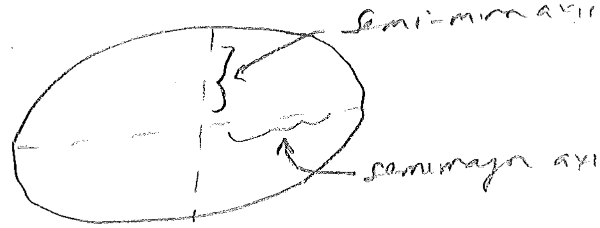
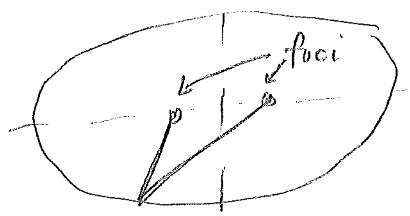
Planets

[after copernicus made clear that
makes more sense to think of planets as orbiting the sun
Kepler's observations over many years showed]

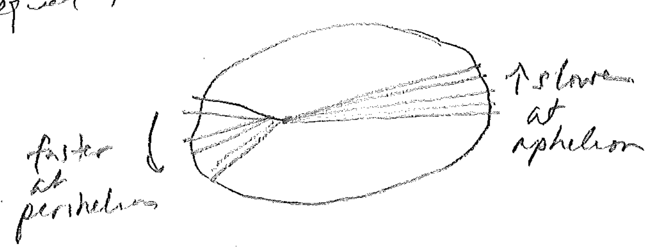
Kepler's laws of planetary motion (1609)

① planets move in elliptical orbits w/ sun at one focus





② a line connecting the sun and planet
sweeps out equal areas in equal time



[Kepler: xkcd cartoon]

③ $(\text{period } T)^2$ is proportional to $(\text{semimajor axis } a)^3$

[all based on empirical observations]


Nice store. How do you keep the floors so clean?
Oh, we hired this dude named Kepler, he's really good. Hard worker. Doesn't mind the monotony. Sweeps out the same area every night.

[Newton realized that gravity acts on apples & planets]

Newton's law of universal gravitation

$V^{grav} = -\frac{GMm}{r}$ $\qquad G = 6.67 \times 10^{-11} \frac{m^3}{kg \cdot s}$

$F_r^{grav} = -\frac{dV}{dr} = -\frac{GMm}{r^2}$

$\vec{F}^{grav} = -\frac{GMm}{r^2}\hat{r}$

Consider circular orbits of radius R

$\Rightarrow$ gravity is centripetal $\Rightarrow$ uniform circular motion

no other force acting

$\Rightarrow \vec{a} = -\frac{v^2}{R}\hat{r}$

$\vec{F}^{net} = m\vec{a} = \vec{F}^{grav}$

$-\frac{mv^2}{R}\hat{r} = -\frac{GMm}{R^2}\hat{r}$

$v^2 = \frac{GM}{R}$

easy [N12.T4] A

$V = \text{const} = \frac{2\pi R}{T}$ so

$$\left(\frac{2\pi R}{T}\right)^2 = \frac{GM}{R}$$

$$\frac{4\pi^2}{GM} R^3 = T^2$$

Since for circular orbit $a = R \Rightarrow$ Kepler's 3rd Law

$$T^2 = \left(\frac{4\pi^2}{GM}\right) a^3$$

easy [N12. T6]$_B$

harder [N12. T7]
↓
discuss.
↳ depends on M, mass of primary

[Use Kepler's 3rd law to determine mass]

$$M = \frac{4\pi^2 R^3}{GT^2}$$

sun

$R_{earth-sun} = 1.5 \times 10^8$ km $= 1.5 \times 10^{11}$ m $= 8.3$ lt min. $= 1$ AU

$T = 1$ yr $= 3.1 \times 10^7$ s

$\Rightarrow M_\odot = 2 \times 10^{30}$ kg

earth

$R_{earth-moon} = 384{,}000$ km $= 3.8 \times 10^8$ m $= 1.25$ ls

$T = 27$ days $= 2.4 \times 10^6$ sec

$M_\oplus = 6.0 \times 10^{24}$ kg

[show video clip of Sagittarius A] → highly elliptical

↳ near center of galaxy
25,000 ly from sun

$\left\{ \begin{array}{l} a = 5.5 \text{ light days} = 950 \text{ AU} \\ T = 15 \text{ years} \end{array} \right.$

$$M_\odot = \frac{4\pi^2}{G} \frac{(1 \text{ AU})^3}{(1 \text{ year})^2}$$

$$M_? = \frac{4\pi^2}{G} \frac{(950 \text{ AU})^3}{(15 \text{ yrs})^2}$$

$$\frac{M_?}{M_\odot} = \frac{(950)^3}{(15)^2} = 3.8 \times 10^6$$

Super massive BH

geosynchronous orbit $R \sim 6.6 R_\oplus$

<u>Nasa site</u>
JTrack 3D to see 900 satellites in earth orbit

including a lot at geosynchronous radius..

if unavailable

k: nasaback/ satellites.jpg

N12-5T

$v = \frac{200\,km}{s} = \frac{2E5\,m}{s}$

$a = 170{,}000\ ly = 1.6E21\ m$

$\Rightarrow\ m = \frac{v^2 r}{G}$

$= \frac{(2E5)^2(1.6E21)}{(6.67E{-11})}$

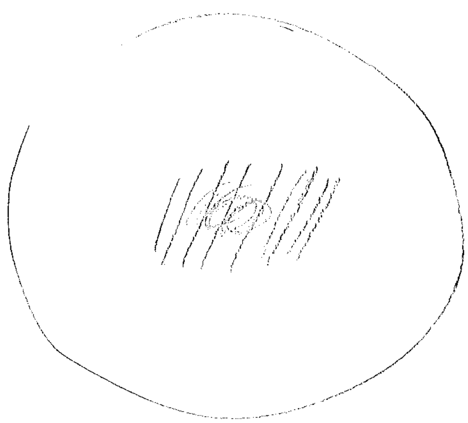
$= \cancel{9.5E41\ kg}$

$\cancel{= E42}$

$= 1.0\ E42\ kg$

= 500 billion solar mass

but visible ~~matter~~ matter
100 billion solar masses

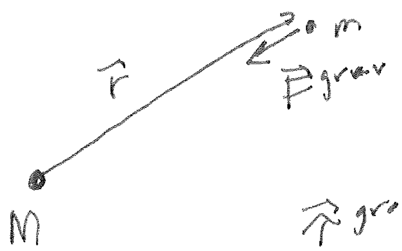


5% visible (protons, neutrons ...)
25% dark matter ← clustered

75% dark energy
↳ causes acceleration of universe

Central forces conserve angular momentum

The gravitational force is centripetal
& ∴ the torque w.r.t. center is zero



$$\vec{\tau}^{grav} = \vec{r} \times \vec{F}^{grav} = 0$$

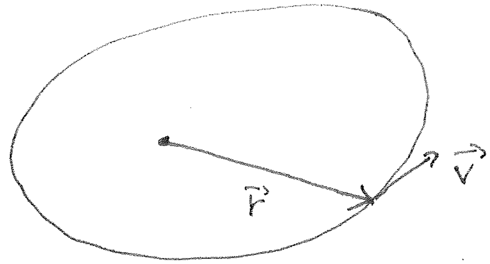
If the only force acting is gravitational the net torque is zero

$$\vec{\tau}^{net} = 0$$

But $\frac{d\vec{L}}{dt} = \vec{\tau}^{net}$ so $\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{const}$

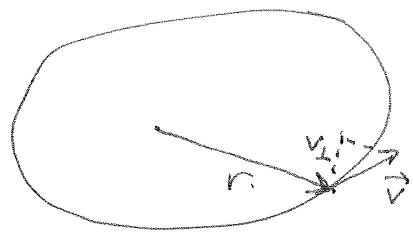
angular momentum is conserved

Planet in an elliptical orbit

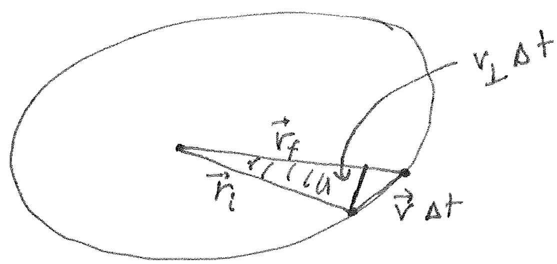


$\vec{L} = \vec{r} \times m\vec{v}$

$L = r m v_\perp$ where $v_\perp$ = component of $\vec{v}$ $\perp$ to $\vec{r}$



Motion diagram



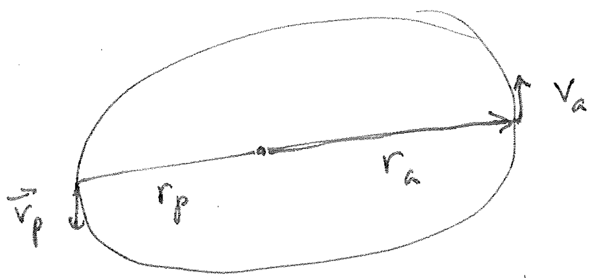
What is area of triangle? $\Delta A = \frac{1}{2} r\, v_\perp\, \Delta t$

What is area swept out per unit time

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r v_\perp = \frac{L}{2m} = \text{const, because } L \text{ conserved}$$

Equal areas swept out in equal times (Kepler 2nd)
is a consequence of cons of $\vec{L}$

Compare speeds at perihelion & aphelion :



$$L = r_a m v_a = r_p m v_p$$

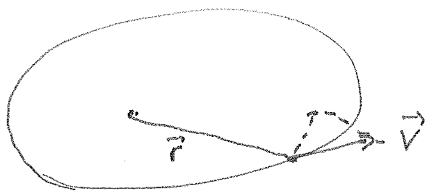
$$v_p = \left(\frac{r_a}{r_p}\right) v_a$$

Planetary motion from an energy perspective

$$K + V = E$$

$$V = -\frac{GMm}{r}$$

$$K = \frac{1}{2} m v^2$$



$$\vec{v} = v_r \hat{r} + v_\theta \hat{\theta}$$

$$= \frac{dr}{dt} \hat{r} + r \frac{d\theta}{dt} \hat{\theta}$$

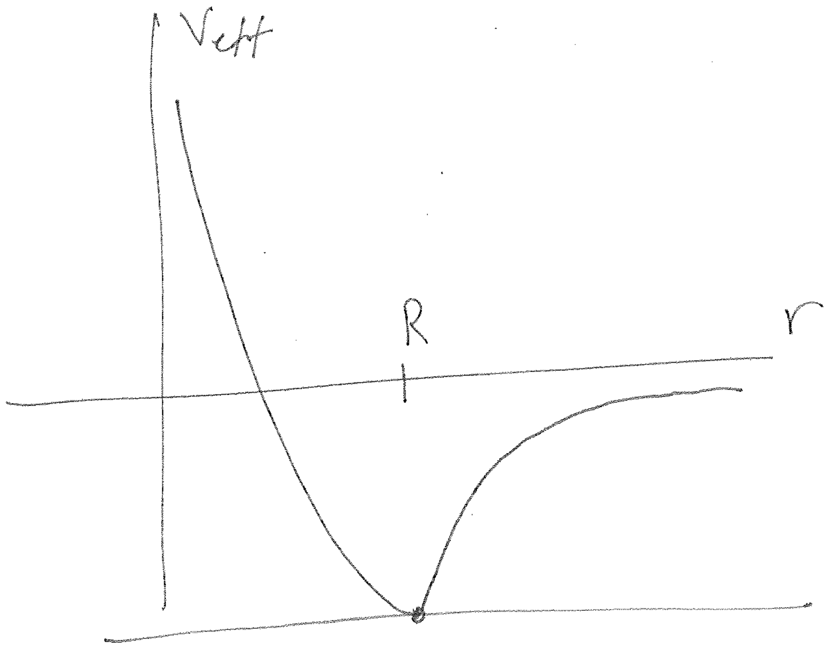
$$v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r \frac{d\theta}{dt}\right)^2$$

But $v_\theta = v_\perp = \frac{L}{mr}$ so

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \frac{L^2}{m^2 r^2}$$

so

$$E = \underbrace{\frac{1}{2} m \left(\frac{dr}{dt}\right)^2}_{\text{"radial" kinetic energy}} + \underbrace{\frac{L^2}{2mr^2} - \frac{GMm}{r}}_{\text{effective potential } V_{eff}}$$



let $r=R$ be minimum of effective potential

$$\left[\frac{dV_{eff}}{dr}=0 \quad \Rightarrow \quad \frac{L^2}{mR^3}=\frac{GMm}{R^2}\right]$$

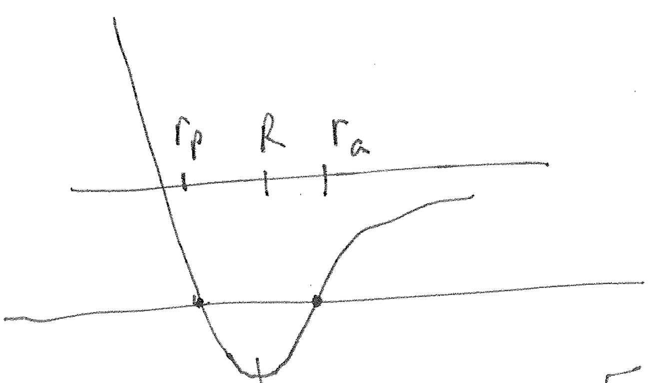
~~Suppose orbit~~

Suppose $E = V_{eff}(R)$

then $\frac{dr}{dt}=0$ ie $r=R$ (const)

circular orbit

Suppose $E > V_{eff}(R)$: planet goes between min radius r_p + max radius r_a + back again $\Rightarrow$ ellipse

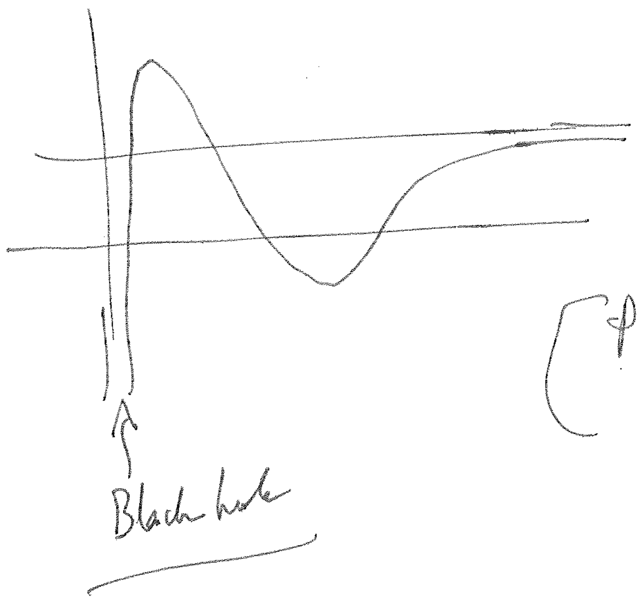


One can show period of oscillation = period of revolution

$\Rightarrow$ closed orbits

[~~GR $\Rightarrow$ not quite closed~~ $\Rightarrow$

GR: additional term in V_{eff}



· oscillation period is slightly different

$\Rightarrow$ orbits not closed

[precession of perihelion of mercury]

↳ several arc-sec per century

$$T = \frac{2\pi R}{v} = \frac{2\pi}{\sqrt{\frac{4\pi}{3} G\rho}} = \sqrt{\frac{3\pi}{G\rho}}$$

$= 5300\,s = 1.5$ hrs $\leftarrow$ for $\rho = 5000 \frac{kg}{m^3}$

N12T.1 Kepler's second law implies that as a planet's distance from the sun *increases* in an elliptical orbit, its orbital speed

A. Increases.
B. Decreases.
C. Remains the same.
D. Changes in a way we cannot determine.

N12T.2 The sun's mass is about 1000 times that of Jupiter, and the radius of Jupiter's orbit is about 1100 times the sun's radius. The center of mass of the sun/Jupiter system is inside the sun, true (T) or false (F)?

N12T.3 Two stars, one with radius r and the other with radius $3r$, orbit each other so that their centers of mass are $25r$ apart. Assume that the stars have the same uniform density. The center of mass of this system is inside the larger star, T or F?

N12T.4 The speed of a satellite in a circular orbit of radius R around the earth is 3.0 km/s. The speed of another satellite in a different circular orbit around the earth is one-half this value. What is the radius of that satellite's orbit?

A. $4R$
B. $2R$
C. $\sqrt{2}R$
D. $R/\sqrt{2}$
E. $R/2$
F. $R/4$
T. Some other multiple of R (specify)

N12T.5 A satellite orbits the earth once every 2.0 h. What is the orbital period of another satellite whose orbital radius is 4.0 times larger?

A. 4.0 h
B. 8.0 h
C. 16 h
D. 64 h
E. Some other period (specify)

N12T.6 The radius of Saturn's orbit is 9.53 times that of the earth. What is the period of Saturn's orbit (assuming that it is nearly circular)?

A. 9.53 y
B. 29 y
C. 91 y
D. 866 y
E. Some other period (specify)

N12T.7 The radius of the earth's (almost circular) orbit around the sun is 150,000,000 km, and it takes 1 y for the earth to go around the sun. Imagine that a certain satellite goes in an almost circular orbit of radius 15,000 km around the earth (this radius is 10,000 times smaller than the earth's orbital radius around the sun). What is the period of this orbit?

A. 10^{-6} y
B. 10^{-4} y
C. 10^{-3} y
D. 10^{6} y
E. These periods are not related in any simple way.