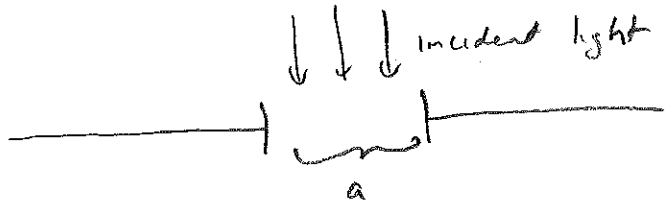
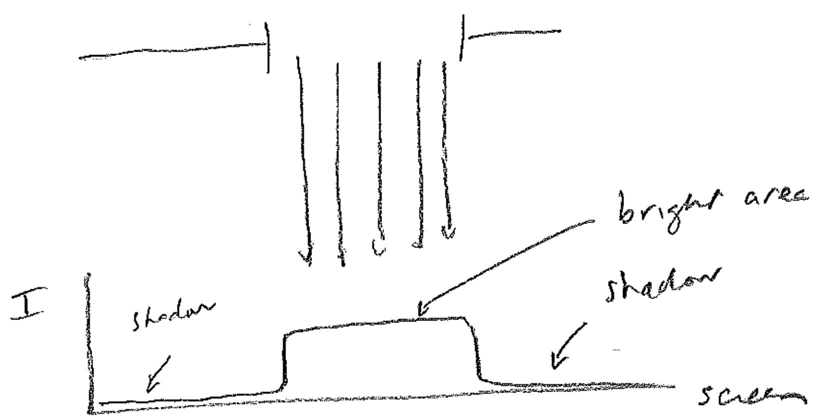
Single slit diffraction

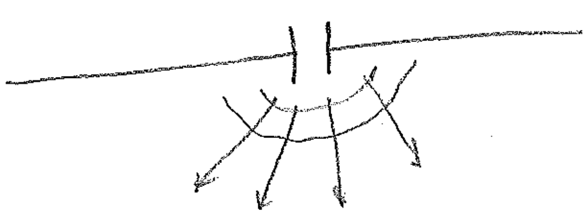
Consider an opaque screen w/ a single slit of width a

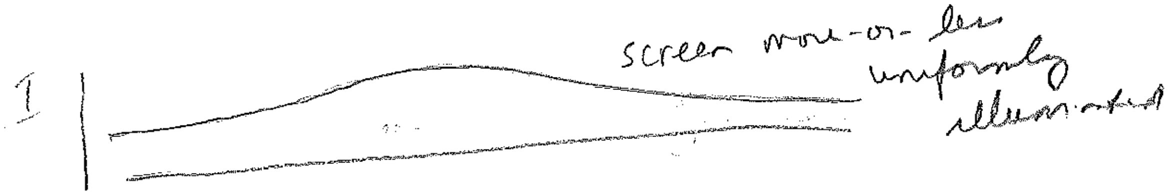


① if $a \gg \lambda$, use geometric optics (light goes in straight lines)

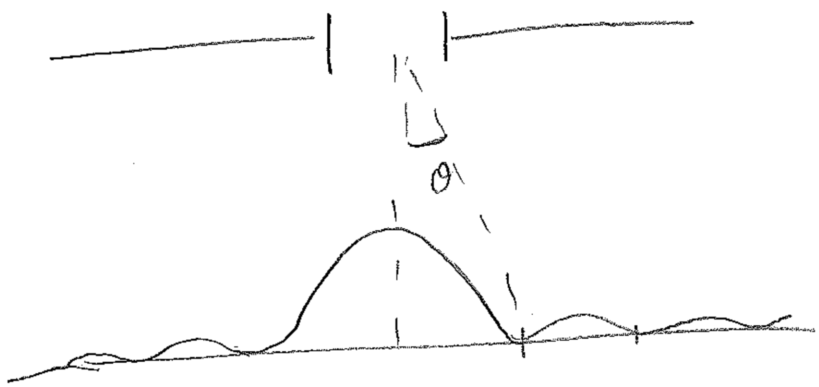


② if $a \ll \lambda$, diffraction





What if $a \gtrsim \lambda$? Single slit diffraction pattern w/ a large central peak



We will derive this pattern.
and show that first minimum occurs when

$$a \sin\theta = \lambda$$

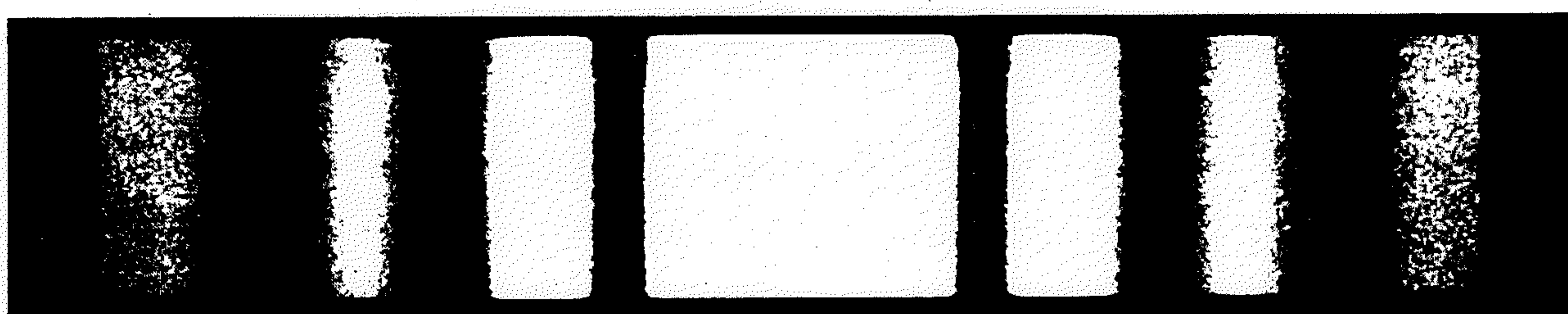


Figure 1 The diffraction pattern produced when light passes through a narrow slit.

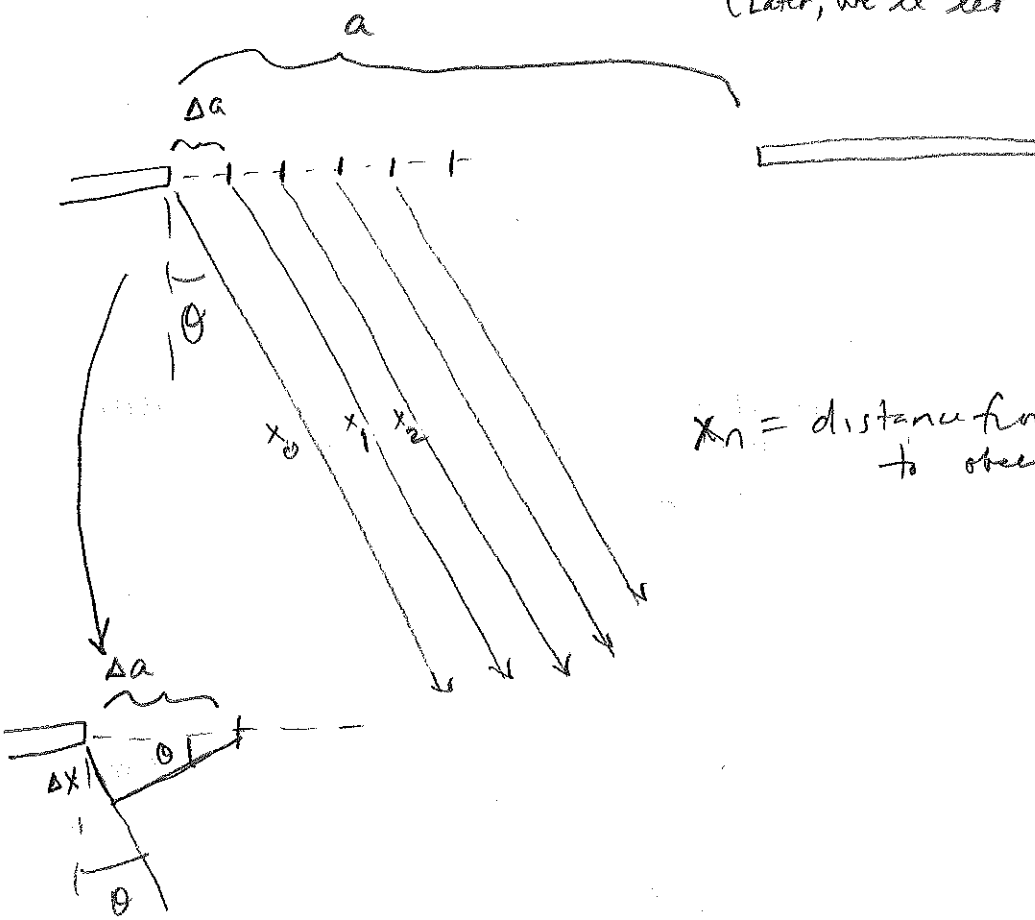
QE3

OH

HR
46, fig 1

Divide slit of width a into N strips of width $\Delta a = \frac{a}{N}$

(Later, we'll let $N \to \infty$)



x_n = distance from nth strip to observation point

$\Delta x = \Delta a \sin\theta$ = path difference between adjacent strips

Each strip produces an EM wave $E_n = \frac{A}{N} \sin(\omega t - k x_n)$

$$x_1 = x_0 - \Delta x$$

$$x_2 = x_1 - \Delta x = x_0 - 2\Delta x$$

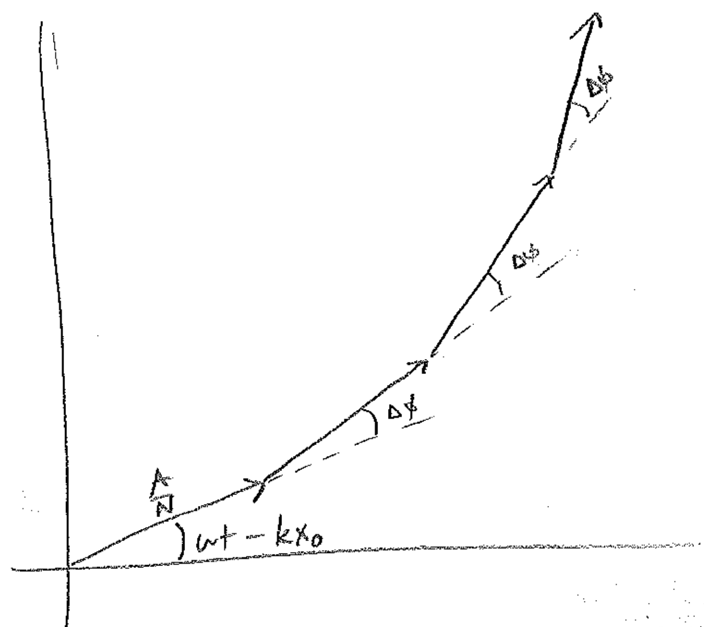
$$\vdots$$

$$x_n = x_0 - n\,\Delta x$$

$$\Rightarrow E_n = \frac{A}{N} \sin(\omega t - k x_0 + n k \Delta x)$$

$$E_{tot} = \sum_{n=0}^{N-1} E_n$$

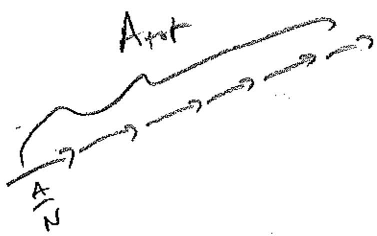
Represent wave by phasors



$\Delta\phi$ = phase difference between adjacent strips $= k\,\Delta x = \frac{2\pi}{\lambda}\Delta a \sin\theta$

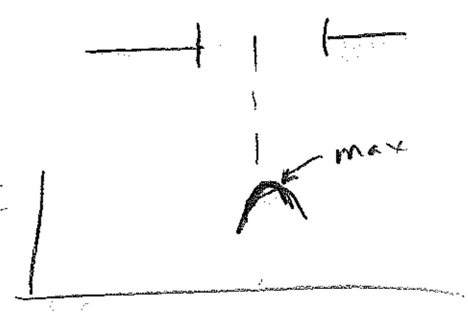
Fully constructive interference occurs if all phasors are parallel

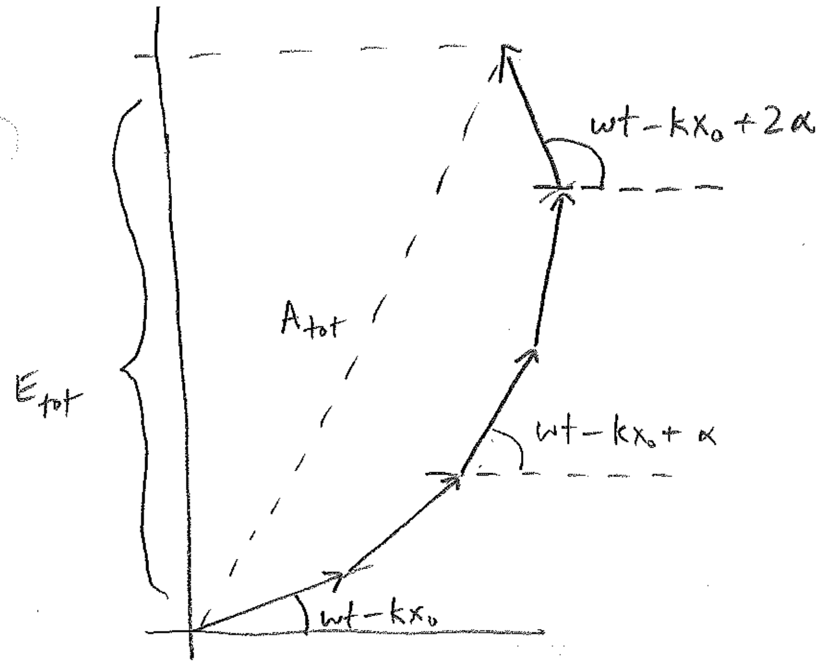
$$A_{tot} = N\left(\frac{A}{N}\right) = A$$



$\Delta\phi = 0$ which requires $\theta = 0$

Maximum of intensity occurs at $\theta = 0$, directly behind slit





Let 2α = phase difference between 1st + last phasors

Then α = phase difference between 1st + "middle" phasor

phasor representing E_{tot} is parallel to middle phasor

$$E_{tot} = A_{tot} \sin(\omega t - kx_0 + \alpha)$$

$$I = \frac{1}{2} \epsilon_0 c A_{tot}^2$$

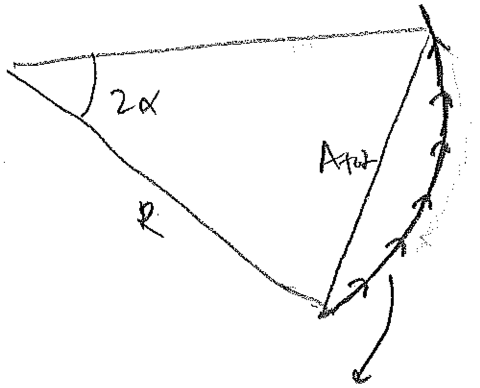
A_{tot} is maximized when all phasors are parallel: $A_{tot} = A$

$$I_{max} = \frac{1}{2} \epsilon_0 c A^2$$

Relative intensity is

$$\frac{I}{I_{max}} = \frac{A_{tot}^2}{A^2}$$

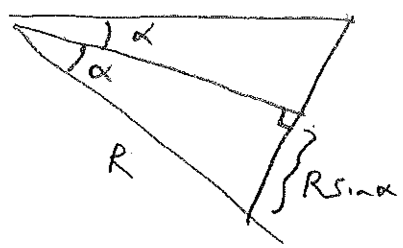
Now we need to find A_{tot} as a function of α



As $N \to \infty$, the phasors lie on an arc of a circle of radius R and angle 2α

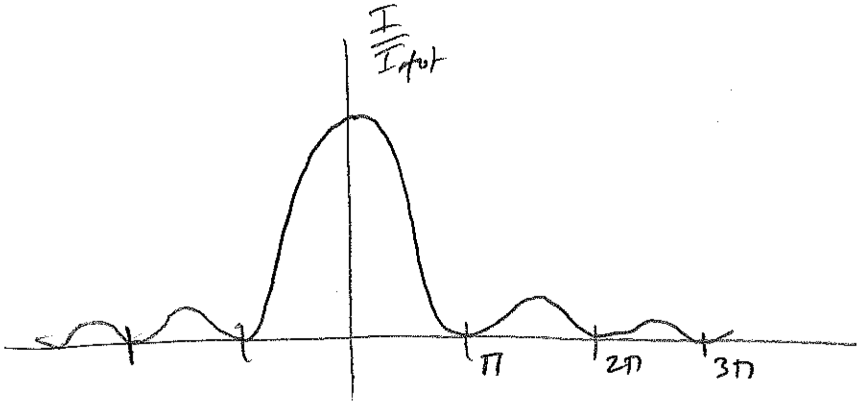
$$\text{arc length} = R \cdot 2\alpha = N \cdot \left(\frac{A}{N}\right) \Rightarrow R = \frac{A}{2\alpha}$$

A_{tot} is a chord on that circle



$$A_{tot} = 2R\sin\alpha = A\,\frac{\sin\alpha}{\alpha}$$

$$\Rightarrow \frac{I}{I_{max}} = \left(\frac{A_{tot}}{A}\right)^2 = \left(\frac{\sin\alpha}{\alpha}\right)^2$$

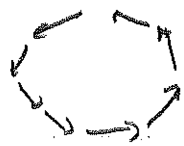


$I = 0$ if $\sin\alpha = 0 \Rightarrow \alpha = m\pi$ (m = nonzero integer)

If $\alpha = 0$ then $\frac{I}{I_{max}} = \lim_{\alpha \to 0}\left(\frac{\sin\alpha}{\alpha}\right)^2 = 1$ (l'Hopital)

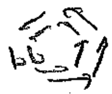
[Recall: small $\alpha \Rightarrow \sin\alpha \approx \alpha$, $\frac{\sin\alpha}{\alpha} \approx 1$.]

Complete destructive interference occurs when phasors form a circle



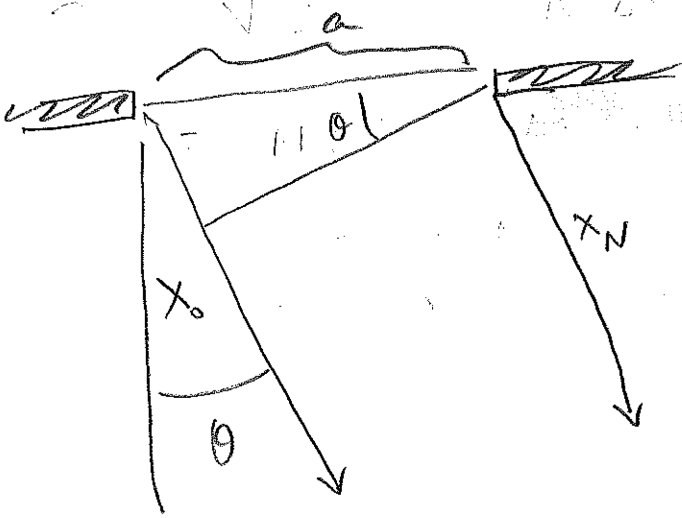
$m=1$

phase difference between last & first phasor is $2\pi m = 2\alpha \Rightarrow \alpha = m\pi$



$m=2$

How is α related to the angle of diffraction θ?

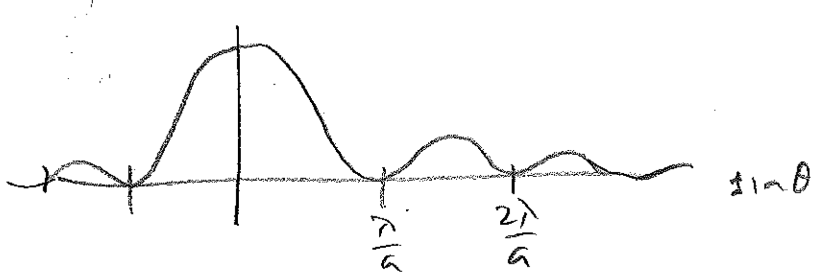


Phase difference between 1st & last phasor

$$2\alpha = k(x_0 - x_N) = \frac{2\pi}{\lambda} a \sin\theta \Rightarrow \alpha = \frac{\pi a \sin\theta}{\lambda}$$

Minima occur when $\alpha = m\pi \Rightarrow a\sin\theta = m\lambda$

m = nonzero integer



Single slit of width a $\Rightarrow \dfrac{I}{I_{max}} = \left(\dfrac{\sin\alpha}{\alpha}\right)^2$

first minimum occurs at $\alpha = \pi$

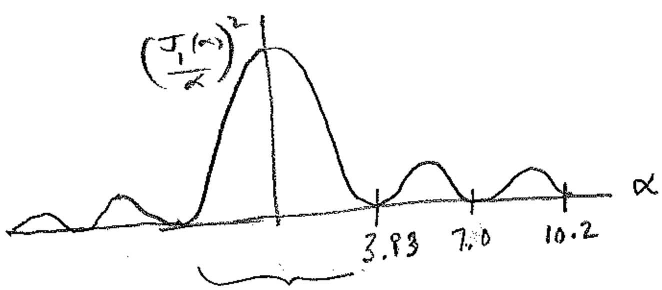
Since $\alpha = \dfrac{\pi a \sin\theta}{\lambda} \Rightarrow \sin\theta = \dfrac{\lambda}{a}$

Circular aperture of diameter a $\Rightarrow \dfrac{I}{I_{max}} = \left(\dfrac{J_1(\alpha)}{\alpha}\right)^2$

J_1 = first Bessel function (similar to $\sin\alpha$)

first minimum occurs at $\alpha = 3.83$

$\Rightarrow \sin\theta = 1.22\,\dfrac{\lambda}{a}$



area inside 1st minimum called "Airy disk"

[OH → QE 10]

story of Fresnel + Poisson spot

[OH → QE 11]

QE10

OH

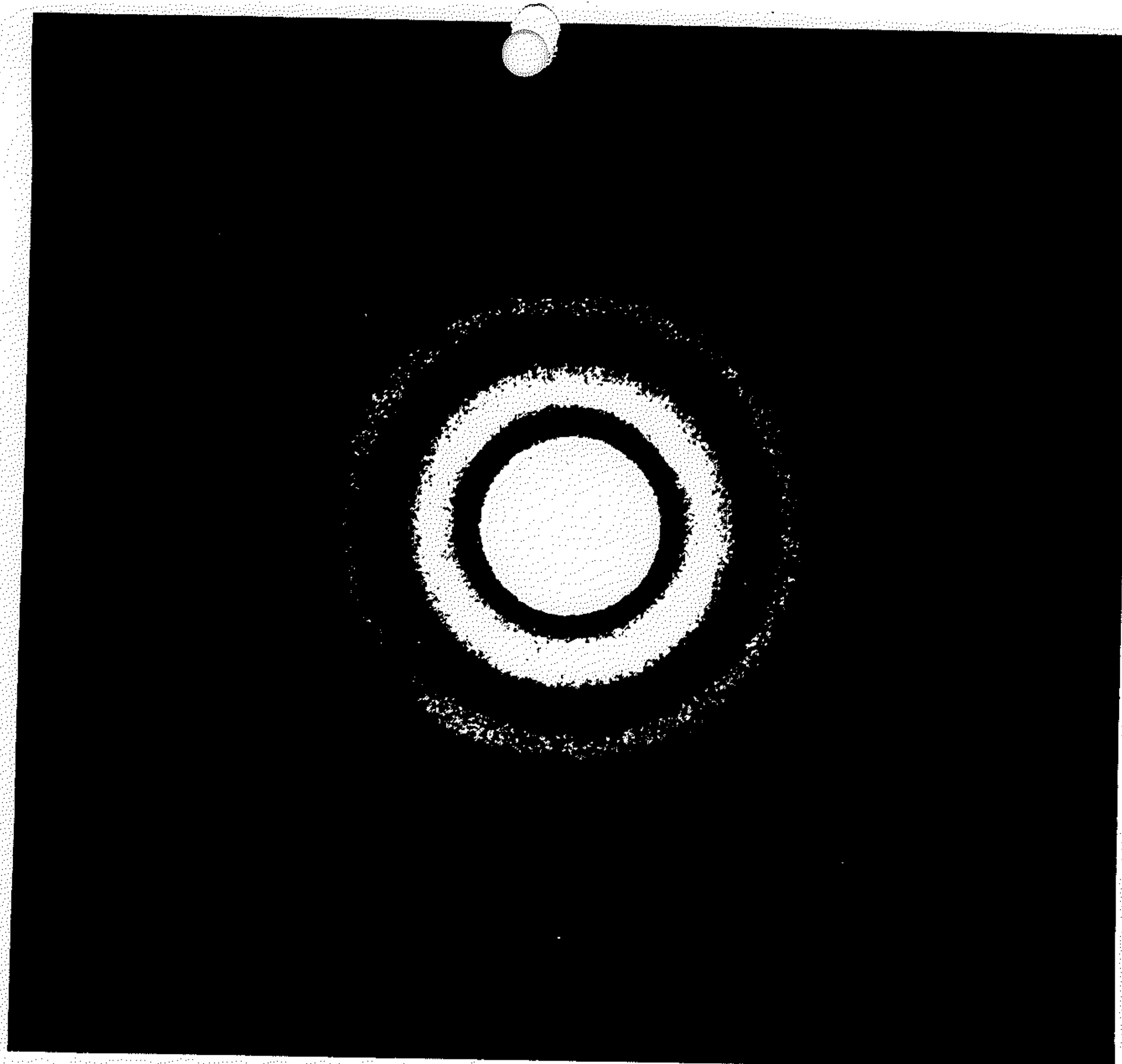


Figure 13 The diffraction pattern of a circular aperture. The central maximum is sometimes called the Airy disk (after Sir George Airy, who first solved the problem of diffraction by a circular aperture in 1835). Note the circular secondary maxima.

HR '96 - fig 13

QE 11

OH

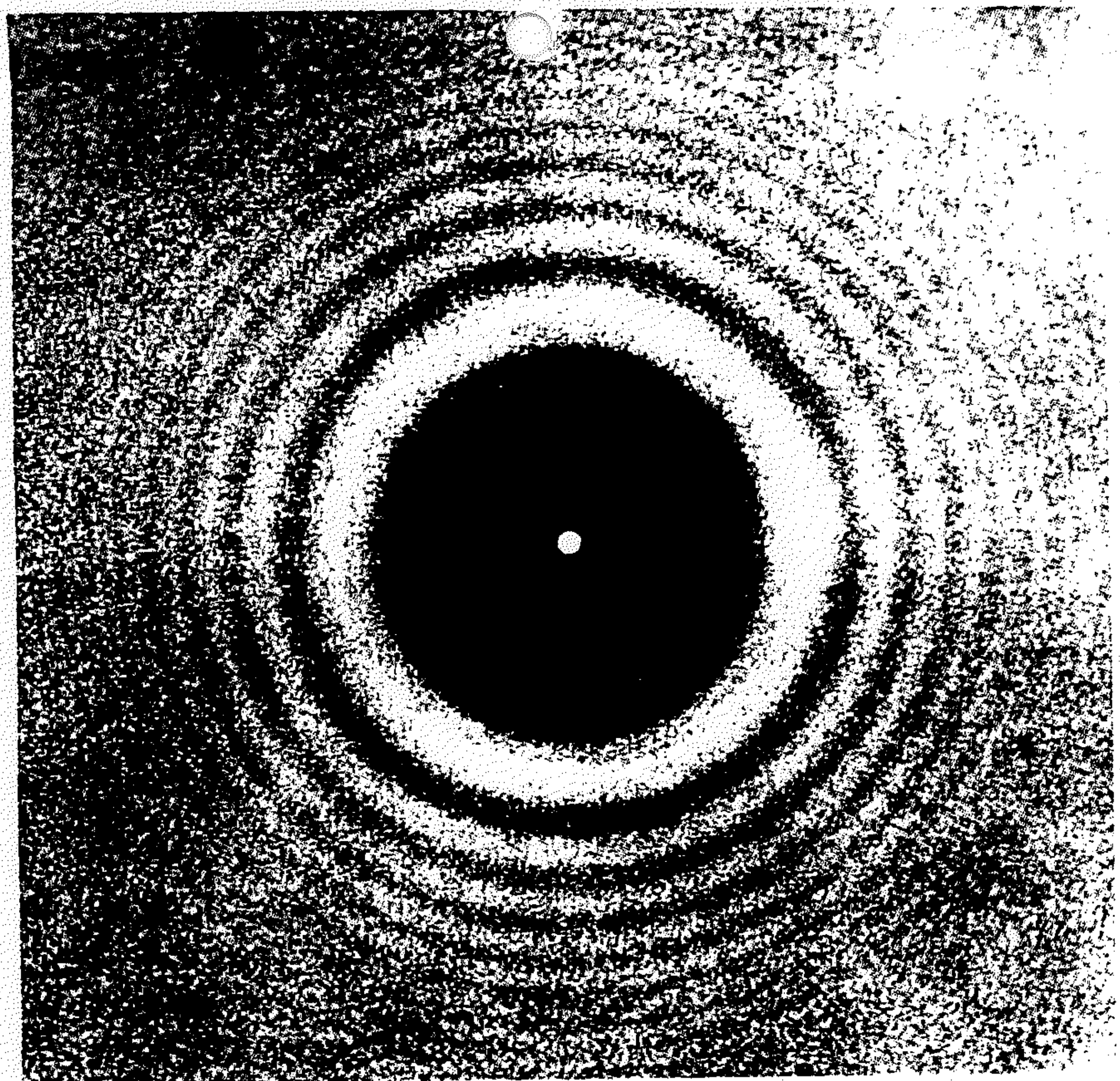
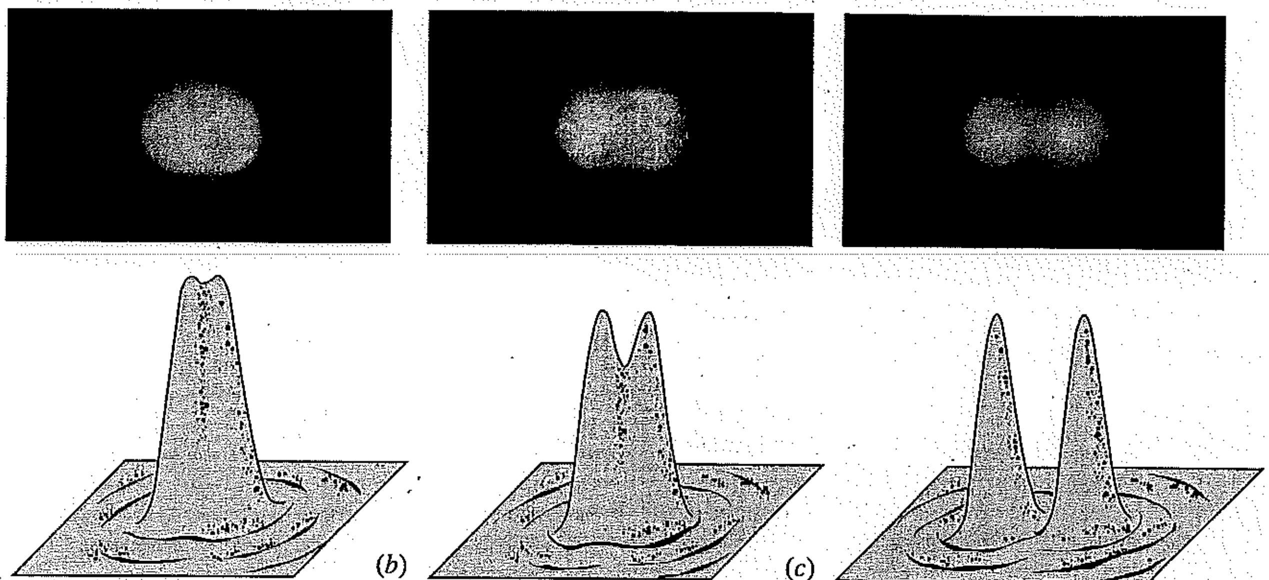


Figure 2 The diffraction pattern of a disk. Note the bright Poisson spot at the center of the pattern.

HR 416, fig 2

QE 12

OM



(a) (b) (c)

FIGURE 42-14. The images of two distant point sources (stars) formed by a converging lens. The diameter of the lens (which is the diffracting aperture) is 10 cm, so that $a/\lambda = 200{,}000$ if the effective wavelength is about 500 nm. In (*a*) the stars are so close together that their images can scarcely be distinguished, due to the overlap of their diffraction patterns. In (*b*) the stars are farther apart and their separation meets Rayleigh's criterion for resolution of their images. In (*c*) the stars are still farther apart and their images are well resolved. Plots of the intensities are shown below the images.