stick]





[Ask, where is E_1 in S' frame?
Well, leading clocks lag so time of E_1 is before E_0 in S' frame]

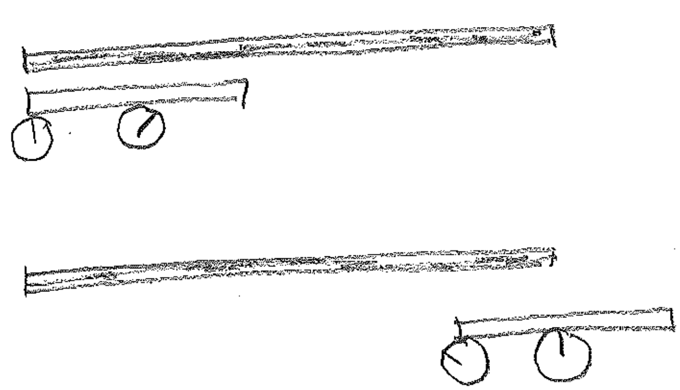




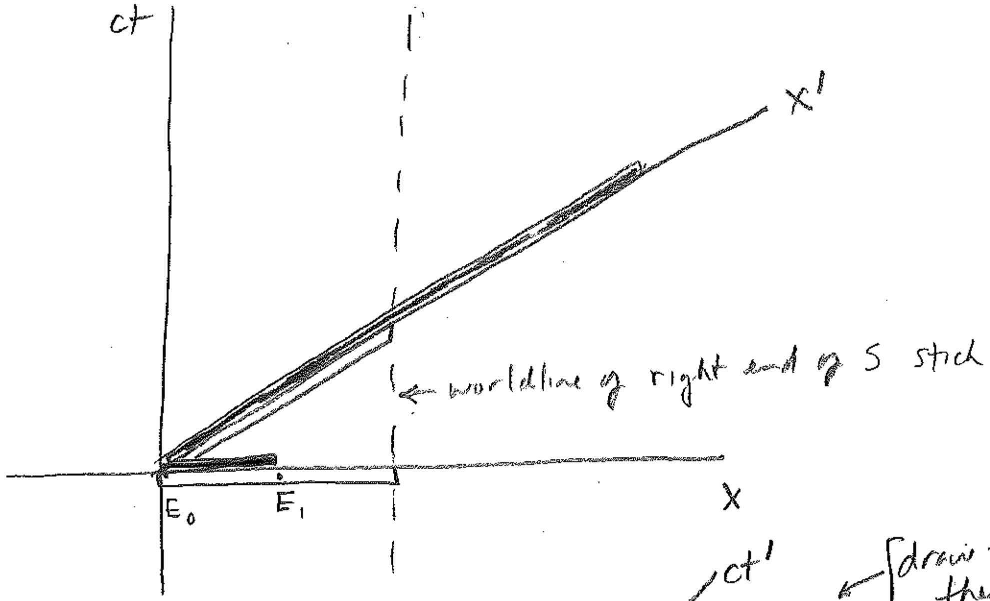
[$t' = -\frac{\sqrt{3}}{2}\left(\frac{1\,\text{m}}{c}\right) = -2.9\,\text{ns}$
[left end of stick travels $\frac{3}{4}$m in t' at speed $\frac{\sqrt{3}}{2}c$]

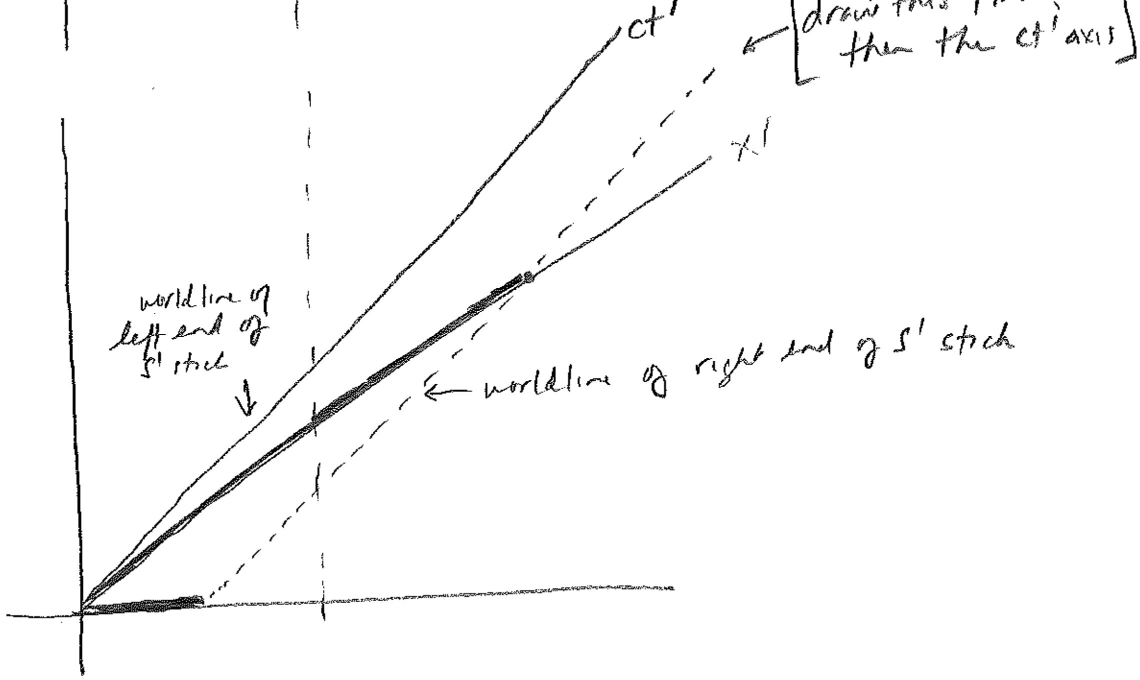
[Add t' axis and earlier snapshot]

[Finally add clocks on bottom stick in S':
clocks run slower by factor of 2.]



[optional: space time diagram of length contraction paradox] k 8





[optional: do spacetime diagram for time dilation paradox]