Mathematical description of travelling waves

wave travelling in +X direction

$$F(x,t) = A \sin(kx - \omega t)$$

Definitions:

A = amplitude

k = wavenumber

ω = angular frequency $\left(\frac{\text{radians}}{\text{sec}}\right)$

we will answer 3 questions:

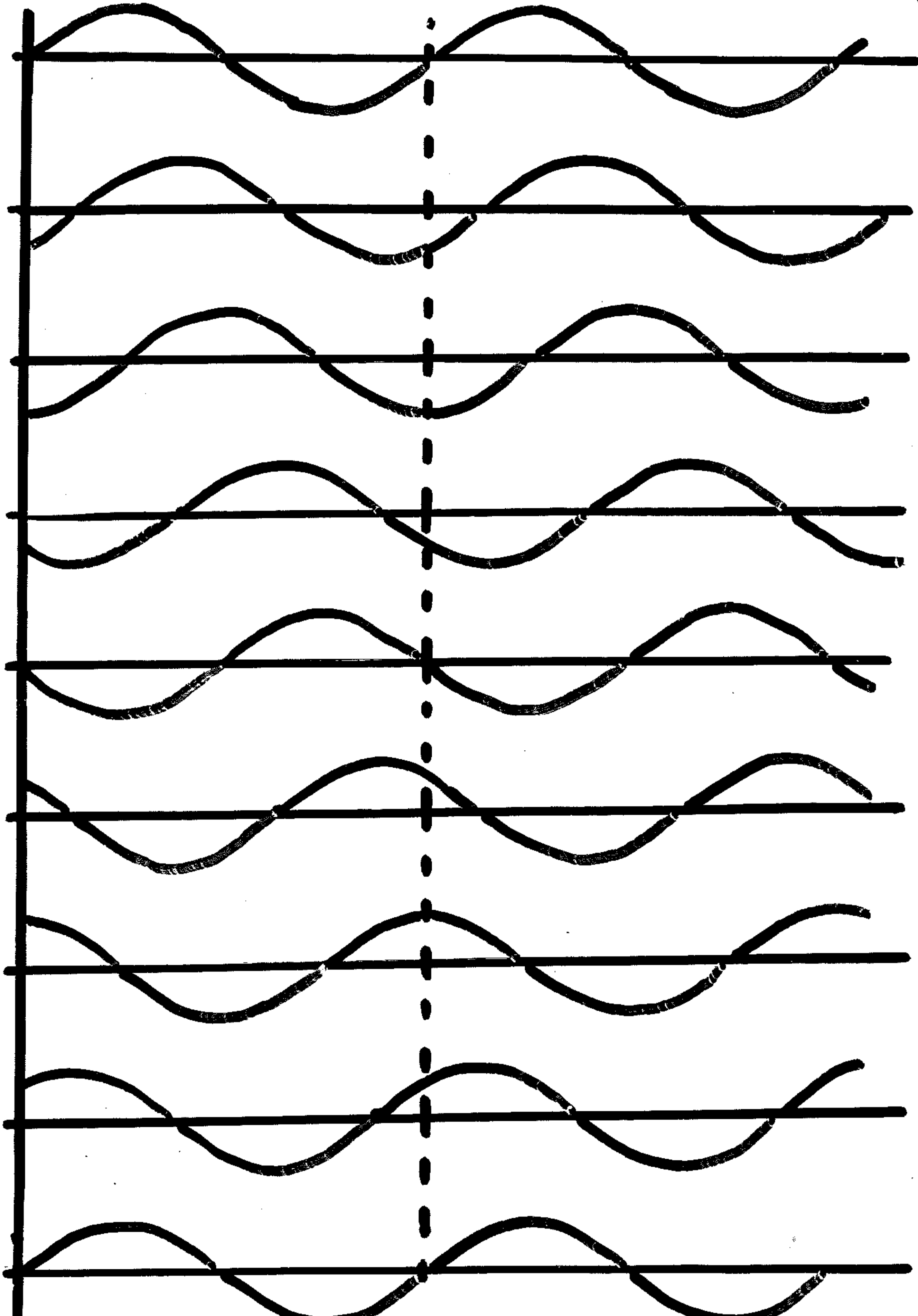
what is the (1) wavelength λ

(2) period T

(3) speed c

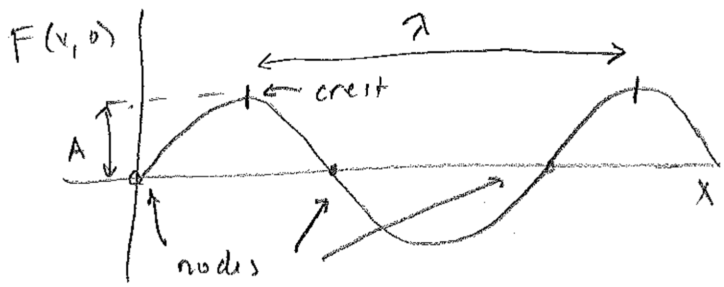
of the wave?

[show transparency of successive snapshots]



Freeze frame of wave at $t = 0$

$$F(x, 0) = A \sin(kx)$$



Wave repeats itself every λ in space

$$\Rightarrow \quad F(x + \lambda, 0) = F(x, 0)$$

$$A \sin(kx + k\lambda) = A \sin(kx)$$

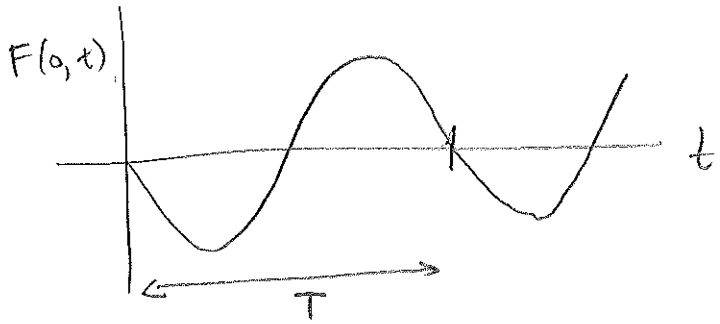
But $\sin\theta = \sin(\theta + 2\pi)$ so

$$\boxed{k\lambda = 2\pi}$$

$$\Rightarrow \quad \lambda = \frac{2\pi}{k} \quad \text{and} \quad k = \frac{2\pi}{\lambda}$$

Examine wave at one location, e.g. $x = 0$

$$F(0,t) = A\sin(0 - \omega t) = -A\sin(\omega t)$$



wave Repeats in time every period T

$$F(0, t+T) = F(0,t)$$

$$-A\sin(\omega t + \omega T) = -A\sin(\omega t)$$

$$\Rightarrow \boxed{\omega T = 2\pi}$$

ω = angular frequency (rad/sec) $= \dfrac{2\pi}{T}$

$$T = \frac{\#\text{ seconds}}{\text{cycle}}$$

f = frequency $= \dfrac{\#\text{ cycles}}{\text{sec.}} = \dfrac{1}{T}$ (units $= s^{-1}$ or Hz)

$\therefore \boxed{\omega = 2\pi f}$ angular frequency not the same as frequency!

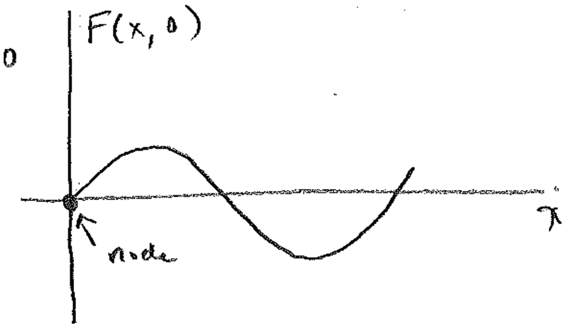
e.g. $f = 60\text{ Hz} = 60\ s^{-1}$ ← same units
$\omega = 377\ \frac{\text{rad}}{\text{sec}} = 377\ s^{-1}$

To measure the phase speed of a wave
(ie speed of each crest, or each node)

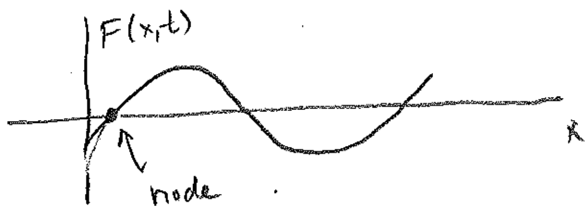
compare two successive snapshots

[show transparency of waves.
draw a line through the nodes
on a *separate* transparency.]

At $t=0$



$x_{node} = 0$ at $t=0$



$$F(x,t) = A \sin(kx - \omega t) = A \sin(k\underbrace{[x - \tfrac{\omega}{k}t]}_{\text{vanishes at the node}})$$

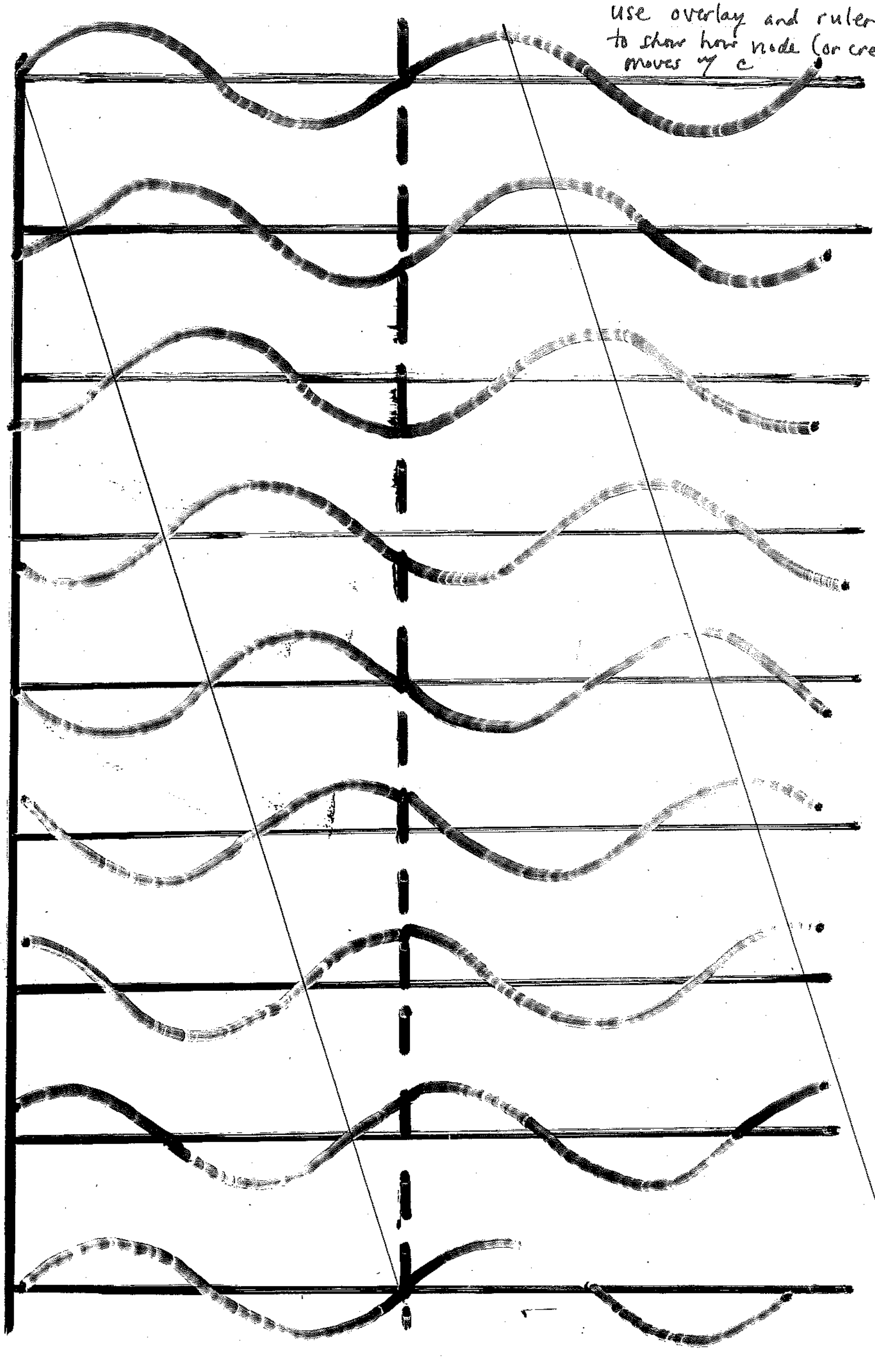
$$x_{node} - \frac{\omega}{k}t = 0$$

$$x_{node} = \left(\frac{\omega}{k}\right)t$$

Node moves with speed $\boxed{c = \frac{\omega}{k}}$

Since $\omega = \frac{2\pi}{T}$ and $k = \frac{2\pi}{\lambda}$, this implies $\boxed{c = \frac{\lambda}{T}}$

Use overlay and ruler
to show how node (or crest)
moves w/ c



Wave vector

Define the wavevector $\vec{k}$ as a vector (k_x, k_y, k_z) whose direction is the direction of wave travel (parallel to $\vec{E} \times \vec{B}$) and whose magnitude is the wavenumber

$$|\vec{k}| = k = \frac{2\pi}{\lambda}$$

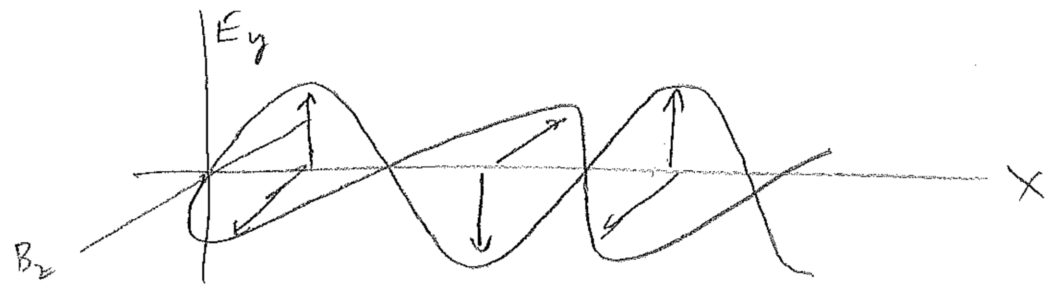
The travelling wave can be described by

$$F = A \sin(\underbrace{\vec{k} \cdot \vec{r}}_{k_x x + k_y y + k_z z} - \omega t)$$

So if wave travels in $+x$ direction

then $\vec{k} = (k, 0, 0)$

so $F = A \sin(kx - \omega t)$ as before electromagnetic wave travelling in +x direction



$$\vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 \\ A \sin(kx - \omega t) \\ 0 \end{pmatrix}$$

$$\vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{A}{c} \sin(kx - \omega t) \end{pmatrix}$$

use $|\vec{B}| = \frac{|\vec{E}|}{c}$ for an EM wave

$\longrightarrow$ [do question]

$$\vec{S} = \epsilon_0 c^2 \vec{E} \times \vec{B} = \begin{pmatrix} \epsilon_0 c A^2 \sin^2(kx - \omega t) \\ 0 \\ 0 \end{pmatrix}$$

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Question:

An electromagnetic wave has a magnetic field

$$\mathbf{B} = ((A/c)\sin(kz - \omega t), 0, 0)$$

What is the corresponding electric field?

a)

$$\mathbf{E} = (0, A\sin(kz + \omega t), 0)$$

b)

$$\mathbf{E} = (0, A\sin(kz - \omega t), 0)$$

c)

$$\mathbf{E} = (0, -A\sin(kz - \omega t), 0)$$

d)

$$\mathbf{E} = (0, 0, A\sin(kz - \omega t))$$

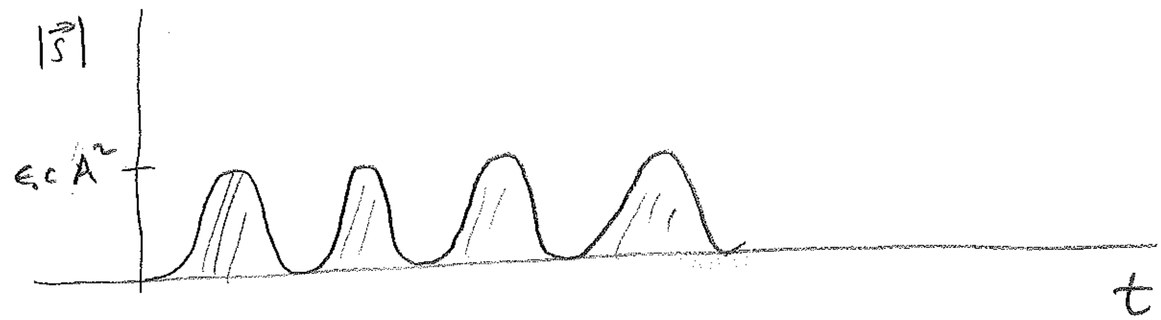
e)

$$\mathbf{E} = ((A/c^2)\sin(kz - \omega t), 0, 0)$$

[ans: c]

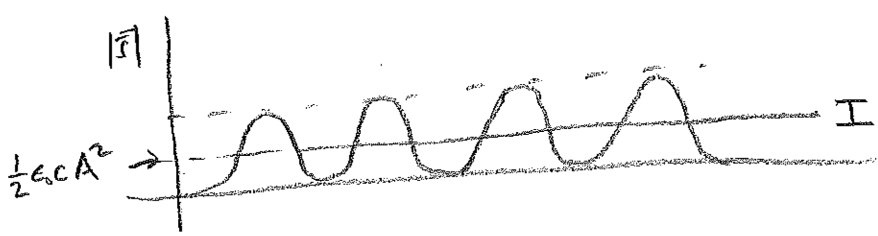
At the origin $(x=0)$, the Poynting vector has magnitude

$$|\vec{S}| = \epsilon_0 c A^2 \sin^2(\omega t)$$



$|\vec{S}|$ oscillates rapidly in time.

A more useful quantity is the intensity I = time avg of $|\vec{S}|$



To compute I, integrate area under $|\vec{S}|$ over one period T
then divide by T

$$I = \frac{1}{T}\int_0^T |\vec{S}|\, dt$$

In HW problem, you will find $I = \frac{1}{2}\epsilon_0 c A^2$