CSCI 3210: Computational Game Theory

Market Equilibria: An Algorithmic Perspective

Ref: Ch 5 [AGT]
Ch 7 [Kleinberg-Tardos]

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Many of the slides are adapted from Vazirani's and Kleinberg-Tardos' textbooks.

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Market

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Study of markets

- **General equilibrium (GE) theory**
  - Seeks to explain the behavior of supply, demand and prices in an economy
  - Partial equilibrium vs GE

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Competitive equilibrium (CE)

- AKA Walrasian equilibrium
  - Formal mathematical modeling of markets by Leon Walras (1874)
- CE consists of prices and allocations
- Equilibrium pricing: demand = supply
- GE $\Longrightarrow$ CE, but CE $\not\Rightarrow$ GE

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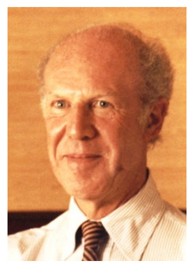
Background

- Good news
  - CE **exists** in Walrasian economy
  - Proved by Arrow and Debreu (1954)
- Bad news
  - Existence proof is not algorithmic

Arrow



Debreu

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Background

- 1st Welfare Theorem
  - Any **CE** (Walrasian equilibrium) leads to a "***pareto optimal***" allocation of resources

Nobody can be better off without making somebody else worse off

- Social justification
  - Let the competitive market do the work (everybody pursuing self-interest)
  - It will lead to pareto optimality (socially maximal benefit)

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Timeline

- 1954 – 2001
  - We are happy. Equilibrium exists.
    Why bother about computation?
  - Sporadic computational results
    - Eisenberg-Gail convex program, 1959
    - Scarf's computation of approximate fixed point, 1973
    - Nenakov-Primak convex program, 1983

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Today's markets

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Electronic marketplaces

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Need for algorithms

- New types of markets
  - The internet market
  - Massive computational power available
  - Need to "compute" equilibrium prices
- Effects of
  - Technological advances
  - New goods
  - Changes in the tax structure
- Deng, Papadimitriou and Safra (2002)- Complexity of finding an equilibrium; polynomial time algorithm for linear utility case
- Devanur, Papadimitriou, Saberi, Vazirani (2002) – polynomial time algorithm for Fisher's linear case

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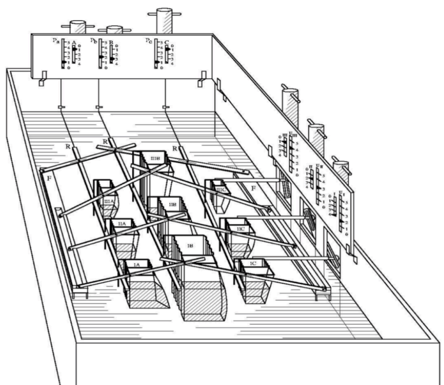
Fisher economy

- Irving Fisher (1891)
  - Mathematical model of a market



Fisher's apparatus to compute equilibrium prices

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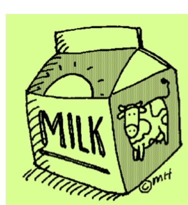
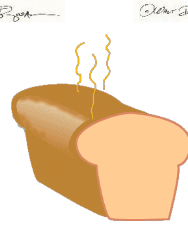
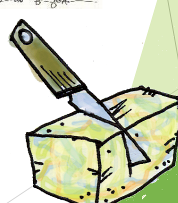
Fisher economy







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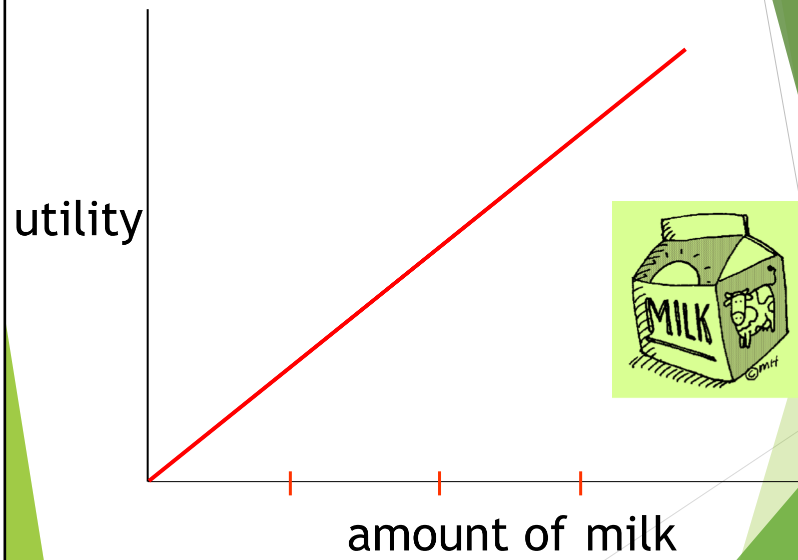
Utility function



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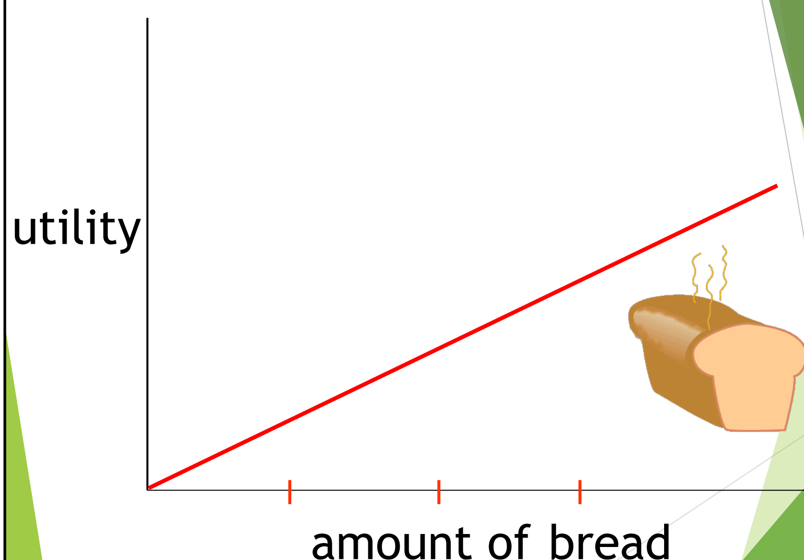
Utility function



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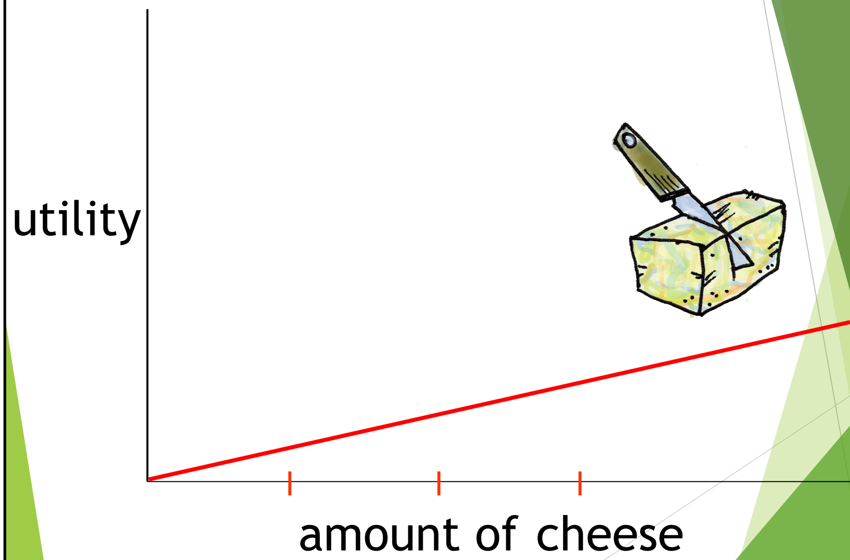
Utility function



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Total utility

- Total utility of a “bundle” of goods
  = Sum of the utilities of individual goods

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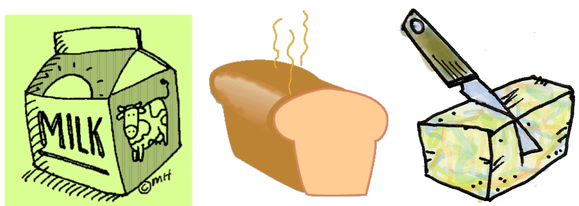
Easy problem

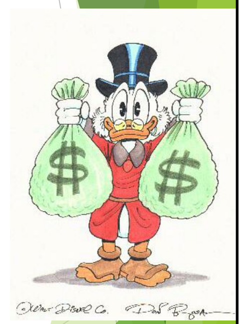
- Prices given
- What would be the optimal bundle of goods for a buyer?

Bang-per-buck (BPB)

Example: $u_2/p_2 > u_1/p_1 > u_3/p_3$



p_1 p_2 p_3



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Fisher market – setup

- Multiple buyers, with individual budgets and utilities
- Multiple goods, fixed amount of each good
- Equilibrium/market-clearing prices
  - Each buyer maximizes utility at these prices
    - Buyers will exhaust their budgets
  - No excess demand or supply

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Fisher's linear case

- Model parameters (what's given)
  - ***n divisible goods*** (1 unit each wlog) and ***n' buyers***
  - e_i = buyer *i*'s budget (integral wlog)
  - u_{ij} = buyer *i*'s utility per unit of good *j* (integral wlog)
  - *Linear* utility functions
- Want (not given): equilibrium allocations
  - x_{ij} = amount of good *j* that *i* buys to maximize

    his/her utility $u_i(\mathbf{x}) = \sum_{j=1}^{n} u_{ij} x_{ij}$

    - No excess demand or supply

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Dual (proof later)

- Want (not given): equilibrium/market-clearing prices
- Prices: $p_1, p_2, \ldots, p_n$
  - After each buyer is assigned an optimal basket of goods (x_{ij}'s) w.r.t. these prices, there's no excess demand or supply
  - x_{ij}'s at these prices: equilibrium/market-clearing allocations

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Can we formulate an optimization routine?

- Does LP work?
- Anything else?

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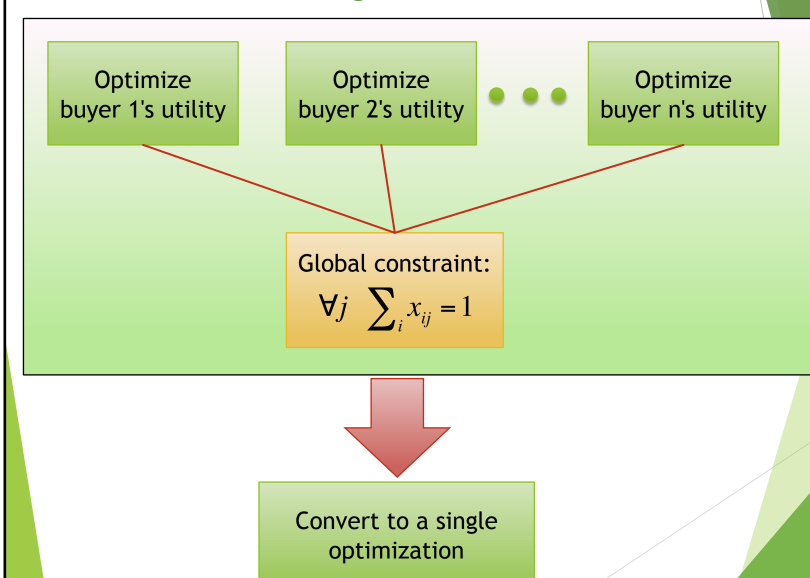
Main challenge



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Eisenberg-Gale Formulation of Fisher Market

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How to devise duals of nonlinear programs?

Lagrange function

KKT conditions

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Eisenberg-Gale convex program (1959)

- Equilibrium allocations captured as
  - Optimal solutions to the Eisenberg-Gale convex program
- Objective function
  - Money weighted geometric mean of buyers' utilities

$$\max\left(\prod_i u_i^{e_i}\right)^{1/\sum_i e_i} \Leftrightarrow \max\left(\prod_i u_i^{e_i}\right) \Leftrightarrow \max \sum_i e_i \log u_i$$

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Eisenberg-Gale convex program

$$\max \sum_i e_i \log\left(\sum_j u_{ij}\, x_{ij}\right)$$

$$subject\ to$$

$$\sum_i x_{ij} \leq 1, \forall j$$

$$x_{ij} \geq 0, \forall i, j$$

- Lagrange function

$$L(x,\lambda,\mu) = -\sum_i e_i \log \sum_j u_{ij} x_{ij} + \sum_j \lambda_j \left(\sum_i x_{ij} - 1\right) + \sum_i \sum_j \mu_{ij}(-x_{ij})$$

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KKT conditions

- Stationary condition

$$\frac{e_i u_{ij}}{\sum_j u_{ij} x_{ij}^*} = \lambda_j^* - \mu_{ij}^* \qquad (1)$$

$$\frac{u_{ij}}{\lambda_j^*} \leq \frac{\sum_j u_{ij} x_{ij}^*}{e_i}$$

- Primal feasibility

$$\sum_i x_{ij}^* \leq 1, \forall j$$
$$x_{ij}^* \geq 0, \forall i, j$$

- Dual feasibility

$$\lambda_i^*, \mu_{ij}^* \geq 0, \forall i, j$$

- Complementary slackness

$$\lambda_j^* \left(\sum_i x_{ij}^* - 1 \right) = 0 \quad \Leftrightarrow \quad \lambda_j^* > 0 \Rightarrow \sum_i x_{ij}^* = 1$$
$$\mu_{ij}^* (-x_{ij}^*) = 0 \quad \Leftrightarrow \quad x_{ij}^* > 0 \Rightarrow \mu_{ij}^* = 0$$

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Does Eisenberg-Gail convex program work for Fisher market?

- Prove: There exist market-clearing prices **iff** each good has some interested buyer (someone who gets positive utility for that good)

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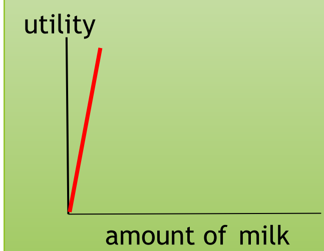
Example

- 2 buyers, 1 good (1 unit of milk)

Buyer 1

Budget, e_1 = \$100

u_{11} = 10/unit of milk

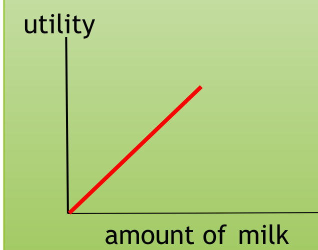


x^*_{11} = ?

Buyer 2

Budget, e_2 = \$50

u_{21} = 1/unit of milk



x^*_{21} = ?

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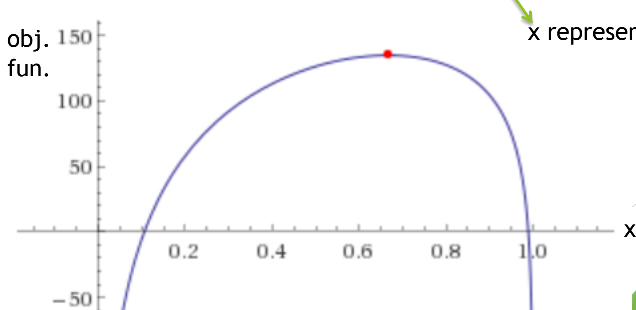
Solution

- $x_{11} = 2/3$, $x_{21} = 1/3$

maximize	function	$100 \log(10x) + 50 \log(1 - x)$
	domain	$0 \leq x \leq 1$

x represents x_{11}



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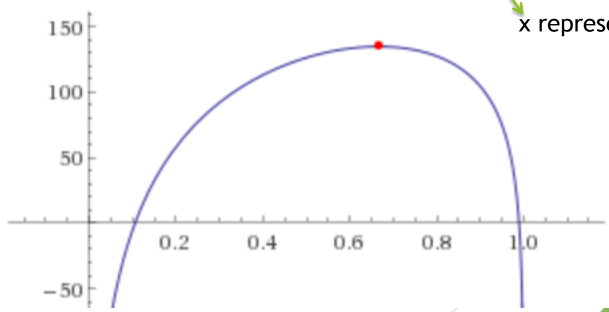
Solution

- Why $x_{11} = 2/3$, $x_{21} = 1/3$?
- Set price of milk = \$150/unit

maximize	function	$100 \log(10\,x) + 50 \log(1 - x)$
	domain	$0 \le x \le 1$

x represents x_{11}



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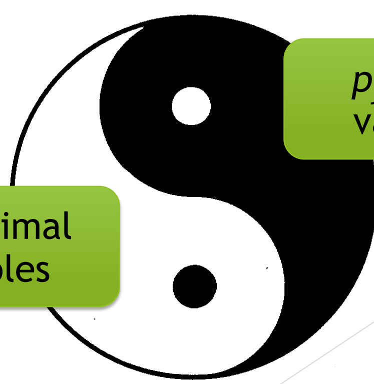
Primal-dual

- p_j
  = The price of good j at an equilibrium
  = Dual variable corresponding to the primal constraint for good j: $\sum_i x_{ij} \le 1$



p_j's: dual variables

x_{ij}'s: primal variables

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Interesting properties

- The set of equilibria is convex
- Equilibrium prices are unique!
- All entries rational => equilibrium allocations and prices rational

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Flow

Max Flow & Min Cut
Ref: Ch 7 of Kleinberg-Tardos

Slides adapted from the Algorithm Design textbook slides
[Kleinberg, Tardos, K. Wayne, P. Kumar]

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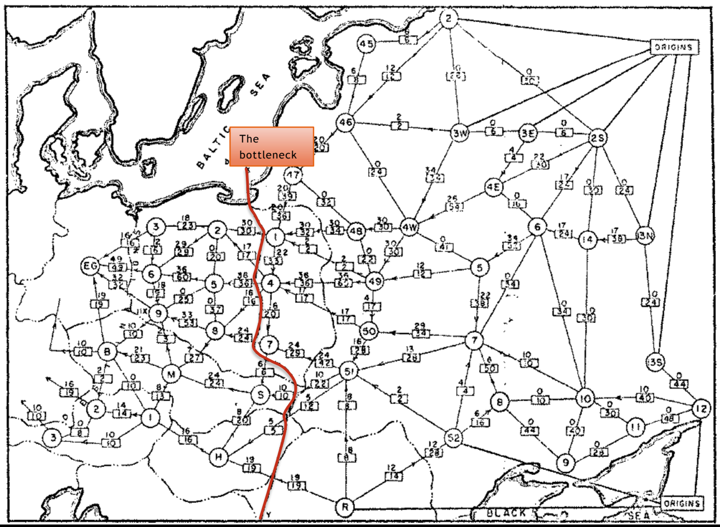
History: Schrijver (2002)

- http://homepages.cwi.nl/~lex/files/histtrpclean.pdf
- Soviet rail network: Harris and Ross [1955] (declassified 1999)



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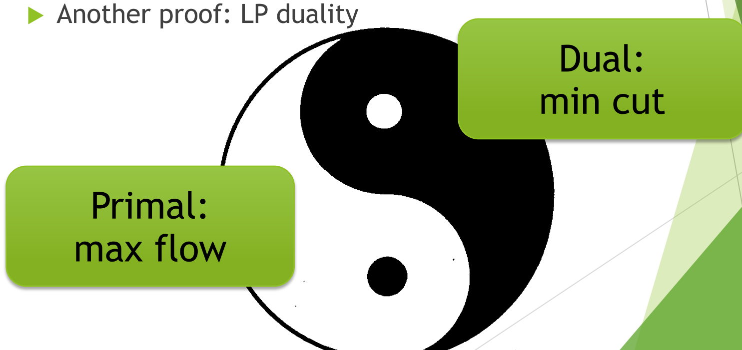
Big picture

- Tolstoi (1930): Find max flow
- Harris & Ross (1955): Find min cut
- Ford & Fulkerson (1956): They are the same
  - Their proof: combinatorial
  - Another proof: LP duality



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Applications

Network	Nodes	Arcs	Flow
communication	telephone exchanges, computers, satellites	cables, fiber optics, microwave relays	voice, video, packets
circuits	gates, registers, processors	wires	current
mechanical	joints	rods, beams, springs	heat, energy
hydraulic	reservoirs, pumping stations, lakes	pipelines	fluid, oil
financial	stocks, currency	transactions	money
transportation	airports, rail yards, street intersections	highways, railbeds, airway routes	freight, vehicles, passengers
chemical	sites	bonds	energy

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Applications

- **Fisher market**
- Network connectivity
- Bipartite matching
- Data mining
- Open-pit mining
- Airline scheduling
- Image processing
- Project selection
- Baseball elimination
- Network reliability
- Security of statistical data
- Distributed computing
- Egalitarian stable matching
- Distributed computing
- Many many more . . .

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Max flow problem

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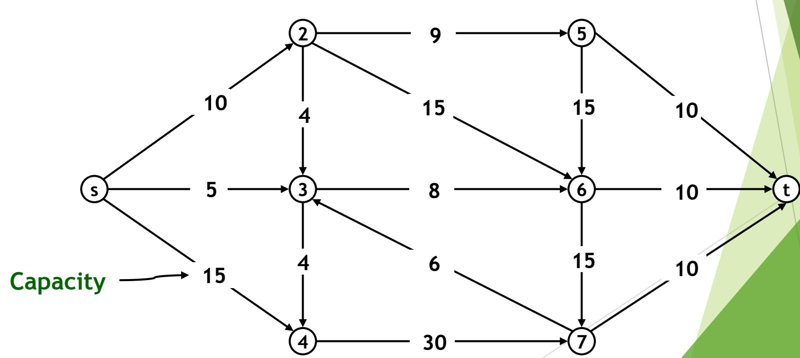
Max flow network

- Directed graph (may have cycles)
- Two distinguished nodes: s = source, t = sink
- c(e) = capacity of arc e (integer)



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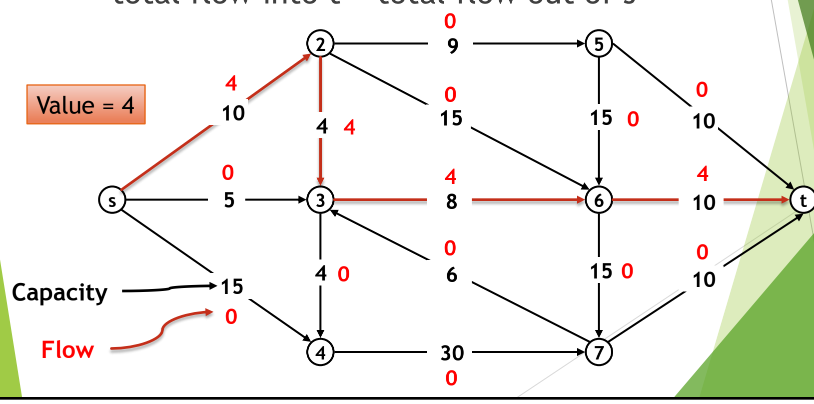
Definition: s-t flow

- Assignment of integer "flow" >= 0 on each arc:
  - (Capacity) Can't exceed arc's capacity
  - (Conservation) flow in = flow out at any node ≠ s, t
- Flow value
  = total flow into t = total flow out of s



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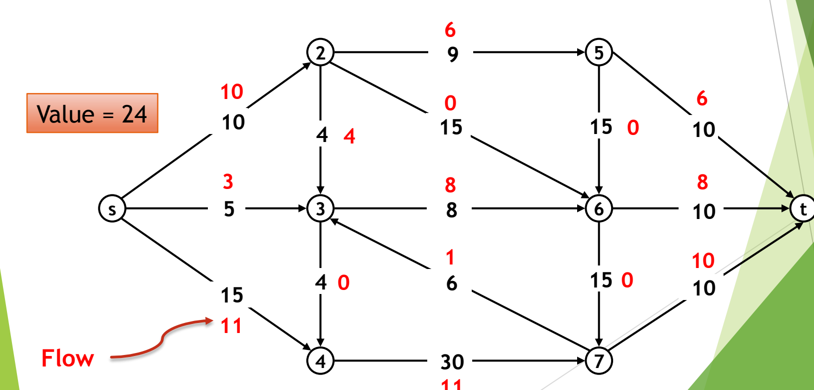
Can we increase the flow value?

- Capacity and flow conservation constraints are satisfied
- Further increase in flow value?



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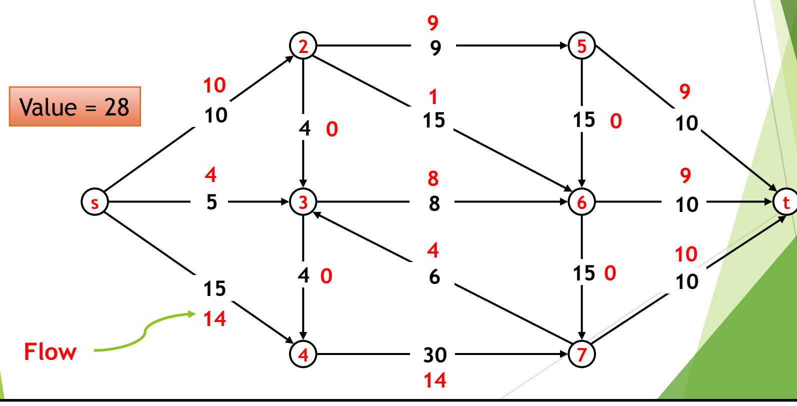
Max flow problem

- Compute the maximum value of an s-t flow



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Algorithms for max flow

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First try: greedy

- Start with f(e) = 0 for all arcs e
- Repeat until stuck:
  - Find an s-t path where each edge has f(e) < c(e)
  - Push more flow along that path

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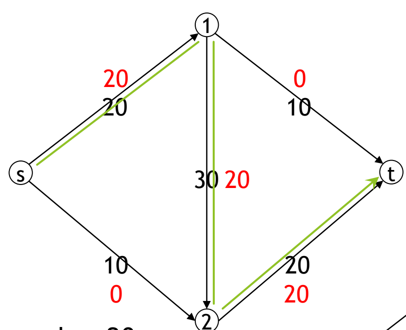
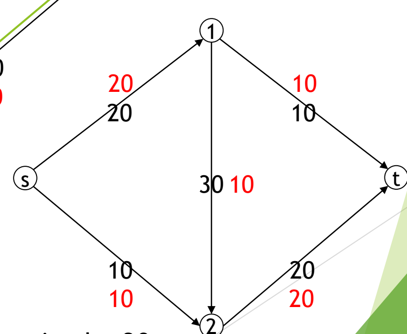
Greedy doesn't work– why?



greedy = 20



optimal = 30

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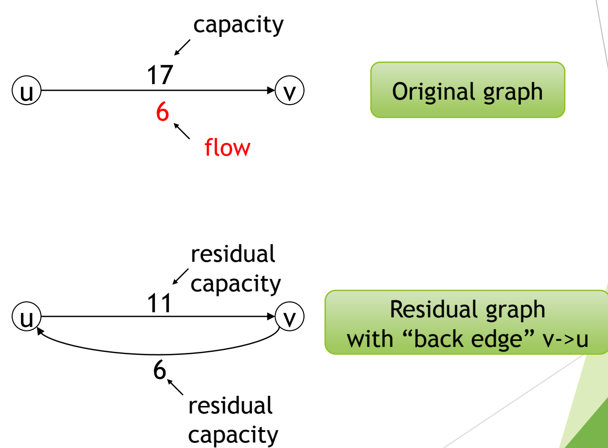
Fix: residual graph

- A way of undoing previous flows



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Ford-Fulkerson algorithm

- Iteratively find s-t paths that admit more flow in the residual graph
  - Such s-t paths: augmenting paths
- Push more flow along augmenting paths
- No further augmenting path?
  - Optimal solution!

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Ford-Fulkerson Demo

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Running time of Ford-Fulkerson

- At most nC iterations
- Total running time: O(mnC)
  - n = # of nodes
  - m = # of edges
  - C = max capacity of any edge
- Not strongly polynomial
  - There are strongly polynomial algorithms

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Min cut problem

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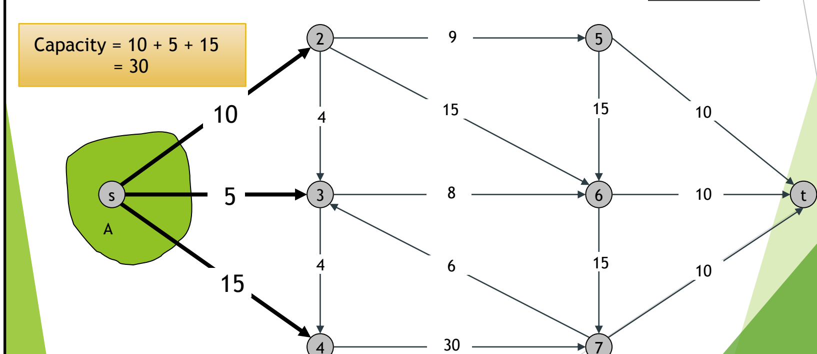
s-t cut

- Partition the nodes into two sets A and B such that s is in A and t is in B
- (A, B) is called an s-t cut
- Capacity of s-t cut (A, B)
  cap(A, B) = sum of capacities of arcs out of A



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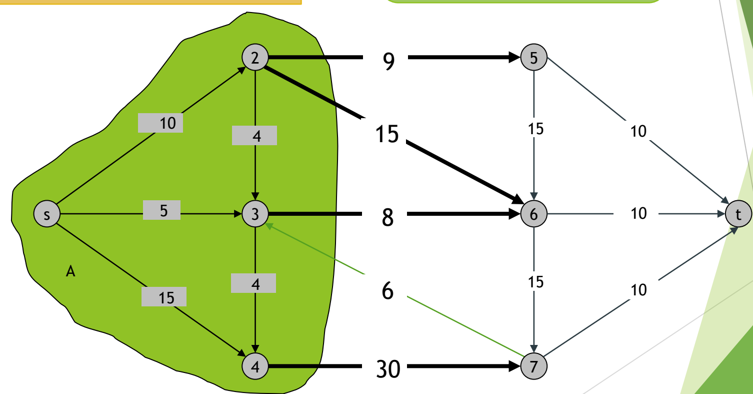
s-t cut: more example

Capacity = 9 + 15 + 8 + 30
= 62

Note: there's no flow here!



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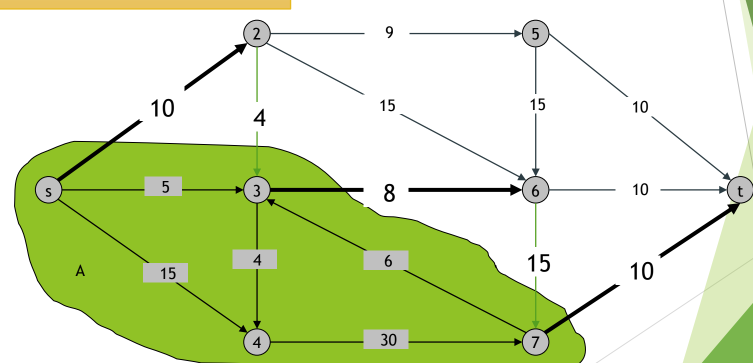
Min cut problem

- Find an s-t cut of minimum capacity

Capacity = 10 + 8 + 10
= 28



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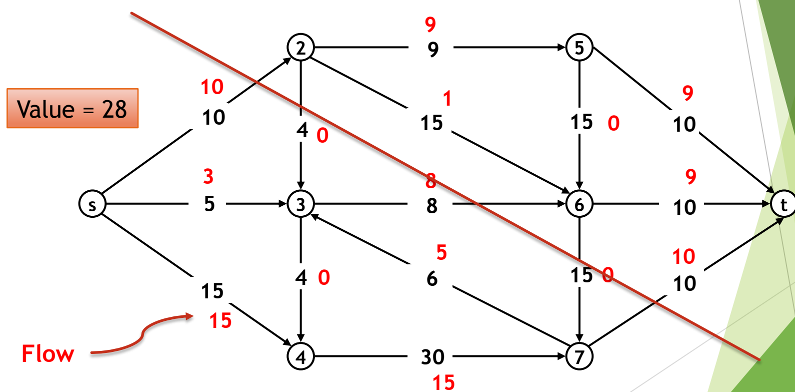
Max flow solution

- Max flow value is also 28!



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Max flow vs. min cut

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LP formulation: max flow

- Maximize $v(f) = \sum_{e \text{ out of } s} f(e)$ (flow)
- Subject to

$$0 \le f(e) \le c(e), \quad \forall e$$ (capacity)

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e), \quad \forall v \text{ except } s,t$$

Integrality theorem: if all capacities are integers, then there exists a max flow with all integer flows.

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LP formulation: min cut

- Dual of max flow
- Weak duality: any flow <= any cut capacity
  - Proof (on board) without using LP duality
- Strong duality: max flow = min cut capacity
  - Ford-Fulkerson's proof without using LP duality

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Max-flow min-cut theorem

- Ford & Fulkerson (1956)
- In any network, the value of the max flow is equal to the value of the min cut.

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How to: max flow → min cut

- Want an s-t cut or partition (A, B)
- A = s and all nodes reachable from s in the final residual graph
- B = rest of the nodes

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Algorithm for Fisher Market

Max Flow

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Reminder: Fisher's linear case

- Model parameters (what's given)
  - *n* ***divisible goods*** (1 unit each wlog) and *n'* ***buyers***
  - e_i = buyer *i*'s budget (integral wlog)
  - u_{ij} = buyer *i*'s utility per unit of good *j* (integral wlog)
  - *Linear* utility functions
- Want (not given): equilibrium allocations
  - x_{ij} = amount of good *j* that *i* buys to maximize his/her utility $u_i(\mathbf{x}) = \sum_{j=1}^{n} u_{ij} x_{ij}$
- Want (not given): equilibrium prices $p_1, p_2, \ldots, p_n$
- No deficit or surplus of any good
- No deficit or surplus of buyers' budgets

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Reminder: KKT conditions of Eisenberg-Gale convex program

- Optimal solutions x_{ij}'s and p_j's must satisfy:

1. $\forall j \in A: \; p_j \geq 0.$
2. $\forall j \in A: \; p_j > 0 \Rightarrow \sum_{i \in A} x_{ij} = 1.$
3. $\forall i \in B, \forall j \in A: \frac{u_{ij}}{p_j} \leq \frac{\sum_{j \in A} u_{ij} x_{ij}}{e_i}.$
4. $\forall i \in B, \forall j \in A: x_{ij} > 0 \Rightarrow \frac{u_{ij}}{p_j} = \frac{\sum_{j \in A} u_{ij} x_{ij}}{e_i}.$

- No deficit or surplus of goods
- Can show no deficit or surplus of buyers' budgets

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Idea of the algorithm

- Look at individual optimization problem
- Buyer i's optimization program:

$$\max \sum_j u_{ij} x_{ij}$$
$$\text{s.t. } \sum_j p_j x_{ij} \leq e_i$$

- Global constraint:

$$\forall j \;\; \sum_i x_{ij} = 1$$

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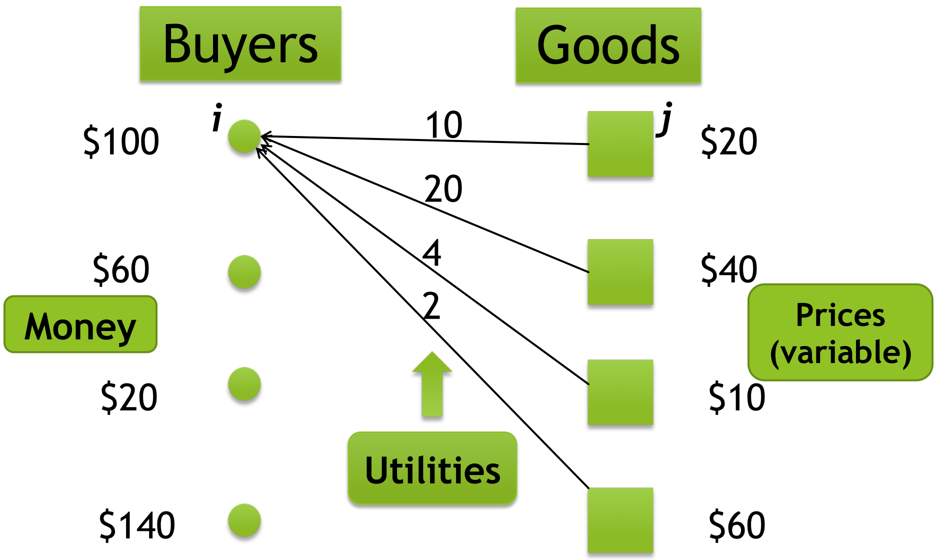
Example



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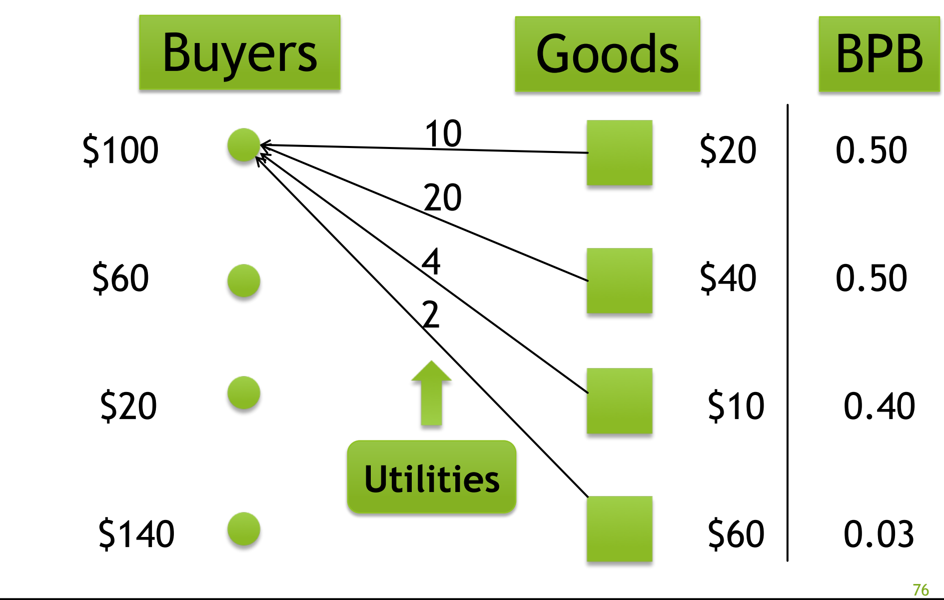
Bang-per-buck = utility/price



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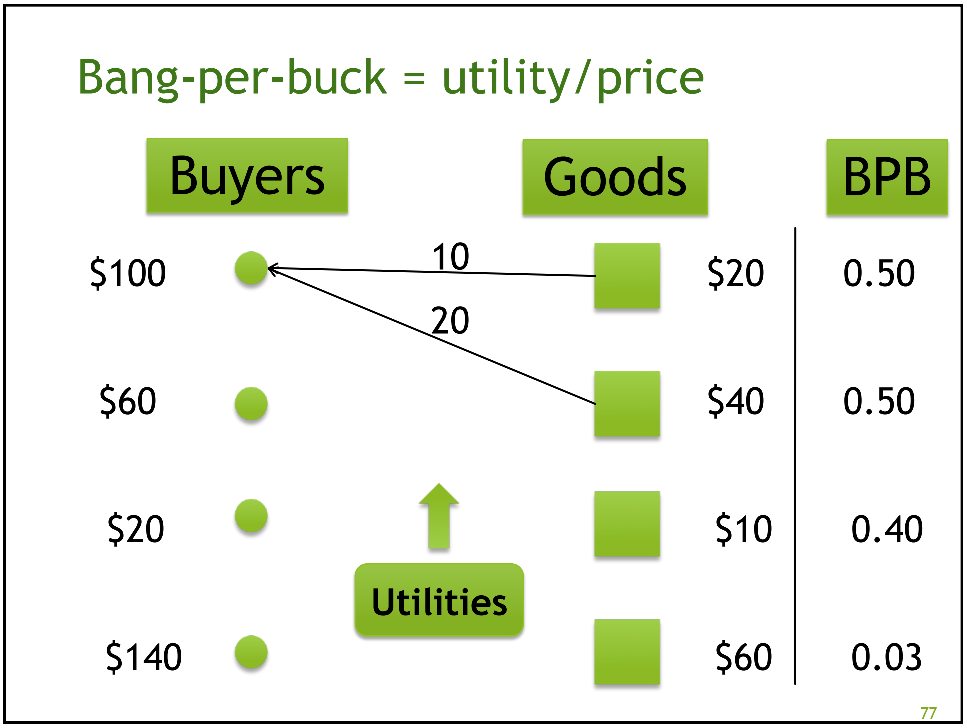
Bang-per-buck = utility/price
Buyers
Goods
BPB
$100
$60
$20
$140
10
20
Utilities
$20
$40
$10
$60
0.50
0.50
0.40
0.03
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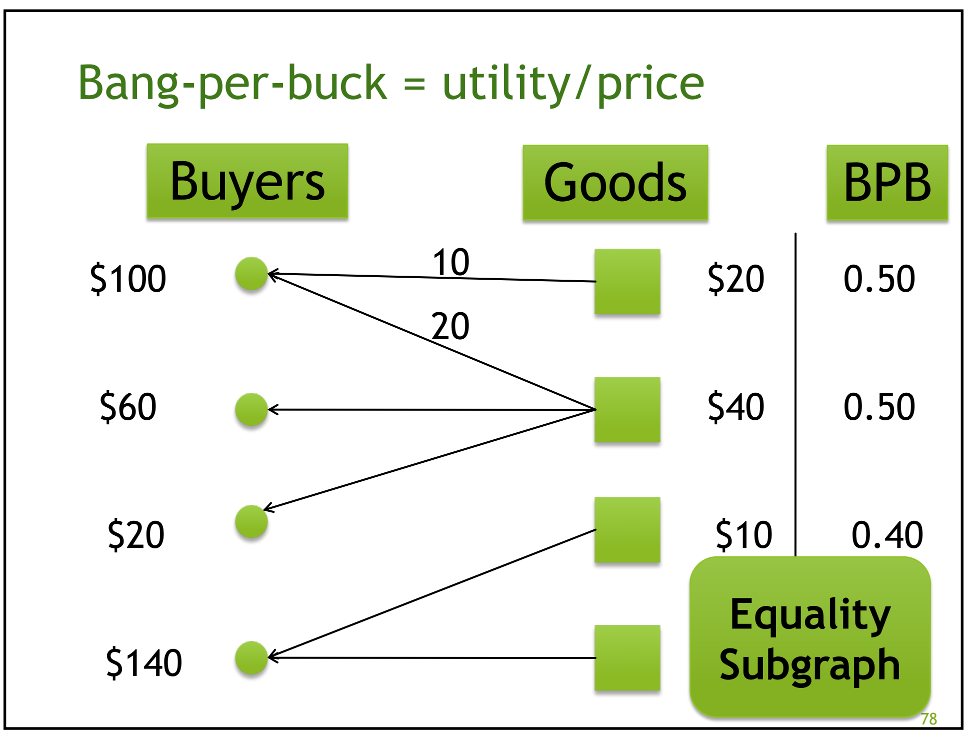
Bang-per-buck = utility/price
Buyers
Goods
BPB
$100
$60
$20
$140
10
20
$20
$40
$10
0.50
0.50
0.40
Equality Subgraph
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Equality subgraph

- Buyer is happiest when she can buy goods in equality subgraph
- How to maximize sales (market clearance) in the equality subgraph at a given price?

Use max flow!
("balanced flow" here)

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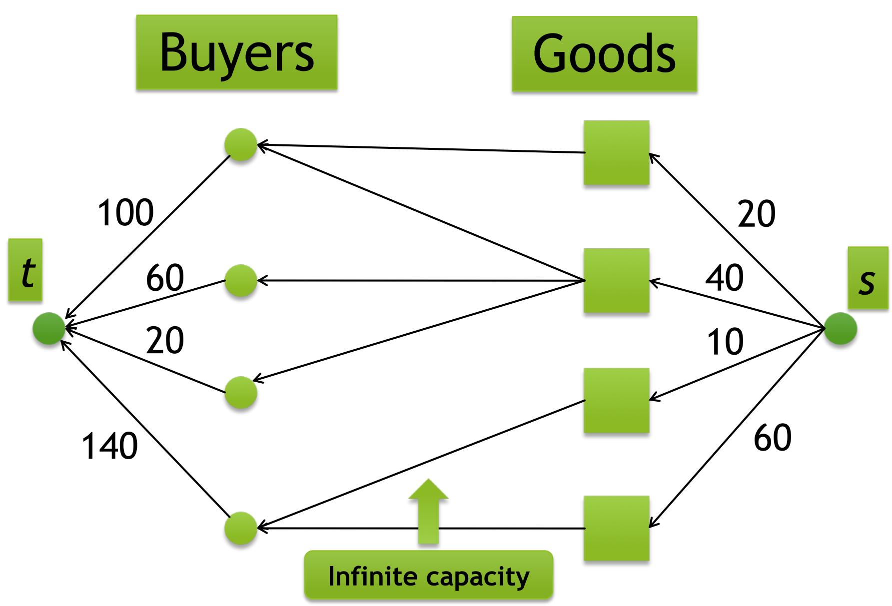
Max flow



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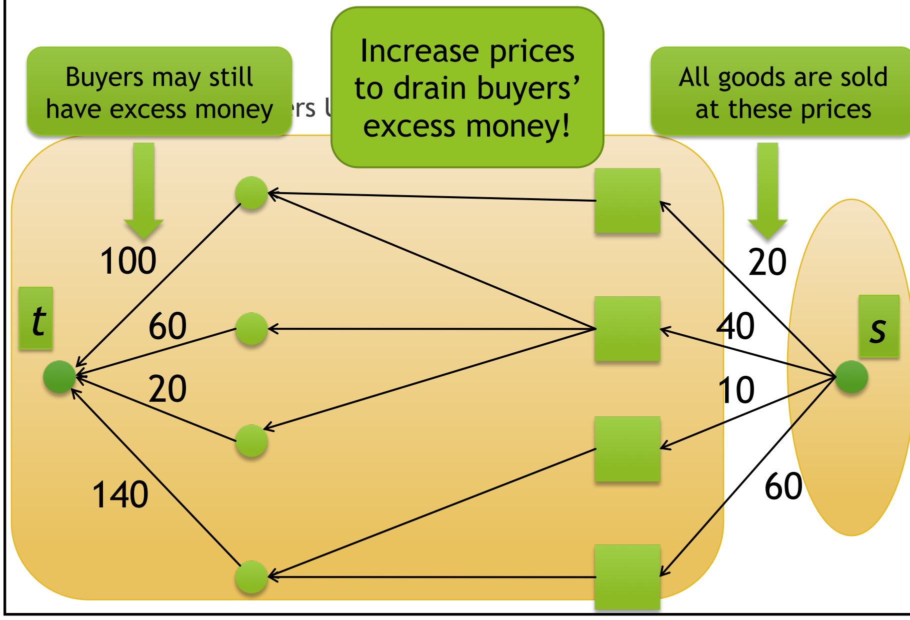
Idea of the algorithm



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Initialize

- All prices = 1/n, n = # of goods
- Assume
  - Each buyer has integral amount of money

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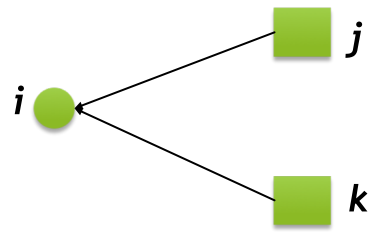
How to raise prices?

- We do not want to kill off any edge from the equality subgraph



$$\frac{u_{ij}}{p_j} = \frac{u_{ik}}{p_k}$$

$$\Rightarrow \frac{p_k}{p_j} = \frac{u_{ik}}{u_{ij}}$$

- Multiply prices by the same number ***x***
  - Initially *x* = 1

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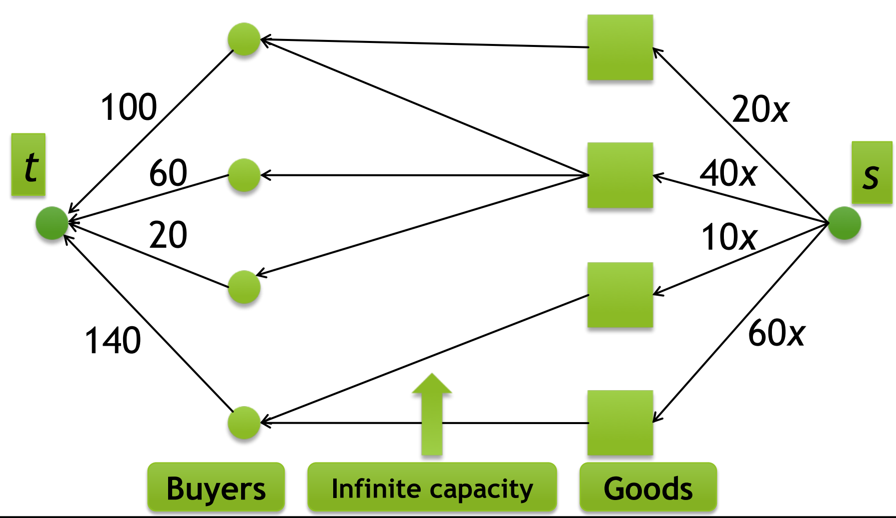
Algorithm

- Initially *x* = 1, then *x*↑
  - How to increase *x*? (Later)



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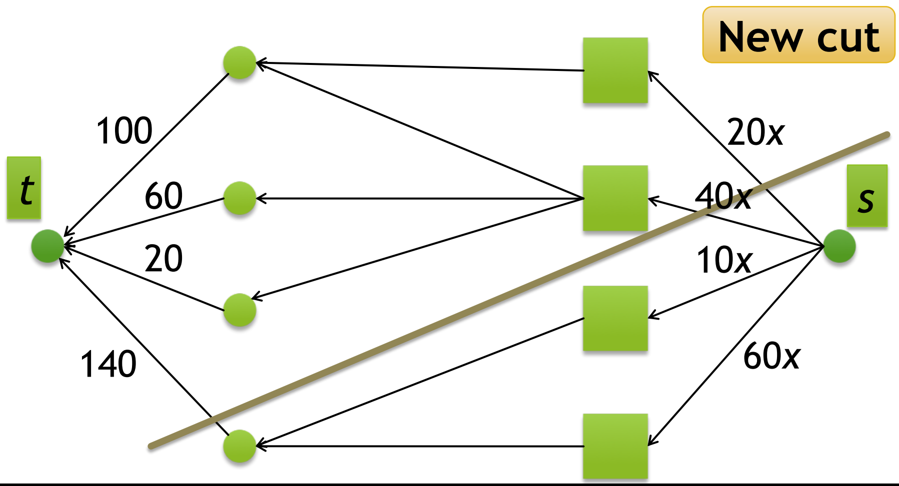
Algorithm

- x = 2: A buyer's surplus money = 0 (Event #1)



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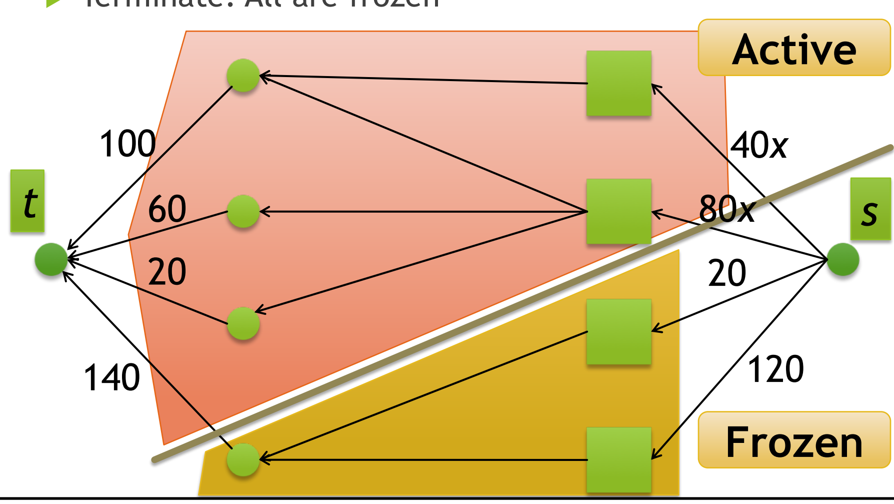
Algorithm

- Reinitialize x = 1, work the active part
- Terminate: All are frozen



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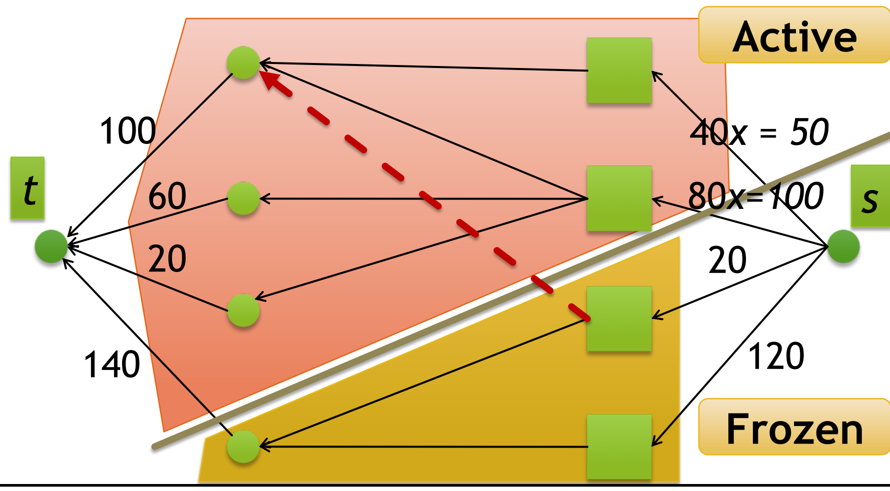
Algorithm

- x = 1.25 (Event #2)

Note: In the original algorithm, the active buyers are partitioned into sets: max-surplus buyers and the rest (not shown here for presentation, but x is relevant to the max-surplus buyers).



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Algorithm

- In case of event #2:"Unfreeze" the relevant part and recurse

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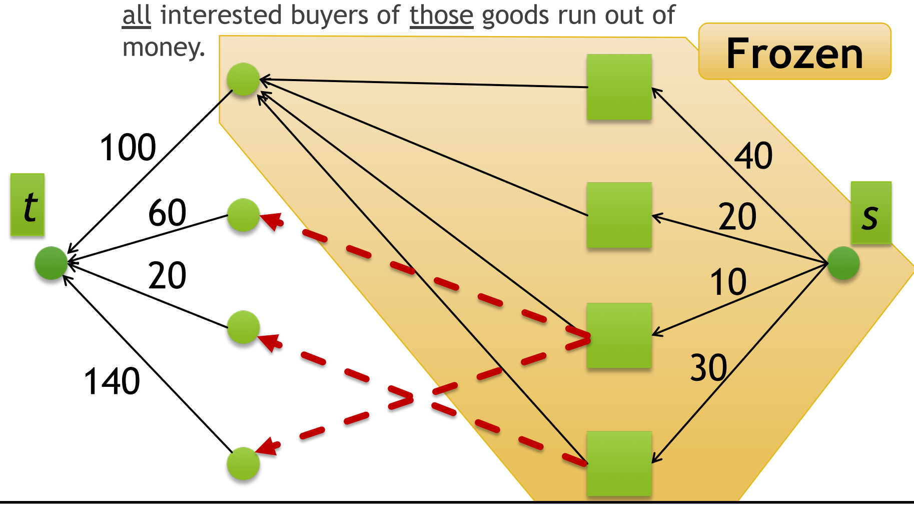
Question

- Is this scenario possible?
  - No. Every buyer must have some goods that maximize BPB. Can't freeze a set of goods unless all interested buyers of those goods run out of money.



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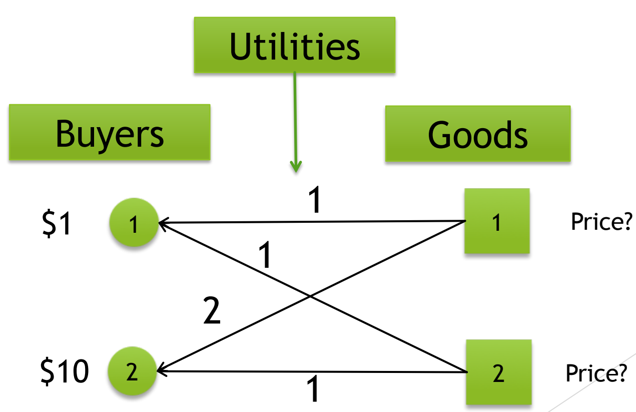
More example

- 2 buyers, 2 goods



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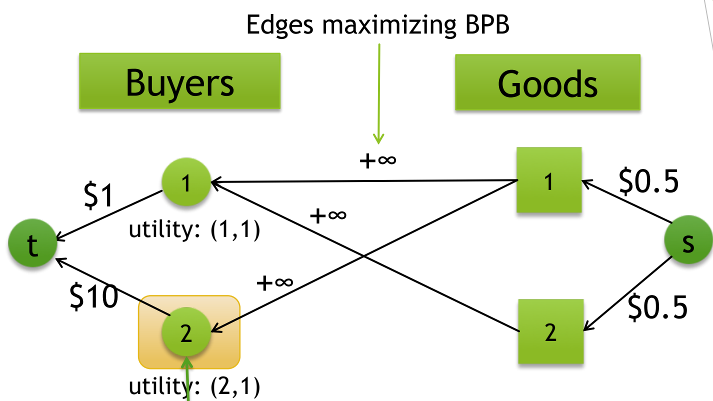
Example (continued)



- Initialize all prices to ½
- Compute a "balanced flow" (not shown here)
- Find the buyer(s) with the max surplus money
- Nothing frozen: No subset of goods for which all interested buyers run out of money

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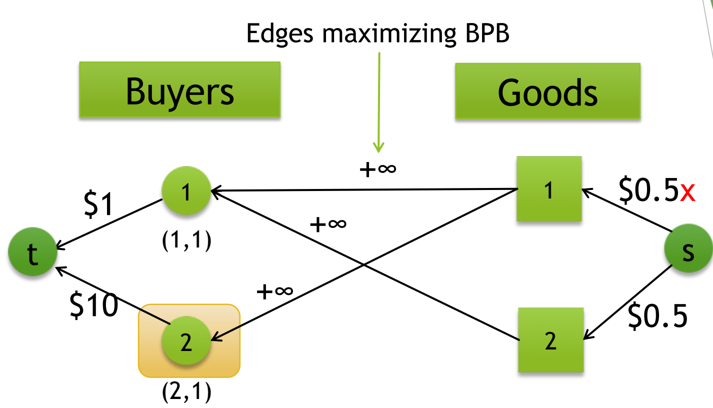
Example (continued)



- Increase prices of goods that the max-surplus buyers are interested in
- x > 0 ➔ edge from 1 to 1 disappears
- x = 2 ➔ New edge (event 2) from 2 to 2

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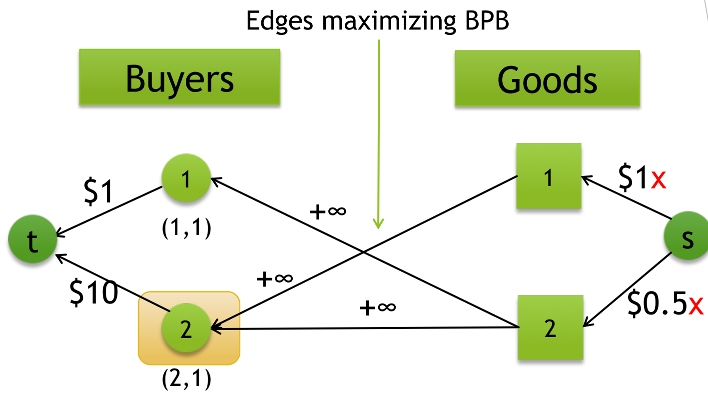
Example (continued)



- Increase prices of goods that the max-surplus buyers are interested in
- x = 7⅓ ➔ All goods frozen ➔ equilibrium
  - Note: all buyers run out of money

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Running time

- We can compute x values at event #1 and #2 efficiently
- Balanced flow is polynomial, but not strongly polynomial
  - Running time depends on the amount of money each buyer has
  - $O(n^4 (\log n + n\log U + \log M))$ applications of max-flow
    - n = # of goods
    - U = max u_{ij} for any i and j
    - M = total amount of money of all buyers

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