Mathematical description of travelling waves

wave travelling in +x direction

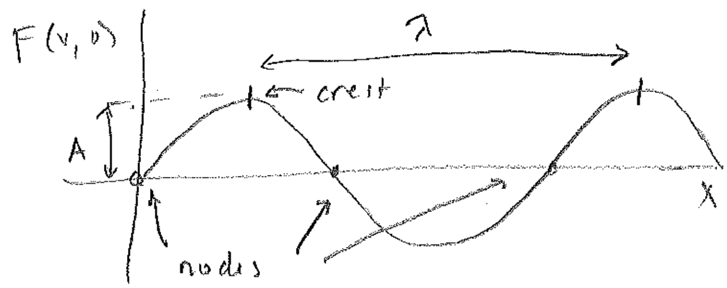
$$F(x,t) = A \sin(kx - \omega t)$$

A = amplitude
k = wavenumber
ω = angular frequency $\left(\frac{\text{radians}}{\text{sec}}\right)$

we will answer 3 questions:
what is the ① wavelength λ
② period T
③ speed c
of the wave?

Freeze frame of wave at $t = 0$

$$F(x, 0) = A \sin(kx)$$



Wave repeats itself every λ in space

$$\Rightarrow \quad F(x+\lambda, 0) = F(x, 0)$$

$$A \sin(kx + k\lambda) = A \sin(kx)$$

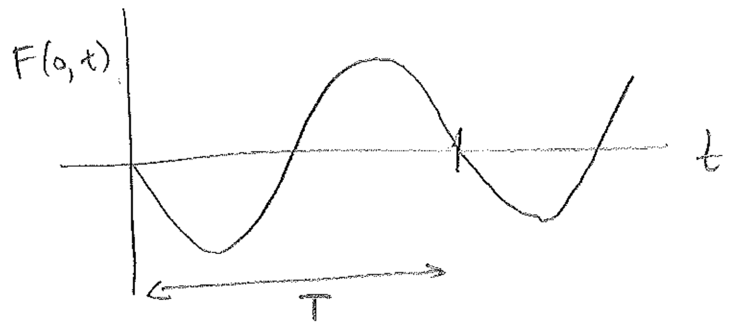
But $\sin\theta = \sin(\theta + 2\pi)$ so

$$\boxed{k\lambda = 2\pi}$$

$$\Rightarrow \quad \lambda = \frac{2\pi}{k} \quad \text{and} \quad k = \frac{2\pi}{\lambda}$$

Examine wave at one location, e.g. $x = 0$

$$F(0,t) = A \sin(0 - \omega t) = -A \sin(\omega t)$$



wave Repeats in time every period T

$$F(0, t+T) = F(0,t)$$

$$-A \sin(\omega t + \omega T) = -A \sin(\omega t)$$

$$\Rightarrow \boxed{\omega T = 2\pi}$$

ω = angular frequency (rad/sec) $= \frac{2\pi}{T}$

$$T = \frac{\#\text{ seconds}}{\text{cycle}}$$

f = frequency $= \frac{\#\text{ cycles}}{\text{sec.}} = \frac{1}{T}$ (units $= s^{-1}$ or Hz)

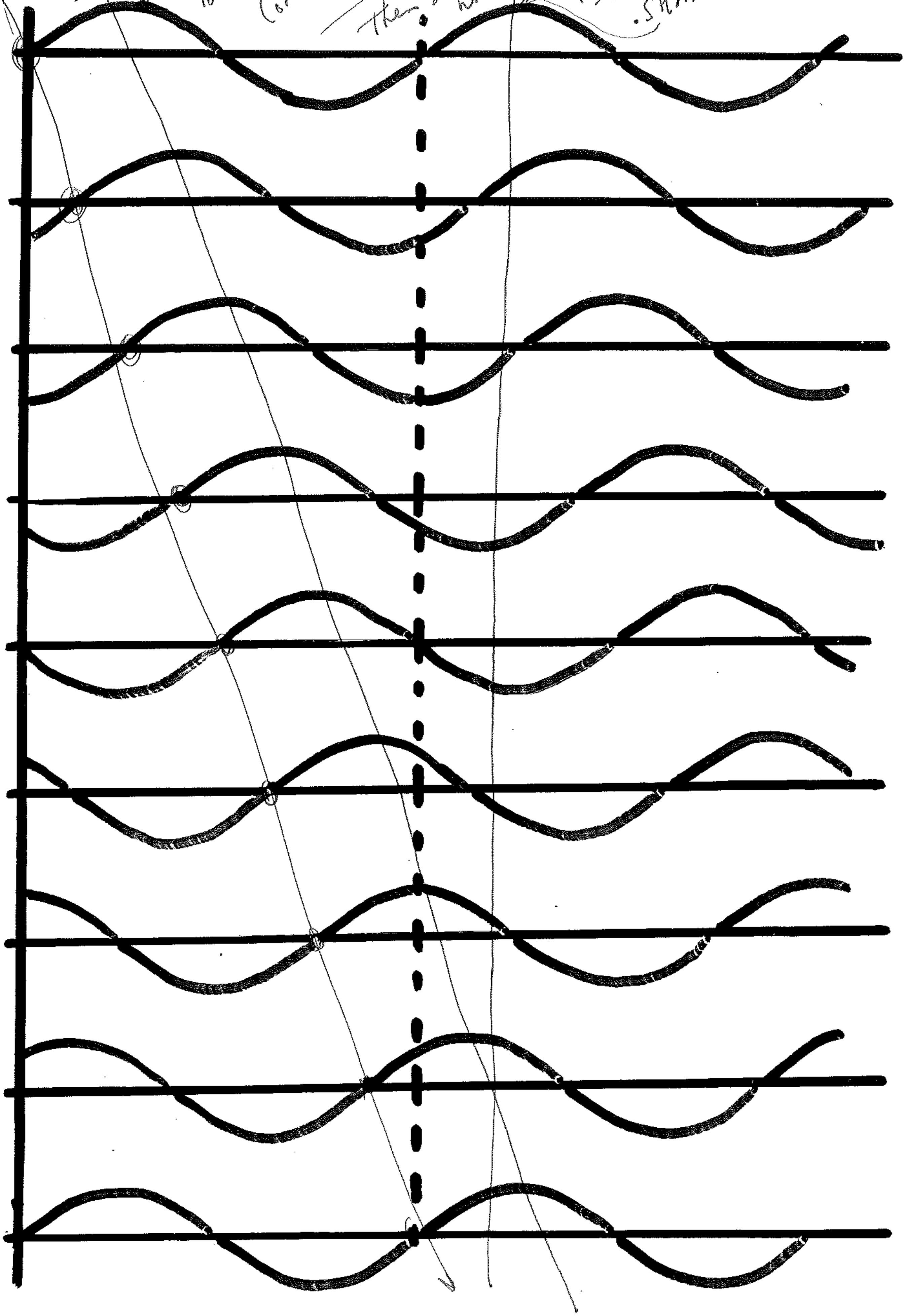
$\therefore \ \omega = 2\pi f$

e.g. $f = 60\text{ Hz} = 60\ s^{-1}$ ⟵ same units

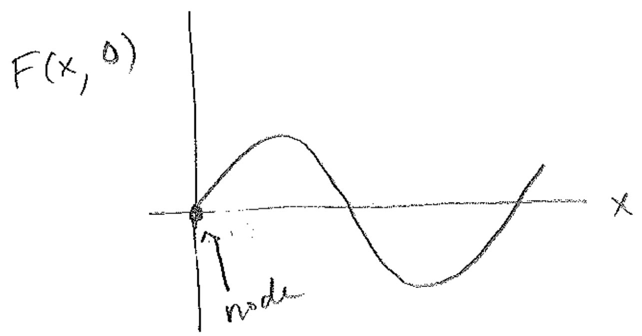
$\omega = 377 \frac{\text{rad}}{\text{sec}} = 377\ s^{-1}$

Using an overlay, show how node moves (or crest)

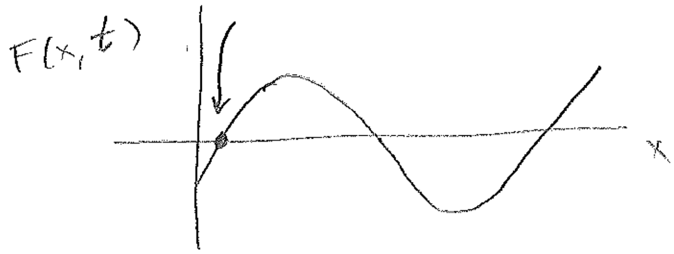
Then show how any single ~~pt~~ executes SHM



To measure the phase speed of the wave
(ie the speed of each crest, or each node)
compare ~~two~~ successive snapshots at time 0 and t
[show transparency]



$x_{node} = 0$ at $t = 0$



$$F(x,t) = A \sin(kx - \omega t) = A \sin\left(k\underbrace{\left[x - \frac{\omega}{k}t\right]}_{\text{vanishes at the node}}\right)$$

$$x_{node} - \frac{\omega}{k}t = 0$$

$$x_{node} = \left(\frac{\omega}{k}\right)t$$

Node moves w/ speed $c = \dfrac{\omega}{k}$

Since $\omega = \dfrac{2\pi}{T}$ and $k = \dfrac{2\pi}{\lambda}$, we can write $c = \dfrac{\lambda}{T}$

Also $f = \dfrac{1}{T}$ so $c = \lambda f$, $f = \dfrac{c}{\lambda}$

[Demo: travelling wave at different freq.]

[sometimes $f \to \nu$ (nu)]

Define the wavevector $\vec{k}$ as a vector (k_x, k_y, k_z) whose direction is the direction of wave travel (parallel to $\vec{E}\times\vec{B}$) and whose magnitude is the wavenumber

$$|\vec{k}| = k = \frac{2\pi}{\lambda}$$

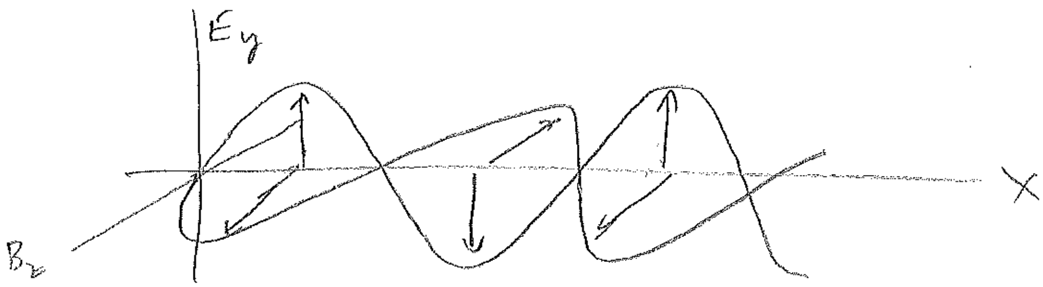
The travelling wave can be described by

$$F = A \sin(\underbrace{\vec{k}\cdot\vec{r}}_{k_x x + k_y y + k_z z} - \omega t)$$

so if wave travels in $+x$ direction

then $\vec{k} = (k, 0, 0)$

so $F = A \sin(kx - \omega t)$ as before electromagnetic wave travelling in +x direction



$$\vec{E} = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \begin{pmatrix} 0 \\ A \sin(kx-\omega t) \\ 0 \end{pmatrix}$$

$$\vec{B} = \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{A}{c} \sin(kx-\omega t) \end{pmatrix}$$

because $|\vec{B}| = \frac{|\vec{E}|}{c}$ for an EM wave.

$$\vec{E} \times \vec{B} = \begin{pmatrix} \frac{A^2}{c} \sin^2(kx-\omega t) \\ 0 \\ 0 \end{pmatrix}$$

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Question:

An electromagnetic wave has a magnetic field

$$\mathbf{B} = ((A/c)\sin(kz - \omega t), 0, 0)$$

What is the corresponding electric field?

a)

$$\mathbf{E} = (0, A\sin(kz + \omega t), 0)$$

b)

$$\mathbf{E} = (0, A\sin(kz - \omega t), 0)$$

c)

$$\mathbf{E} = (0, -A\sin(kz - \omega t), 0) \quad \leftarrow$$

d)

$$\mathbf{E} = (0, 0, A\sin(kz - \omega t))$$

e)

$$\mathbf{E} = ((A/c^2)\sin(kz - \omega t), 0, 0)$$