M.Sc. Engg. Thesis

Computing β-Drawings of 2-Outerplane Graphs

by
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Candidate's Declaration

It is hereby declared that this thesis or any part of it has not been submitted elsewhere for the award of any degree or diploma.

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Acknowledgments

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Abstract

One of the relatively new graph drawing problems is the proximity drawing of graphs. A proximity drawing of a plane graph G is a straight-line drawing of G with the additional geometric constraint that two vertices of G are adjacent if and only if the well-defined "proximity region" of these two vertices does not contain any other vertex. In general, a proximity drawing of a graph has some appealing features. For example, the group of vertices which are adjacent to each other tend to stay close together in the proximity drawing and the vertices that are nonadjacent tend to stay relatively far apart from each other. These underlying features of a proximity drawing of a graph have made it useful in many practical application areas. One class of parameterized proximity drawings is the β-drawing, where the value of the parameter β can be any nonnegative real number including ∞. The problem of whether a class of graphs is β-drawable, for some value of β, has been studied for two classes of graphs, namely trees and outerplanar graphs. However, for larger classes of graphs the problem of β-drawability is still an open problem. In this thesis we concentrate on the problem of β-drawings of 2-outerplane graphs for $1 < \beta < 2$. We provide a characterization of a subclass of biconnected 2-outerplane graphs for having a β-drawing for the specified range of β values. We provide a drawing algorithm as well. We also identify a subclass of biconnected 2-outerplane graphs that are not β-drawable for $1 < \beta < 2$.

Chapter 1

Introduction

In general, a graph represents relationships among a set of entities. When each vertex v of a graph represents a point p in space, the graph takes upon some geometric properties. There are many graphs that have interesting geometric properties. Such graphs are known as *geometric graphs*. Well-known geometric graphs such as Voronoi diagrams, Delaunay triangulations, convex hulls, visibility graphs, etc. have evolved over time for modeling and solving various practical problems. Compared to the mentioned geometric graphs, proximity graphs are relatively new in the field of computational geometry. Still, proximity graphs have been used in a wide range of applications. In this chapter, first we introduce proximity graphs. Then we specify several application areas where proximity graphs are being used. We introduce a parameterized family of proximity graphs, known as the β-proximity graphs, for $0 \leq \beta \leq \infty$. Then we review the literature and finally, state the results of the thesis.

1.1 Proximity Graphs

Proximity graphs are geometric graphs. Although the notion of "proximity graphs" was coined in much later, essentially the foundation of this fast expanding field of study has been the *Gabriel graph*, introduced by Gabriel and Sokal in the context of geographic variation analysis [GS69]. A Gabriel graph is a plane graph in which two vertices are adjacent if and only if the closed circle having these two vertices as its two antipodal points contains no other vertex of the graph. Here the closed circle just mentioned is also known as the *proximity region*, specifically the *Gabriel region*, of the two vertices. Like Gabriel graphs other types of proximity graphs also have a well-defined proximity region. Proximity regions are also termed as *regions of influence* by many authors. A definition of the proximity region is the heart of a proximity graph. In fact, this is the prime reason why a proximity graph is known as a geometric graph. All the geometric properties of any proximity graph are just results of the definition of its proximity region. For different proximity regions we get different proximity graphs although they might have the same set of points in the plane, each point being represented by a vertex of the corresponding graph. We clarify this idea in the next section by an example from the application domain of pattern recognition, more specifically instance-based learning.

1.2 Applications of Proximity Graphs

In this section we specify several applications of proximity graphs with the aim of clarifying the idea of constructing a proximity graph of a plane graph.

1.2.1 Dataset Thinning in NN-rule

The nearest neighbor classification rule, also known as the NN-rule, is one of the most famous classification rules in pattern recognition. It has gained its popularity because of its simplicity and also because of Cover and Hart's theorem that the probability of a classification error using the nearest neighbor rule is at most twice the Bayes probability of classification error [CH67]. It may be mentioned here that the Bayes probability of classification error is the minimum among all classification rules. The nearest neighbor classification rule says that given a set of N training vectors $\{x_i, \theta_i\}_{i=1}^{N}$, where $x_i = \{x_{i1}, x_{i2}, ..., x_{il}\}$ is the l-dimensional feature vector of the i-th object and θ_i is the class of that object, classify the test feature vector x_{test} to the class θ_k if $\mathtt{min}_{1 \leq i \leq N} d(x_i, x_{test}) = d(x_k, x_{test})$, where d denotes Euclidean distance. Simply stating, classify a test feature vector to the class of its nearest training vector. Now what's the problem of this very simple approach of classification? The first problem is that for a large value of dataset size N we need huge storage space. The second and related problem is that of computational requirement, since we need to find distance of the test feature vector from every training feature vector to find the minimum distance. For these two reasons researchers have been looking for ways to reduce the size of the training dataset without incurring too much degradation in performance. Bhattacharya *et. al.* has achieved it using two types of proximity graphs, namely the Gabriel graph and the relative neighborhood graph [BPT92]. They have also provided experimental results showing that training dataset reduction using Gabriel graph introduces a very low margin of additional error. Next we illustrate how this reduction in dataset is achieved.

Figure 1.1: Training dataset for nearest neighbor decision rule– filled circles and unfilled circles represent points of two different classes.

Let us consider 2-dimensional feature vectors and two classes for simplicity. The training data points are shown in fig. 1.1. We represent training data points of class 1 by filled circles and those of class 2 by unfilled circles. Taking these data points as the vertices, we can compute the Gabriel graph by adding an edge between any two distinct vertices if and only if the proximity region, i.e. the closed circle having the points corresponding to these two vertices[1] as its antipodal points, does not contain any other point. The corresponding Gabriel graph is shown in fig. 1.2. According to the training dataset reduction algorithm of [BPT92], we have to delete every vertex from the Gabriel graph whose neighbors all belong to its own class. The vertices that will remain after deletion are shown by double circles in fig. 1.2.

Figure 1.2: Gabriel graph corresponding to points shown in fig. 1.1. The dotted circles explain why the edges (a, b) and (c, d) have been added and why the edge (e, f) has not been added.

Figure 1.3: Relative neighborhood graph corresponding to points shown in fig. 1.1. The dotted lunes (intersection of two circles) explain why the edges (a, b) and (c, d) have been added and why the edge (e, f) has not been added.

[1] Henceforth we use the notations of a vertex and its physical location as a point equivalently for the purpose of simplification.

Another way to reduce the training dataset, as proposed by Bhattacharya *et. al.*, is by constructing the relative neighborhood proximity graph [BPT92]. In this type of proximity graph the proximity region of two points x and y is the intersection of two open disks of radius $d(x, y)$ centered at x and y. In a relative neighborhood graph there is an edge between x and y if and only if the relative neighborhood proximity region of x and y is empty. Fig. 1.3 shows the relative neighborhood graph resulting from the training dataset shown in fig. 1.1. After constructing the relative neighborhood graph we apply the same vertex deletion rule as described above and the remaining vertices after deletion are shown as double circles in fig. 1.3.

1.2.2 Other Applications

Given a set of points and a definition of the proximity region we can construct a proximity graph. An important property of this graph is that it provides us a description of the internal structure of the set of points. For some other definition of the proximity region, namely the γ-proximity region, the resulting proximity graph can describe the external shape of the point set [Velt92]. The ability of proximity graphs to describe the internal or external structure of a set of points has found its application in *computational morphology*, which is concerned with the analysis of the shape of a set of points. Research findings in computational morphology have industrial applications in *computer vision*. Proximity graphs have also been used in graph-based methods of clustering and manifold learning [CZ05]. Besides that, proximity graph-based methods have been applied in data mining [Tous05], topology control in wireless sensor networks [LCWW03, LSW04, Li03] and in many other diverse fields.

1.3 Parameterized Family of Proximity Graphs

We have just seen two types of proximity graphs– the Gabriel graph and the relative neighborhood graph. One might wonder how many other types of proximity graphs are there. In fact, there is an infinite number of different types of proximity graphs. This is due to an infinite family of parameterized proximity graphs introduced by Kirkpatrick and Radke [KR85]. This family of proximity graphs is called β-skeletons, where β stands for the parameter that can take any real number value in $[0, \infty]$. Interestingly, Gabriel graph and relative neighborhood graph both belong to this family of proximity graphs. Gabriel graph is the closed proximity graph for the value of $\beta = 1$ and relative neighborhood graph is the open proximity graph for $\beta = 2$.

1.4 Literature Review

Being brought to light in the year of 1969, proximity graphs might seem to be old geometric graphs. But the most interesting thing is that it has been providing new research trends quite regularly. In this section we review two basic problems concerning proximity graphs: firstly, how one can compute a proximity graph when a set of points is given as input and secondly, how a given plane graph be proximity-drawn.

1.4.1 Computation of Proximity Graphs

The oldest research direction concerning proximity graphs is– given a set of points and a definition of the proximity region, how can we compute the proximity graph efficiently and what are the properties of this graph? This research area has been explored and reviewed very nicely in a paper by Jaromczyk and Toussaint [JT92]. One research problem in this direction is– what are the lower bounds and upper bounds of the number of edges in different types of proximity graphs? This would have direct application in the dataset thinning problem of k-nearest neighbor classification rule, as outlined in the example above. There has been a great deal of research on this problem primarily for Gabriel graphs and relative neighborhood graphs. Taking n as the number of vertices, upper bounds of $3n-8$ edges for Gabriel graphs [MS84] and $3n-10, n \geq 8$, for relative neighborhood graphs [Urqu83] have been known for 2-dimensional Euclidean space. For higher dimensions, proximity graphs are known to become thicker in the number of edges. For relative neighborhood graphs, the maximum number of edges in 3-dimensional or higher dimensional Euclidean space is $O(n^{3/2+\epsilon})$, for any $\epsilon > 0$ [JK91]. And for Gabriel graphs, the lower bound on the maximum number of edges in 3- or more dimensional Euclidean space is $\Omega(n^2)$ [Smit89, CEG90, JT92]. Moreover, there have been studies on the expected number of edges of a Gabriel graph and a relative neighborhood graph. Interested readers are referred to [Smit89, Devr88, JT92].

There have also been studies on how to compute a proximity graph on a given set of points efficiently. For the 2-dimensional case, The brute-force approach to computing a proximity graph would have time complexity of $\theta(n^3)$. The brute-force algorithm is: for each potential edge and for each point in the given point set, check whether the point is in the proximity region of the two endpoints of the potential edge in consideration. If the answer of this checking is yes then this edge is discarded, otherwise this edge is added to the graph. Here, the number of potential edges is $\theta(n^2)$ since every vertex has the potential to be neighbor of any of the other $n-1$ vertices. The brute-force algorithm would be very expensive in case of large values of n. So a substantial amount of research efforts has been made to reduce this $\theta(n^3)$ time complexity. For relative neighborhood graphs, Toussaint reduced this to $O(n^2)$ using the fact that the relative neighborhood graph is a subgraph of the Delaunay triangulation on the same set of points [Tous80]. Given a point set, the Delaunay triangulation can be found in $O(n\log n)$. Since the Delaunay triangulation has $O(n)$ edges, once the Delaunay triangulation has been found some of the edges can be deleted from the Delaunay triangulation in $O(n^2)$ time. Thus this two-step procedure has an overall complexity of $O(n^2)$. The same algorithm would also work for Gabriel graphs, since Gabriel graphs are also subgraphs of corresponding Delaunay triangulations. Later on, further reduction in the complexity leading to $O(n\log n)$ algorithms has been accomplished for both Gabriel graphs [MS84] and relative neighborhood graphs [Supo83]. Algorithms have also been developed to compute Gabriel graphs and relative neighborhood graphs in 3- or more dimensional space. Interested readers are referred to [JK91, SC91, AM92, Supo83, Smit89, JT92].

1.4.2 Proximity Graph Drawing

Apart from the problem of computing a proximity graph and analyzing its underlying properties, another problem that has been receiving a lot of attention from the graph drawing community of late is– given a planar embedding of a graph and a definition of the proximity region, is it possible to achieve a *straight-line drawing* of the graph maintaining the proximity constraints? If yes then how can it be drawn? This has emerged as one of the relatively new graph drawing challenges. Today it is widely known as the *proximity drawability problem*, or the *β-drawability problem*, when the proximity regions are parameterized by β. Intuitively, the initial research effort in this direction has focused on proximity drawability of trees. The problem of proximity drawability of trees can be framed in this way– given a class of trees and a value of β, does this class of trees admit a β-drawing? Here we can classify the trees according to the maximum vertex degree. This problem has been studied in [BDLL95, BLL96] and the classes of trees that admit open β-drawings for $0 \leq \beta \leq \frac{1}{1-cos(\frac{2\pi}{5})}$ and $\frac{1}{cos(\frac{2\pi}{5})} < \beta < \infty$ have been found. In the same paper the classes of trees that admit closed β-drawings for $0 \leq \beta < \frac{1}{1-cos(\frac{2\pi}{5})}$ and $\frac{1}{cos(\frac{2\pi}{5})} \leq \beta \leq \infty$ have also been been found. Outside these ranges of β values, for $\beta = 2$ it has been found that the class of finite trees with maximum vertex degree at most 5 are β-drawable [BLL96]. However, there is some gray area for $\frac{1}{1-cos(\frac{2\pi}{5})} \leq \beta \leq \frac{1}{cos(\frac{2\pi}{5})}$ and $\beta \neq 2$ in the sense that determining which classes of trees are β-drawable for this range of β values is still an open problem.

Another class of graphs that has been studied from the graph drawing perspective is the class of outerplanar graphs. It has been shown that all biconnected outerplanar graphs can be β-drawn for $1 \leq \beta \leq 2$ [LL96]. In the same paper it has been shown that β-drawability of connected outerplanar graphs for some value of $\beta \in [0, \infty]$ depends on the maximum vertex degree in the block-cut-vertex tree of that graph. One of the open problems left in [LL96] is to extend the problem of β-drawability of graphs to other nontrivial classes of graphs apart from trees and outerplanar graphs. In this thesis we characterize a subclass of biconnected 2-outerplane graphs that can be β-drawn for $\beta \in (1, 2)$. We also show that all biconnected 2-outerplane graphs are not β-drawable for this range of β values.

The β-drawability problem we have discussed so far is also known as the *strong β-drawability* problem. In this problem, two points x and y will be connected by a straight line segment (i.e. there is an edge xy in the corresponding graph) if and only if the β-proximity region of x and y is devoid of other points. A more relaxed version of this problem is known as the *weak β-drawability* problem. In the weak β-drawability problem the only condition is that if x and y are adjacent then their proximity region must not contain any other point. Due to this relaxation, if x and y are not adjacent then their proximity region may be or may not be empty. It has been shown that although weak β-drawability is a relaxed problem, yet it is more practically applicable than the strong β-drawability problem [DLW96]. In the same paper it has been shown that a weak β-drawing of any graph G can be obtained in linear time for $\beta \in [0, sin(\frac{\pi}{(d+1)}))$, where d is the maximum vertex degree of G. Since the weak proximity drawability problem is a relaxed version of the strong proximity drawability problem, any strong proximity drawing also implies a weak proximity drawing. This gives us the impression that finding a weak proximity drawing of a graph would be easier compared to finding a strong proximity drawing of it. A more challenging problem regarding weak proximity drawings is to embed each vertex at an integer coordinate and find a polynomial area drawing. Extending the hv-drawing algorithm to satisfy the weak proximity constraint, Penna and Vocca have given an $O(n^2)$-area weak β-drawing of binary trees for any $\beta \in [0, \infty)$ [PV98]. The algorithm places every vertex at an integer coordinate. Penna and Vocca have further extended their procedure to obtain polynomial volume (i.e. in 3-dimension space) weak Gabriel drawings of trees of any degree [PV04]. Their results imply that graphs that do not admit proximity drawings in 2-dimension space might still admit proximity drawings in 3-dimension space. However, polynomial area or volume strong proximity drawings of graphs is still an open problem, even for drawing binary trees for $\beta = 1$.

1.5 Summary

In this thesis we are concerned with the problem of β-drawability of biconnected 2-outerplane graphs. Previous results related to our studies and new findings in this thesis are summarized in table 1.1. We have used several abbreviations in table 1.1. *BC* denotes *biconnected* and *BC2O* denotes *biconnected 2-outerplane*.

The results of the thesis are summarized as follows:

- We have given a necessary condition for β-drawability of biconnected 2-outerplane graphs, for $1 < \beta < 2$. The necessary condition comes from the geometric property that irrespective of the positions of the four vertices of a quadrilateral, if we draw four circles, each circle having two endpoints of each side of the quadrilateral as its antipodal points (i.e. $\beta = 1$), then each point inside the quadrilateral is covered by at least one circle.

- We have given sufficient conditions for β-drawability of biconnected 2-outerplane graphs, for $1 < \beta < 2$. The sufficient conditions induce a large and nontrivial class of biconnected 2-outerplane graphs. In table 1.1, this class of biconnected 2-outerplane graphs is indicated as a subset of the β-drawable class, for $1 < \beta < 2$.

Table 1.1: Summary of previous and new results

Value of β	**Classes of β-drawable graphs**	
$\beta = 1$	$[\beta]$**-drawable:** trees with the maximum degree 4, except a forbidden class of trees	[BLL96]
	$[\beta]$**-drawable:** all BC 1-outerplanar graphs	[LL96]
$1 < \beta < \frac{1}{1-cos\frac{2\pi}{5}}$	(β)**-drawable:** trees with the maximum degree 4 $[\beta]$**-drawable:** trees with the maximum degree 4	[BDLL95]
	(β)**-drawable:** all BC 1-outerplanar graphs $[\beta]$**-drawable:** all BC 1-outerplanar graphs	[LL96]
	(β)**-drawable:** a proper subset of BC2O graphs $[\beta]$**-drawable:** a proper subset of BC2O graphs **Forbidden:** BC2O graphs with less than 5 external vertices	Ours
$\beta = \frac{1}{1-cos\frac{2\pi}{5}}$	(β)**-drawable:** trees with the maximum degree 4 $[\beta]$**-drawable:** subset of trees with the maximum degree 5, superset of trees with the maximum degree 4	[BDLL95]
	(β)**-drawable:** all BC 1-outerplanar graphs $[\beta]$**-drawable:** all BC 1-outerplanar graphs	[LL96]
	(β)**-drawable:** a proper subset of BC2O graphs $[\beta]$**-drawable:** a proper subset of BC2O graphs **Forbidden:** BC2O graphs with less than 5 external vertices	Ours
$\frac{1}{1-cos\frac{2\pi}{5}} < \beta < 2$	(β)**-drawable:** subset of trees with the maximum degree 5, superset of trees with the maximum degree 4 $[\beta]$**-drawable:** subset of trees with the maximum degree 5, superset of trees with the maximum degree 4	[BDLL95]
	(β)**-drawable:** all BC 1-outerplanar graphs $[\beta]$**-drawable:** all BC 1-outerplanar graphs	[LL96]
	(β)**-drawable:** a proper subset of BC2O graphs $[\beta]$**-drawable:** a proper subset of BC2O graphs **Forbidden:** BC2O graphs with less than 5 external vertices	Ours
$\beta = 2$	(β)**-drawable:** trees with the maximum degree 5	[BLL96]

- For a biconnected 2-outerplane graph that satisfies the sufficient conditions, we have provided an $O(n^2)$ drawing algorithm for β-drawing the graph, for $1 < \beta < 2$. This drawing algorithm is obtained by the constructive proof of the sufficient conditions. The drawing algorithm requires to find an external vertex that is described as the "apex". Finding an apex takes $O(n^2)$ time and once an apex is found the remaining drawing algorithm needs $O(n)$ time.

- The specified necessary condition implies a forbidden class of biconnected 2-outerplane graphs which cannot be β-drawn, for $1 < \beta < 2$. In table 1.1, this class of biconnected 2-outerplane graphs is indicated as a subset of the forbidden class. The sufficient conditions imply a subclass of biconnected 2-outerplane graphs that are β-drawable for the same range of β values. We have shown that if a biconnected 2-outerplane graph does not belong to the forbidden class and also not to the above mentioned β-drawable class, then this graph is not necessarily β-drawable.

1.6 Organization of the Thesis

The remainder of the thesis is organized as follows. In chapter 2 we state preliminary definitions, notations and basic properties of proximity drawings. Then in chapter 3 we characterize a subclass of biconnected 2-outerplane graphs that admits β-drawings for $\beta \in (1, 2)$. In chapter 4, we prove that all biconnected 2-outerplane graphs do not admit β-drawings for the specified range of β values and we identify a forbidden class of biconnected 2-outerplane graphs. In chapter 5, we conclude the thesis with an outline of future research directions related to our studies.

Chapter 2

Preliminaries

In this chapter we first present definitions of graphs, planar and plane graphs, straight-line drawings of planar graphs, 1-outerplanar and 2-outerplanar graphs and then we define β-proximity graphs and state several properties of β-drawings of graphs.

2.1 Graphs

A *graph* is used to model relationships among a set of objects or entities. Mathematically, a graph G is a tuple (V, E), where V denotes the set of vertices and E denotes the set of edges, each edge being an unordered pair of vertices. The set of vertices of G is also denoted by $V(G)$ and the set of edges by $E(G)$.

Figure 2.1: A graph representing a road network among seven cities.

Fig. 2.1 shows the graph-based model of a fictitious road network among seven cities. For this graph we have $V(G) = \{$Dhaka, Chittagong, Feni, Comilla, Rajshahi, Khulna, Bogra$\}$. Each edge, denoted by a straight-line segment between the corresponding vertices, indicates a direct road link between two cities.

If two vertices u and v are endpoints of an edge, then u and v are called *adjacent* and are *neighbors*. We denote "u is adjacent to v" by $u \leftrightarrow v$. If vertex v is an endpoint of edge e, then e is *incident* to v. The degree of vertex v is the number of edges incident to v, assuming that v is not connected to itself by means of an edge.

A *path* is a simple graph whose vertices can be ordered in a list so that two vertices are adjacent if and only if they are consecutive in the list of vertices. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.

A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. A graph G is called *connected* if each pair of vertices in G belongs to a path; otherwise, G is called *disconnected*. An *induced subgraph* $G[T]$ is a subgraph obtained by deleting a set of vertices $\overline{T}$ from G, where $\overline{T} = V(G) - T$. We denote $G[T] = G - \overline{T}$.

The *components* of a graph G are its maximal connected subgraphs. A *separating set* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The *connectivity* of G is the minimum size of a separating set S such that $G - S$ is disconnected or has only one vertex. A graph is k-connected if its connectivity is at least k.

2.2 Planar Graphs and Plane Graphs

A graph is *planar* if it has an embedding in the plane without any edge-crossing, except at vertices on which two or more edges are incident. For example, the graph in fig. 2.2(i) is a planar graph, because it has an embedding in the plane without any edge-crossing, namely the embedding shown in fig. 2.2(ii).

Figure 2.2: (i) A planar graph. (ii) A plane graph: planar embedding of the planar graph in (i).

A *plane graph* G is defined as a planar graph with a fixed embedding in the plane without any edge-crossing, except at vertices on which two or more edges are incident. For example, the graph shown in fig. 2.2(ii) is a plane graph, where the embedding is fixed and there is no edge-crossing. In fact, a planar graph can have many planar embeddings and each of these embeddings is a plane graph.

2.3 Straight-line Drawings

The discipline of *graph drawing* is concerned with visualizing a graph "nicely". While there are many factors related to this objective, i.e. aesthetic visualization, one of the main factors is crossing of edges. We know that a planar graph has an embedding in the plane without any edge crossings. Thus a planar graph can be drawn in a plane without incurring any edge-crossings. Here, each vertex is drawn as a point in the plane. However, there is a question of how the edges will be drawn. If each edge is drawn as a straight-line segment and in this way there occurs no edge-crossing, then the resulting drawing is called a *straight-line drawing* of the given planar graph. Besides straight-line drawings, there are many other ways of drawing a graph. However, in this thesis we are concerned mainly with straight-line drawings.

2.4 1-Outerplanar Graphs and 2-Outerplanar Graphs

An *outerplanar graph* is a graph that has a planar embedding such that all the vertices lie on the external face. This graph is also known as a *1-outerplanar graph*.

Figure 2.3: (a) A straight-line drawing of a 1-outerplanar graph. (b) Another straight-line drawing of the same graph.

Fig. 2.3(a) and (b) show two straight line drawings of a 1-outerplanar graph. Although in the embedding shown in fig. 2.3(a) all vertices are not on the external face, yet the graph is 1-outerplanar since it has an embedding where all the vertices lie on the external face as shown in fig. 2.3(b). For a specific embedding of a graph if all the vertices are on the external face (as in fig. 2.3(b)) we say that the embedded graph is a *1-outerplane graph*, otherwise the embedded graph is not 1-outerplane (as in fig. 2.3(a)). So the definition of 1-outerplanar graph is independent of embeddings, whereas the definition of 1-outerplane graph is concerned with a specific embedding.

Figure 2.4: A straight-line drawing of a graph of outerplanarity 2.

These definitions can be generalized as follows. For an integer $k > 1$, an embedded graph[1] is *k-outerplane* if the embedded graph obtained by removing all the vertices of

[1] In this thesis whenever we use the term *embedded graph* we mean *planar embedded graph*.

the external face is a $(k-1)$-outerplane graph. On the other hand, we call a graph k-*outerplanar* if it has an embedding that is k-outerplane. A related notion is the *outerplanarity* of a graph which is defined as follows. For an integer $k > 0$, a graph has outerplanarity k if k is the least positive integer such that the graph is k-outerplanar.

For example, the graph corresponding to fig. 2.3 is 1-outerplanar as it has a 1-outerplane embedding (fig. 2.3(b)) and the same graph is also 2-outerplanar as it has a 2-outerplane embedding (fig. 2.3(a)). Therefore, $k = 1$ is the least positive integer such that it is k-outerplanar and thus the outerplanarity of the graph is 1. Fig. 2.4 shows a straight-line drawing of a graph that has no 1-outerplane embedding, but has a 2-outerplane embedding. So the outerplanarity of this graph is 2.

We now define a few notations related to a biconnected 2-outerplane graph.

Figure 2.5: Straight-line drawing of a 2-outerplane graph and related notations: a,b,c and d are internal vertices; ab, bc, cd and da are internal edges; e, f, g, h, i and j are external vertices; ef, fg, gh, hi, ij and je are external edges and all the edges except internal and external edges are mixed edges.

For a biconnected 2-outerplanar graph G, let Γ be a 2-outerplane embedding of G. We call the vertices of G that are in the external face in Γ the *external vertices*. The remaining vertices are called the *internal vertices*. Each edge between two external vertices is called an *external edge*. Similarly each edge between two internal vertices is called an *internal edge*. The remaining edges each connecting an external vertex with an internal vertex are called *mixed edges*. These definitions are illustrated in fig. 2.5.

2.5 Fan of a Vertex

In this section we define the notion of *fan of a vertex* of a biconnected outerplanar graph.Let $G = (V, E)$ be a biconnected outerplanar graph. For any vertex $u \in V$, the fan of u, denoted by F_u, is the subgraph of G induced by the vertices in V that share an internal face with u in a 1-outerplanar embedding of G. Here, the vertex u is called the *apex* of F_u. Since G is outerplanar, F_u is also outerplanar. Let Γ be a 1-outerplanar embedding of G in which F_u has the 1-outerplanar embedding Φ. Let $u_1, u_2, \ldots, u_k$ be the vertices of neighbors of u in clockwise order in Φ. The edge (u, u_1) is called the *first edge* of F_u and (u, u_k) the *last edge* of F_u for that embedding. We call each edge (u, u_i) a *radial edge* of F_u, for $i = 2, \ldots, k-1$. Apart from the first edge, the last edge and the radial edges, all other edges of F_u are called *fan edges*. We denote the m vertices on the boundary of Φ in between u_i and u_{i+1} by $u_{i,1}, u_{i,2}, \ldots, u_{i,m}$ in clockwise order, where $1 \leq i \leq k-1$. These notations are illustrated in fig. 2.6.

In this thesis, we adopt the notion of fan of a vertex in a biconnected 2-outerplane graph by allowing the apex of the fan to be an external vertex of the graph.

2.6 Complex Cycle

Let C be a cycle in a plane graph G. C is called a *complex cycle* if there is a vertex $v \in V(G)$ located inside C. If there are k vertices on the complex cycle C then C is called a complex k-cycle.

Fig. 2.7 shows a complex 4-cycle C in a plane graph G.

Figure 2.6: Fan of a vertex u of a biconnected outerplanar graph and related notions: u is the apex; uu_1 is the first edge, uu_3 is the last edge; uu_2 is a radial edge; $u_{1,2}u_2$ and $u_2u_{2,1}$ are two fan edges and the shaded subgraph is the fan of apex u, denoted by F_u.

Figure 2.7: A complex 4-cycle C in a plane graph G: C is shaded in the figure.

2.7 β-Regions

In this section, we define β-proximity regions or β-regions in short. For any two distinct points in the plane there is an associated region parameterized by β, which is called the β-region of the two points. Kirkpatrick and Radke introduced β-regions in two variants– lune-based and circle-based β-regions [KR85]. In this thesis we study only the lune-based variant. This proximity region can be further subdivided into two types– *open β-regions* (also denoted by (β)-regions) and *closed β-regions* (also denoted by [β]-regions). In the (β)-region, the boundary of the region is considered to be outside the region. However, in the [β]-region, the boundary of the region is included in the region of interest.

Figure 2.8: $R[x, y, \beta]$ for several values of $\beta \in [0, \infty]$.

For two distinct points x and y in the plane the associated lune-based open β-region $R(x, y, \beta)$ and closed β-region $R[x, y, \beta]$ are defined as follows.

- For $\beta = 0$, $R(x, y, \beta)$ is the empty region and $R[x, y, \beta]$ is the straight line segment connecting x and y.

- For β in $(0, 1)$, $R(x, y, \beta)$ is the intersection of two open disks of radius $\frac{d(x,y)}{2\beta}$ passing through both x and y and $R[x, y, \beta]$ is the intersection of the two corresponding closed disks.[2] If the coordinates of x are (a, b) and the coordinates of y are (c, d) then these two disks are centered at the points $(\frac{a^2-c^2+b^2-d^2-2(b-d)\frac{-B+\sqrt{B^2-4AC}}{2A}}{2(a-c)}, \frac{-B+\sqrt{B^2-4AC}}{2A})$

[2] $d(x, y)$ denotes Euclidean distance between the points x and y.

and $(\frac{a^2-c^2+b^2-d^2-2(b-d)\frac{-B-\sqrt{B^2-4AC}}{2A}}{2(a-c)}, \frac{-B-\sqrt{B^2-4AC}}{2A})$, where $A = 1 + \frac{(b-d)^2}{(a-c)^2}$, $B = -2b - \frac{(b-d)(a^2-c^2+b^2-d^2)}{(a-c)^2} + \frac{2a(b-d)}{a-c}$ and $C = a^2+b^2+\frac{(a^2-c^2+b^2-d^2)^2}{4(a-c)^2} - \frac{a(a^2-c^2)+a(b^2-d^2)}{a-c} - (\frac{d(x,y)}{2\beta})^2$.

- For β in $[1, \infty)$, $R(x, y, \beta)$ is the intersection of two open disks or radius $\frac{\beta d(x,y)}{2}$, centered at the points $(1-\frac{\beta}{2})x + \frac{\beta}{2}y$ and $\frac{\beta}{2}x + (1-\frac{\beta}{2})y$. $R[x, y, \beta]$ is the intersection of the two corresponding closed disks.

- For $\beta = \infty$, $R(x, y, \beta)$ is the open infinite strip perpendicular to the line segment xy and $R[x, y, \beta]$ is the corresponding closed infinite strip.

Examples of $R[x, y, \beta]$ are shown in fig. 2.8 for several values of β. It can be observed that as the value of β increases, the corresponding β-region also increases and for $0 \leq \beta_1 < \beta_2 \leq \infty$, $R[x, y, \beta_1]$ is contained inside $R[x, y, \beta_2]$.

2.8 Notions of β-Drawings

In this section we define β-drawings and the β-drawability problem, which is the problem of interest in this thesis. We also state several properties of a β-drawing of a graph.

2.8.1 β-drawings

Given a straight-line drawing Γ of a graph $G = (V, E)$ and the value of parameter β, we say that Γ is an *open β-drawing*, also written as *(β)-drawing*, of G if Γ maintains the following proximity constraint: $uv \in E$ if and only if $R(u, v, \beta)$ does not contain any vertex of $V - \{u, v\}$ in the drawing Γ. We can define *closed β-drawings*, written as *$[\beta]$-drawings*, similarly by considering closed β-regions.

2.8.2 The β-drawability problem

A graph is *(β)-drawable* (or *$[\beta]$-drawable*) if it admits a (β)-drawing (or $[\beta]$-drawing). The *(β)-drawability problem* asks the question of whether an input plane graph G is (β)-

drawable or not for a specified value of β. Similarly, the $[\beta]$-*drawability problem* can be defined.

A class of graphs $\mathcal{G}$ is called (β)-drawable if every graph $G \in \mathcal{G}$ admits a (β)-drawing [BDLL95]. If there is a graph $G \in \mathcal{G}$ that does not admit a (β)-drawing then we say that $\mathcal{G}$ is not (β)-drawable. We use similar definition for closed β-drawable class of graphs. In this thesis we simply use the notations *β-regions*, *β-drawings* or *β-drawable* graphs whenever the discussion applies for both (β)-regions and $[\beta]$-regions.

2.8.3 Angular measurements related to β-drawings

The definitions of β-regions and β-drawings imply that β-drawings are essentially scaling-independent. That is, if we scale a β-drawing maintaining the aspect ratio, that is the ration of width and height, then the resulting drawing still maintains the β-proximity constraints. As a result, lengths of edges or distances among vertices are seldom used in the analysis of β-drawings of graphs. What actually matters is some form of angular measurement. In this thesis we use two angular measurements $\alpha(\beta)$ and $\gamma(\beta)$ that are defined as follows:

- For $\beta \geq 0$, $\alpha(\beta) = inf\{\angle xzy \parallel z \in R[x, y, \beta], z \neq y\}$. This angle is illustrated in fig. 2.9(a).
- For $\beta \geq 2$, $\gamma(\beta) = \angle zxy$, where $z \neq y$ is a point on the boundary of $R[x, y, \beta]$ and $d(x, y) = d(x, z)$. This angle is shown in fig. 2.9(b).

It can be observed from fig. 2.9(a) that as the value of β increases the angle $\alpha(\beta)$ decreases. For example, $\alpha(0) = \pi$, $\alpha(1) = \frac{\pi}{2}$, $\alpha(2) = \frac{\pi}{3}$ and $\alpha(\infty) = 0$. On the other hand, with the increase of β the angle $\gamma(\beta)$ also increases as shown in fig. 2.9(b). For example, $\gamma(2) = \frac{\pi}{3}$ and $\gamma(\infty) = \frac{\pi}{2}$.

The following property expresses the relationship between β and either of $\alpha(\beta)$ and $\gamma(\beta)$. The property can be proved starting from the definitions of $\alpha(\beta)$ and $\gamma(\beta)$ and

Figure 2.9: (a) The angle $\alpha(\beta)$ for $\beta = 2$ and $\beta = 4$. (b) The angle $\gamma(\beta)$ for $\beta = 2$ and $\beta = 4$.

using elementary geometry. For brevity we denote $\alpha(\beta)$ by α and $\gamma(\beta)$ by γ.

Property 2.8.1 [BDLL95, LL96]

- $\beta = sin\alpha$ *for* $0 \leq \beta \leq 1$ *and* $\frac{\pi}{2} \leq \alpha \leq \pi$.
- $\beta = \frac{1}{1-cos\alpha}$ *for* $1 < \beta \leq 2$ *and* $0 \leq \alpha < \frac{\pi}{2}$.
- $\beta = \frac{1}{cos\gamma}$ *for* $2 \leq \beta \leq \infty$ *and* $\frac{\pi}{3} \leq \gamma < \frac{\pi}{2}$.

2.8.4 What values of β are most interesting?

The following two properties state that given a point set P, $G(P, \beta)$ and $G[P, \beta]$ are planar for $\beta > 1$ and are connected for $\beta < 2$. These properties explain the choice of $\beta \in (1, 2)$ in our study.

Property 2.8.2 [BDLL95] *For* $\beta > 1$*,* $G(P, \beta)$ *and* $G[P, \beta]$ *are planar. Furthermore,* $G[P, 1]$ *is planar, but* $G(P, 1)$ *is not necessarily planar. For* $\beta < 1$*,* $G(P, \beta)$ *and* $G[P, \beta]$ *are not necessarily planar.*

Property 2.8.3 [BDLL95] *For $\beta < 2$, $G(P, \beta)$ and $G[P, \beta]$ are connected. Furthermore, $G(P, 2)$ is connected, but $G[P, 1]$ is not necessarily connected. For $\beta > 2$, $G(P, \beta)$ and $G[P, \beta]$ are not necessarily connected.*

2.8.5 The β-boundary curve

We now define the *β-boundary curve* that has been introduced by Lenhart and Liotta [LL96]. For $\beta \in [1, 2]$, any distinct pair of points u and v in the plane is associated with a β-boundary curve denoted by $\mathcal{C}_{u,v,\beta}$ such that for every point z on the curve $\mathcal{C}_{u,v,\beta}$, v is on the boundary of $R[u, z, \beta]$. Each point z on the curve $\mathcal{C}_{u,v,\beta}$ is parameterized by the angle $\theta \equiv \angle vuz$ and positive values of θ correspond to the occurrence of u, v and z in clockwise order.

Figure 2.10: $\mathcal{C}_{u,v,\beta}(\theta)$ for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and for the values of $\beta = 1.6$ and $\beta = 3.0$.

Fig. 2.10 illustrates $\mathcal{C}_{u,v,\beta}(\theta)$ for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ and for several values of β, where u is plotted at the point $(0, 0)$ and v at $(0, 1)$. It can be observed that with the decrease of the value of β the curve becomes more and more flat and for $\beta = 1$, the curve becomes a straight line through v and perpendicular to the line uv. The physical significance of the curve $\mathcal{C}_{u,v,\beta}$ is that it implies the region where we can place the endpoint x of the edge

(u, x) so that the point v does not lie in the β-region of u and x. For example, if we place x outside the curve $\mathcal{C}_{u,v,\beta}$ (w.r.t. u) then v will be inside $R[u, x, \beta]$. To the contrary, if we place x inside $\mathcal{C}_{u,v,\beta}$ then $R[u, x, \beta]$ will not contain v.

The following lemma presents the mathematical definition of the β-boundary curve.

Lemma 2.8.1 [LL96] *For $\theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, if $u = (0, 0)$ and $v = (0, 1)$ then*
$\mathcal{C}_{u,v,\beta}(\theta) = \frac{2(\sqrt{1-(2/\beta-1)^2 sin^2\theta} - (2/\beta-1)cos\theta)}{\beta(1-(2/\beta-1)^2)}(sin\theta, cos\theta)$.

2.9 Chapter Summary

In this chapter we have defined various notations and terminologies that are used in this thesis. We have defined graphs, planar and plane graphs, 1-outerplanar and 2-outerplanar graphs and various other notions. We have also defined the β-drawability problem, on which we focus in this thesis. We have illustrated the concepts of β-regions and described several properties of proximity graphs parameterized by β. In the next chapter we are going to shed light on one of the main works of this thesis– β-drawings of biconnected 2-outerplane graphs. We are going to give a necessary condition and a set of sufficient conditions for β-drawability of a biconnected 2-outerplane graph.

Chapter 3

β-Drawability of Biconnected 2-Outerplane Graphs

In this chapter we give characterization of biconnected 2-outerplane graphs that can be β-drawn. We provide a constructive proof of our claim. The constructive prove gives us an $O(n^2)$ drawing algorithm for β-drawing a biconnected 2-outerplane graph that satisfies a set of sufficient conditions.

3.1 A Necessary Condition for β-Drawability

There are biconnected 2-outerplane graphs that are not β-drawable for $\beta \in (1, 2)$. For example the graph shown in fig. 2.4 is not β-drawable. In this biconnected 2-outerplane graph there are four external vertices and one internal vertex. Suppose that we want to achieve a [1]-drawing of this graph. No matter where we place the four external vertices, the internal vertex will be inside the proximity region of at least one of the four pairs of neighboring external vertices. As a result the graph is not [1]-drawable. The graph will not be β-drawable for $\beta > 1$ as well, since the proximity regions will only increase and we will have no place to position the internal vertex. The same situation will arise for a

2-outerplane graph with three external vertices. Thus the following lemma holds.

Lemma 3.1.1 *Let G be a planar embedded graph. If G has a complex 4-cycle or a complex 3-cycle then G cannot be β-drawn for $1 < \beta < 2$.*

From lemma 3.1.1, we can arrive at the following corollary expressing a necessary condition for β-drawability of a biconnected 2-outerplane graph, for $1 < \beta < 2$.

Corollary 3.1.2 *Let G be a biconnected 2-outerplane graph. G has no β-drawing, for $1 < \beta < 2$, if G has less than five external vertices.*

3.2 Sufficient Conditions

By definition, corollary 3.1.2 implies that not all of the graphs in the class of biconnected 2-outerplane graphs are β-drawable, for $\beta \in (1, 2)$. We are now interested in finding a subclass of biconnected 2-outerplane graphs that are β-drawable for $1 < \beta < 2$. The following theorem characterizes a subclass of biconnected 2-outerplane graphs that are β-drawable for $1 < \beta < 2$.

Theorem 3.2.1 *A biconnected 2-outerplane graph G is β-drawable for $\beta \in (1, 2)$ if G satisfies the following conditions:*

1. *There are at least five external vertices; and*
2. *There is an external vertex u such that the fan F_u has all of the following properties:*
   - (a) *F_u is biconnected 1-outerplane;*
   - (b) *F_u contains all the internal vertices; and*
   - (c) *Every vertex in F_u has at most one neighbor outside F_u and every vertex outside F_u has at most one neighbor in F_u.*

3.2.1 Proof of the Sufficient Conditions

In the rest of this chapter we provide a constructive proof of theorem 3.2.1. But before going on to the proof, an interesting question arises regarding a biconnected 2-outerplane graph G that satisfies all the conditions in theorem 3.2.1: is there any possibility of the occurence of a complex 3-cycle or a complex 4-cycle in G? The following lemma confirms that this can never happen.

Lemma 3.2.2 *Let G be a biconnected 2-outerplane graph satisfying the conditions in theorem 3.2.1. Then a 3-cycle or a 4-cycle of G is a face of G.*

Proof. We prove the statement by contradiction. Since G satisfies the conditions in theorem 3.2.1, G has an external vertex u such that F_u is biconnected 1-outerplane, contains all the internal vertices and has the property that every vertex in F_u has at most one neighbor in $G - F_u$ and every vertex in $G - F_u$ has at most one neighbor in F_u.

Suppose that there is a 4-cycle C in G with a vertex v inside the cycle. The following cases can occur regarding C:

Figure 3.1: Case I in the proof of lemma 3.2.2: apex u is on the 4-cycle with a vertex v inside it (all edges are not shown). (a) Vertex x of the subgraph $G - F_u$ has two neighbors in F_u. (b) F_u is not 1-outerplane.

Figure 3.2: Case II in the proof of lemma 3.2.2: apex u is not on the 4-cycle C having a vertex v inside it. C is shown by a dotted closed contour. F_u and $G - F_u$ may contribute 0 or more vertices to C, provided that the total number of vertices on C is 4. In any case, v is an internal vertex of G not contained in F_u.

- *Case I: Apex u is on the cycle C.* In this case, if there is a vertex v inside C then either a vertex of $G - F_u$ will have two neighbors in F_u (violating condition 2(c) of theorem 3.2.1), or F_u will not be 1-outerplane (violating condition 2(a)). This case is illustrated in fig. 3.1.

- *Case II: Apex u is not on the cycle C.* In this case, taking F_u to be 1-outerplane, if C has a vertex v inside it then v is an internal vertex of G not present in F_u, which violates condition 2(b) of theorem 3.2.1. This case is illustrated in fig. 3.2.

Thus every 4-cycle of G is always a face. Using similar arguments it can be proved that every 3-cycle of G is also a face. □

We are now going to present a constructive proof of theorem 3.2.1. The outline of the proof is as follows.

Let G be a biconnected 2-outerplane graph satisfying the conditions specified in theorem 3.2.1. According to the conditions, G has at least five external vertices. In addition to that, G has an external vertex u such that the fan F_u satisfies the conditions 2(a), 2(b) and 2(c). First we have to find this external vertex u. Once such an external vertex u has been found, then we draw the fan F_u. Next we draw the remaining graph $G - (V(F_u))$ and add edges between the vertices of F_u and the vertices of $G - (V(F_u))$. Finally, we prove the correctness of the drawing procedure.

Finding an appropriate apex

Now, let us see how we can find an external vertex of G such that the corresponding fan satisfies the required conditions. In the next lemma we give an algorithm that takes as input a biconnected 2-outerplane graph G that satisfies the conditions stated in theorem 3.2.1 and finds a set of external vertices of G such that for each vertex u in this set, the corresponding fan F_u satisfies conditions 2(a), 2(b) and 2(c) of theorem 3.2.1. We call such a set of external vertex the *set of candidate apices*.

Lemma 3.2.3 *Let G be a biconnected 2-outerplane graph satisfying all the conditions specified in theorem 3.2.1. The set S_a of external vertices u for which F_u satisfies conditions 2(a), 2(b) and 2(c) of theorem 3.2.1 can be found in $O(n^2)$ time, where n is the number of vertices of G.*

Proof. The set S_a, which we call the set of *candidate apices*, can be found by considering each external vertex of G and including only those external vertices of G into S_a for which the corresponding fan satisfies the required conditions. For a given external vertex u of G, verifying whether F_u satisfies the conditions 2(a), 2(b) and 2(c) of theorem 3.2.1 can be done in $O(n)$. Thus, if we consider each external vertex of G and verify the conditions then the total time complexity of this procedure becomes $O(n^2)$. □

Next we give an algorithm named *FindApex*, that takes a biconnected 2-outerplane graph satisfying the conditions of theorem 3.2.1 as input and outputs the set S_a of candidate apices. The algorithm clearly outlines the steps in the proof of lemma 3.2.3. For each external vertex u, the algorithm checks whether the fan F_u satisfies all the required conditions: whether a vertex of the fan has at most one neighbor outside the fan, whether a vertex outside the fan has at most one neighbor in the fan and whether the fan is biconnected 1-outerplane. For a specific fan F_u, checking these three conditions requires $O(n)$ time, where n is the number of vertices of the graph. Since these three conditions are checked for fans of all the external vertices of the graph, the total time complexity of the algorithm is $O(n^2)$.

Algoirthm FindApex (G) returns a set of candidate apices

$S_a = \phi$; /*S_a is the set of candidate apices.*/
For each external vertex u *do*
    if each internal vertex shares a face with u *then*
        flag = true
        $G' = G - V(F_u)$
        for each vertex $v \in V(G')$ *do*
            if v has 2 or more neighbors in F_u *then*
                flag = false
            end
        end
        for each vertex $v \in V(F_u) - \{u\}$ *do*
            if v has 2 or more neighbors in G' *then*
                flag = false
            end
        end
        if flag = true and F_u is biconnected 1-outerplane *then*
            $S_a = S_a \cup \{u\}$
        end
    end
end
return S_a

Drawing the fan

We can obtain a set S_a of candidate apices using the algorithm *FindApex* stated in lemma 3.2.3. We can choose any apex from this set for the purpose of β-drawing of the graph. In the next two lemmas we show how we can β-draw the fan F_u corresponding to an apex $u \in S_a$. These two lemmas are due to Lenhart and Liotta [LL96]. Lemma 3.2.4 specifies a useful result on how to union two β-drawings so that the resulting drawing maintains the proximity constraints. We change lemma 3.2.5 slightly from the original lemma given in [LL96] to fit our purpose.

Lemma 3.2.4 [LL96] *Let G_1 and G_2 be two planar embedded graphs whose intersection consists of a single edge uv, where u and v lie on the external face of both G_1 and G_2. Let Γ_1 and Γ_2 be embedding-preserving β-drawings of G_1 and G_2 respectively, for $1 < \beta < 2$. If $\Gamma_1 \cup \Gamma_2$ lies in a convex region having Γ_1 and Γ_2 on opposite sides of the edge uv then $\Gamma_1 \cup \Gamma_2$ is a β-drawing of $G_1 \cup G_2$.*

Lemma 3.2.5 [LL96] *Let F_u be a biconnected outerplane fan with apex u. F_u can be β-drawn inside a triangle Δabc, for $\beta \in (1, 2)$, $\angle abc > \frac{\pi}{2}$ and $\angle bac < \frac{\pi}{4}$. Furthermore, this drawing has the property that the fan edges form a convex chain such that for any three vertices v_1, v_2 and v_3 on the chain in clockwise order, $\angle v_1v_2v_3 > \frac{\pi}{2}$.*

Proof. The statement can be proved constructively by induction on the number of neighbors of u.

Base Case: u has exactly two neighbors u_1 and u_2

Suppose that the two neighbors of u are connected by the chain of nonneighbors of u: $u_{1,1}$, $u_{1,2}$, $\ldots$, $u_{1,m}$. In this case F_u can be β-drawn by the following procedure (for illustration please see fig. 3.3):

First, u is placed at the point a of Δabc and u_1 at the point b. Then a straight-line ux is computed such that $\angle u_1ux \leq \angle bac$. The line ux intersects the curve $\mathcal{C}_{u,u_1,\beta}$ at the point

Figure 3.3: Base case of drawing a fan F_u, when u has only two neighbors– u_1 and u_2.

p and the curve $\mathcal{C}_{u,u_1,2}$ at the point q. u_2 is placed on the straight-line ux in between the points p and q. This is always possible, since $\mathcal{C}_{u,u_1,2}$ is interior to the curve $\mathcal{C}_{u,u_1,\beta}$. This placement of u_2 ensures that u_2 is not in $R[u, u_1, \beta]$, since u_2 is exterior to the β-boundary curve $\mathcal{C}_{u,u_1,2}$ and u_1 is not in $R[u, u_2, \beta]$, since u_2 is placed interior to the curve $\mathcal{C}_{u,u_1,\beta}$.

Next the nonneighbors of u between u_1 and u_2 are placed. Let w be a point inside Δabc such that $\angle uu_1w = \frac{\pi}{2}$. $u_{1,1}$, $u_{1,2}$, $\ldots$, $u_{1,m}$ are placed on the curve $\mathcal{C}_{u,u_2,2}$ in between u_1w and $\mathcal{C}_{u,u_1,\beta}$. This is possible since u_1w is tangent to the curve $\mathcal{C}_{u,u_1,\beta}$ on the point u_1 for $\beta \in (1, 2)$. This placement has the following properties:

Firstly, being placed on a convex curve, $u_{1,1}$, $u_{1,2}$, $\ldots$, $u_{1,m}$ form a convex chain. Furthermore, $\angle u_{1,m}u_2u < \frac{\pi}{2}$.

Secondly, since u and $u_{1,i}$ are nonadjacent for $1 \leq i \leq m$, there must be at least one vertex in $R[u, u_{1,i}, \beta]$. We find that $u_{1,i}$, for $1 \leq i \leq m$, is exterior to the curve $\mathcal{C}_{u,u_1,\beta}$. So u_1 is inside $R[u, u_{1,i}, \beta]$, for $1 \leq i \leq m$.

Thirdly, since $\angle u_1 u_{1,i} u_2 > \frac{\pi}{2}$, for $1 \leq i \leq m$, we can conclude that for any pair of non-consecutive vertices on the chain, there is an intermediate vertex of the chain in their β-region.

Thus the statement is proved for the base case.

Induction: u has $k > 2$ neighbors

Suppose that a fan is β-drawable if the apex has less than k neighbors. Now, let the apex u have k neighbors. We prove that F_u is β-drawable. F_u can be β-drawn by the following procedure (for illustration please see fig. 3.4):

Figure 3.4: Induction step of drawing a fan F_u: u has more than two neighbors.

Let x be a point on bc such that $\theta = \angle bax = \frac{\angle bac}{k-1}$. As in the base case, we can draw the part of F_u consisting of u, its two neighbors u_1, u_2 and the chain of fan vertices from u_1 to u_2 within the obtuse triangle Δabx. Let the straight-line $u_{1,m}u_2$ intersect ac at the point c'. Since $\angle u_{1,m}u_2u < \frac{\pi}{2}$, $\angle uu_2c' > \frac{\pi}{2}$. As a result, by the induction hypothesis, the remaining part of the fan can be drawn inside the obtuse triangle $\Delta au_2c'$.

By lemma 3.2.4, joining the drawings of the above mentioned two parts of the fan F_u we find a correct β-drawing of F_u as a whole. □

Drawing the remaining graph

Once we find an apex u by lemma 3.2.3, we can β-draw the fan F_u that satisfies the conditions in theorem 3.2.1 according to lemma 3.2.5. Suppose that the fan F_u has been drawn inside an obtuse triangle Δabc such that $\angle abc > \frac{\pi}{2}$ and $\angle bac < \frac{\pi}{4}$. We now show how we can draw the remaining part of the graph so that the graph G, as a whole, is correctly β-drawn.

Regions for drawing F_u and $G - V(F_u)$

First of all, we want to find two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ in the plane such that for any two points x and y in $\mathcal{R}_1$, $R[x, y, \beta]$ never overlaps $\mathcal{R}_2$ and vice versa for any two points in $\mathcal{R}_2$. The intention is to place the drawing of F_u in $\mathcal{R}_1$ and the drawing of $G - V(F_u)$ in $\mathcal{R}_2$. Then edges will be added in between vertices of F_u and vertices of $G - V(F_u)$ so that the proximity constraints are not violated. Of course, there are other issues apart from selection of such regions, which will be considered as well.

Let us compute two non-parallel and non-perpendicular straight lines L_1 and L_2 such that the acute angle δ at the intersection point of the two straight lines satisfies $\frac{\pi}{4} < \delta < \alpha(\beta)$. The two intersecting straight lines divide the plane into four regions. Among these four regions two regions contain the acute angle δ and these two are the regions of our interest: $\mathcal{R}_1$ and $\mathcal{R}_2$. The constraint $\delta < \alpha(\beta)$ ensures that the proximity region of any

Figure 3.5: (a) If $\delta > \alpha$, there exists points x and y in region $\mathcal{R}_1$ such that $R[x, y, \beta]$ overlaps with the region $\mathcal{R}_2$. (b) However, if $\delta < \alpha$, for any two points x and y in region $\mathcal{R}_1$, $R[x, y, \beta]$ never overlaps with the region $\mathcal{R}_2$.

two points of $\mathcal{R}_1$ is outside the region $\mathcal{R}_2$ and vice versa. We illustrate the idea in fig. 3.5. The other constraint $\frac{\pi}{4} < \delta$ will be explained very shortly.

Placement of the fan F_u

We place the triangle Δabc, inside which F_u is drawn, on the plane as follows (please see fig. 3.6 for illustration):

All the vertices of F_u are inside the region $\mathcal{R}_2$ and with respect to the convex chain of fan vertices, u is positioned opposite to the region $\mathcal{R}_1$. Let uu_1 be the first edge and uu_k be the last edge of F_u, for $k \geq 2$. The line segments uu_1 and uu_k, when extended, intersect the lines L_1 and L_2 at points p and q, respectively, on the boundary of the region $\mathcal{R}_1$. Since the acute angle between the lines uu_1 and uu_k is less than $\frac{\pi}{4}$, we need to impose the constraint $\delta > \frac{\pi}{4}$ so that the above mentioned intersections are always possible. Furthermore, the intersections guarantee that if the remaining graph $G - V(F_u)$ is placed inside the portion of the region $\mathcal{R}_1$ in between the extended line segments ap and

Figure 3.6: The regions for positioning the fan F_u and the remaining graph $G - V(F_u)$.

aq (shaded region in fig. 3.6) then it will be possible to draw edges between the vertices of F_u and the vertices of $G - V(F_u)$ on the rays emanating from u. In fact, this property is crucial for our proof.

Placement of the vertices of $G - V(F_u)$

The fan F_u being drawn in the region $\mathcal{R}_2$, the vertices of $G - V(F_u)$ are drawn in the region $\mathcal{R}_1$ as follows:

First, an arc xzy is computed in the region $\mathcal{R}_1$ with a as the center and x and y being the intersection points with the extended line segments ap and aq respectively, such that the length $ax \geq ap$ and the length $ay \geq aq$. This is illustrated in fig. 3.7.

Suppose that ut_1 is the first edge and ut_m is the last edge of the fan F_u and the chain of fan vertices in clockwise order are: $t_1, t_2, \ldots, t_m$. Let us now compute the rays emanating from the point u, passing through the chain of fan vertices $t_1, t_2, \ldots, t_m$ and intersecting

Figure 3.7: An arc xzy is computed in the region $\mathcal{R}_1$ taking a as the center and x and y the intersection points with the extended line segments ap and aq respectively.

the arc xzy at the points $w_1, w_2, \ldots, w_m$ respectively. Please see fig. 3.7 for illustration.

Vertices of $G - V(F_u)$ can be divided into two types- type 1 and type 2. Next we define these two types of vertices and also fix the positions of these vertices. These types of vertices are illustrated by an example in fig. 3.8.

- *Type 1: The vertices of $G - V(F_u)$ that share an internal face with the vertices of F_u*

  The type 1 vertices can further be subdivided into two subtypes:

  - *Type 1A*: Some of the vertices of type 1 are adjacent to the vertices of F_u. We name these type 1 vertices type 1A.
  - *Type 1B*: The remaining type 1 vertices are nonadjacent to any vertex of F_u.

Figure 3.8: Two types of vertices in $G - V(F_u)$, where $V(F_u) = \{u, a, b, c, d, e, f, g\}$: type 1 vertices are those vertices that share an internal face with the vertices of F_u and type 2 vertices are those vertices that do not share an internal face with the vertices of F_u. Here, h, i, j, k, l, m and n are type 1 vertices and o, p, q, r, s and t are type 2 vertices.

We call these type 1 vertices type 1B.

Placement of type 1A vertices:

Note that by the property of the graph G, for each of the fan vertex t_i, for $1 \leq i \leq m$, there can be at most one neighbor in $G - V(F_u)$. For $1 \leq i \leq m$, if $\exists v \in V(G - V(F_u))$ such that $v \leftrightarrow t_i$ then place the vertex v on the arc xzy at the point w_i that has been computed previously. Please see fig. 3.7 for illustration. Let the newly placed vertices be $v_1, v_2, \ldots, v_l$ in clockwise order, where $l \leq m$ since some of the m fan vertices might not have neighbors in $G - V(F_u)$.

Placement of type 1B vertices:

As mentioned above, the type 1B are those type 1 vertices that are not adjacent to

Figure 3.9: Placement of vertices of type 1B.

any vertex of F_u, but share an internal face with a vertex of F_u. We can find these vertices by clockwise traversal of the external face of G. For $1 \leq i \leq l-1$, the vertices on the external face of G that share faces with the vertices of F_u and that are in between nonadjacent vertices v_i and v_{i+1}, form the set of type 1B vertices. We place these type 1B vertices on a convex curve maintaining the relative order by induction as follows. We prove the correctness of the placement later on. Please see fig. 3.9 for illustration.

– *Base case*

Suppose that there are exactly two type 1A vertices: v_i and v_{i+1}, where $v_i \leftrightarrow t_j$ and $v_{i+1} \leftrightarrow t_{j'}$ and $t_j, t_{j'} \in V(F_u)$. Suppose that $v_{i,1}$, $v_{i,2}$, $\ldots$, $v_{i,r}$ are type 1B vertices from v_i to v_{i+1} in clockwise order and $t_{j,1}$, $t_{j,2}$, $\ldots$, $t_{j,s}$ are fan vertices of F_u also in clockwise order from t_j to $t_{j'}$.

Let gg' and g_1g_1' be the tangents to the arc xzy at the points v_i and v_{i+1} respectively. From the proof of lemma 3.2.5, $\angle ut_jt_{j,1} < \frac{\pi}{2}$. So, $\angle t_{j,1}t_jv_i > \frac{\pi}{2}$. Let us draw a straight line t_jd such that $\angle t_{j,1}t_jd \geq \frac{\pi}{2}$ and gg' intersects the line t_jd at the point d' "below" the point d'' at which g_1g_1' intersects t_jd. Note that intersections in this way are always possible since $\angle t_{j,1}t_jv_i > \frac{\pi}{2}$. Furthermore, the line t_jd will intersect the arc xzy at a point in between v_i and v_{i+1}, because of the convex placement of the fan vertices and $\angle t_{j,1}t_jd \geq \frac{\pi}{2}$ and $\angle t_{j,s}t_{j'}v_{i+1} > \frac{\pi}{2}$.

Let w be a point on the arc xzy such that w, v_i, v_{i+1} occur in clockwise order and the straight line wv_iw' intersects the line t_jd at the point d''' in between d' and d'' such that $\angle v_id'''v_{i+1} > \frac{\pi}{2}$. Let hh' be perpendicular to the line ww' at the point v_i. Let us draw an arc $x_1v_iy_1$ with center on the straight line hh' and y_1 being a point on the line t_jd in between d''' and d'. Note that this is always possible since $wd'''w'$ is the tangent to the arc at the point v'. We place the type 1B vertices $v_{i,1}$, $v_{i,2}$, $\ldots$, $v_{i,r}$ on the arc $x_1 v_i y_1$ in between the points v_i and y_1 maintaining their relative order. This placement guarantees that for any three type 1 vertices s_1, s_2 and s_3 in clockwise order around the external face of G, $\frac{\pi}{2} < \angle s_1 s_2 s_3 < \pi$.

– *Induction Step*

Suppose that if there are less than k type 1A vertices, for some positive integer k, then the type 1B vertices can be placed with the property that for any three consecutive type 1 vertices v_1, v_2 and v_3, $\frac{\pi}{2} < \angle v_1 v_2 v_3 < \pi$.

Now assume that there are $k \geq 2$ type 1A vertices with v_i, v_{i+1} and v_{i+2} being the last three type 1A vertices in clockwise traversal of the external face of G. By the induction hypothesis we can place all the type 1B vertices occurring before v_{i+1} in clockwise order maintaining the required constraint. We now have to correctly place the type 1B vertices in between v_{i+1} and v_{i+2}.

Let $v_{i+1} \leftrightarrow t_{j'}$ and $v_{i+2} \leftrightarrow t_{j''}$, where $t_{j'}, t_{j''} \in V(F_u)$ and $t_{j',1}$, $t_{j',2}$, $\ldots$, $t_{j',s'}$ are the fan vertices of F_u from $t_{j'}$ to $t_{j''}$ in clockwise order. Let $h_1 h_1'$ be perpendicular to the line $v_{i,r} v_{i+1}$ at the point v_{i+1}. Suppose that $g_2 g_2'$ be the tangent to the arc xzy at the point v_{i+2}. By lemma 3.2.5, $\angle t_{j',1} t_{j'} u < \frac{\pi}{2}$. Let us compute a straight line $t_{j'} d_1$ such that $\angle t_{j',1} t_{j'} d_1 \geq \frac{\pi}{2}$ and $t_{j'} d_1$ intersects $g_1 g_1'$ at a point "below" its intersection points with $g_2 g_2'$ and with $v_{i,r} v_{i+1} w_1$. Note that this is always possible. Furthermore, using the same argument as in the base case, it can be shown that the line $t_{j'} d_1$ intersects the arc xzy at a point between v_{i+1} and v_{i+2}.

Now, let us compute an arc with center on the line $h_1 h_1'$ and passing through v_{i+1} and a point y_2 on the line $t_{j'} d_1$ such that y_2 is "above" the intersection point of the lines $t_{j'} d_1$ and $g_1 g_1'$, "below" the intersection point of $t_{j'} d_1$ and $g_2 g_2'$ and also "below" the intersection point of $t_{j'} d_1$ and $v_{i,r} v_{i+1} w_1$. Here, the words "above" and "below" indicate relative order of the intersection points on the line $t_{j'}d_1$ with respect to u.

We now prove that all the type 1 vertices have been placed on a convex curve such that for any three type 1 vertices s_1, s_2 and s_3 in clockwise order, $\angle s_1s_2s_3 > \frac{\pi}{2}$. This property of the drawing will be used in proving the correctness of the drawing. We have the following cases:

* *Case I. s_1 and s_3 are of type 1A*

  In this case, s_1 and s_3 are placed on the arc xzy. If s_2 is of type 1A, then s_2 is also placed on the arc xzy and as a result, $\angle s_1s_2s_3 > \frac{\pi}{2}$, since $\angle s_1us_3 < \frac{\pi}{3}$. On the other hand, if s_2 is of type 1B, then there can be two subcases:

  1. *Disregarding s_2, there is no type 1A vertex between s_1 and s_3.*

     In this subcase, s_2 is a type 1B vertex in between two consecutive type 1A vertices s_1 and s_3. Using an argument similar to the base case, $\angle s_1s_2s_3 > \frac{\pi}{2}$.

  2. *Disregarding s_2, there is a type 1A vertex in between s_1 and s_3.*

     Suppose that the vertex s_2 is in between two type 1A vertices s_i and s_j. Here, s_i may be the same vertex as s_1 and s_j may be the same vertex as s_3, but not both; since if both are true then this turns out to be the same subcase as above.

     By the subcase above, $\angle s_is_2s_j > \frac{\pi}{2}$. By the drawing procedure, whether s_2 is of type 1A or type 1B, s_2 has been placed "inside" the straight line s_1s_i, if $s_1 \neq s_i$ and "inside" the straight line s_js_2, if $s_2 \neq s_j$. Here the word "inside" is meant with respect to the apex u. Thus, the vertex s_2 has been placed "inside" the tangent to the arc xzy at the point s_1, if $s_1 \neq s_i$ and "inside" the tangent to the arc xzy at the point s_2, if $s_2 \neq s_j$ (for illustration please see fig. 3.10). As a result, $\angle s_1s_2s_3 > \pi - \phi$, where $\phi = \angle s_1us_3 < \frac{\pi}{3}$. So, $\angle s_1s_2s_3 > \frac{\pi}{2}$.

Figure 3.10: The internal angle between three type 1 vertices

∗ *Case II. s_1 or s_3 or both are of type 1B*

We have to prove that $\angle s_1 s_2 s_3 > \frac{\pi}{2}$. We have the following subcases:

1. s_1 is of type 1B and s_3 is of type 1A

s_1 is placed "above" the arc xzy and s_3 is placed on this arc. Let us_1 intersect the curve xzy at the point r. Thus using the same argument as in case I, $\angle r s_2 s_3 > \frac{\pi}{2}$, which implies that $\angle s_1 s_2 s_3 > \frac{\pi}{2}$.

2. s_1 is of type 1A and s_3 is of type 1B

This subcase is symmetric to the previous subcase.

3. Both s_1 and s_3 are of type 1B

Let us_1 intersect the curve xzy at the point r_1 and us_3 intersect the curve xzy at the point r_2. By the same argument as in case I, $\angle r_1 s_2 r_2 > \frac{\pi}{2}$, which implies that $\angle s_1 s_2 s_3 > \frac{\pi}{2}$.

- *Type 2: The vertices of $G - V(F_u)$ that do not share an internal face with any vertex*

Figure 3.11: Placement of the vertices of type 2.

of F_u

This type of vertices can be found out by clockwise traversal of the external face of G. Since all the internal vertices are contained in $V(F_u)$, the type 2 vertices are the external vertices of G such that for $1 \leq i \leq l-1$, during traversal they occur in between v_i and v_{i+1}, where v_i and v_{i+1} are adjacent type 1 vertices.

Placement of type 2 vertices:

Placement of the vertices of type 2 is illustrated in fig. 3.11. We place the type 2 vertices in a way described by Lenhart and Liotta [LL96] as follows:

Since all the internal vertices of G are in the fan F_u, none of the type 2 vertices can be adjacent to more than two type 1 vertices, otherwise some of the type 1 vertices become internal vertices of G. We have placed the type 1 vertices so that the chain of edges formed by the type 1 vertices is convex.

Let e_1 be an edge between two type 1 vertices w_1 and w_2 and a subset of the type 2 vertices shares an internal face with the vertex w_1. By lemma 3.2.5, we can draw this subset of vertices inside an obtuse triangle $\Delta w_1 w_2 p_1$ with $\angle w_1 w_2 p_1 > \frac{\pi}{2}$. Similarly if e_2 is the next edge with the type 1 vertices w_2 and w_3 as endpoints, then the subset of type 2 vertices sharing an internal face with the vertex w_2 can be drawn inside an obtuse triangle $\Delta w_2 w_3 p_2$, with $\angle w_2 w_3 p_2 > \frac{\pi}{2}$. Here, we can choose the points p_1

and p_2 such that $\angle w_1 w_2 p_1 > \frac{\pi}{2}$ and $\angle p_1 w_2 p_2 > \frac{\pi}{2}$, since every external angle on the chain of type 1 vertices is greater than π because of convexity.

We perform the placement of type 2 vertices recursively until all of them have been placed.

Proof of proximity constraints

Let us investigate whether the proximity constraints are satisfied for any two vertices of G. We have the following cases.

1. *Proximity constraints for any two vertices of F_u*

   By lemma 3.2.5, the fan F_u is correctly β-drawn inside an obtuse triangle. Furthermore, by the selection of the regions for drawing F_u and $G - V(F_u)$, the proximity region of two adjacent vertices of F_u does not contain any vertex of $G - V(F_u)$.

2. *Proximity constraints for any two vertices of type 1*

   We have three subcases. These are as follows:

   (a) *Proximity constraints for any two vertices of type 1A*

   - *β-region of two nonadjacent type 1A vertices contains at least another vertex.*

     Let v_1 and v_2 be two nonadjacent type 1A vertices. As a result, on the chain of type 1 vertices, there is another vertex r in between v_1 and v_2. This intermediate vertex r is either a type 1A vertex or a type 1B vertex. First, suppose that r is a type 1A vertex. The vertices of type 1A have been placed on an arc centered at the apex u such that $\angle v_1 u v_2 < \delta < \frac{\pi}{2}$. As a result, for any three type 1A vertices v_1, r and v_2, where v_1, r and v_2 occur in clockwise order on the external face of G, $\angle v_1 r v_2 > \frac{\pi}{2}$. So, $r \in R[v_1, v_2, 1]$, which implies that $r \in R[v_1, v_2, \beta]$, for $1 < \beta < 2$.

Now consider the other case that r is a type 1B vertex. r is placed on a curve such that v_1, r and v_2 are on a convex chain and $\frac{\pi}{2} < \angle v_1 r v_2 < \pi$. Thus $r \in R[v_1, v_2, 1]$.

Figure 3.12: $R[v_1, v_2, \infty]$ does not contain any vertex of type 1A or type 1B, where v_1 and v_2 are two adjacent vertices of type 1A.

- *β-region of two adjacent type 1A vertices is empty.*

  Let v_1 and v_2 be two adjacent type 1A vertices. It suffices to prove that $R[v_1, v_2, 2]$ is void of vertices of F_u, of types 1A, 1B and 2.

  Due to the choice of the regions $\mathcal{R}_1$ and $\mathcal{R}_2$, the β-region of any two type 1 vertices does not contain any vertex of F_u. Moreover, since the type 1A vertices have been placed on an arc xzy centered at the point u with $\angle xuy < \delta < \frac{\pi}{2}$, $R[v_1, v_2, \infty]$ does not contain any other type 1A vertex (please see fig. 3.12). We now prove that $R[v_1, v_2, \infty]$ does not contain any type 1B vertex. Let v_i and v_{i+1} be two nonadjacent vertices of type 1A and $v_{i,1}$, $v_{i,2}$, $\ldots$, $v_{i,r}$ be the type 1B vertices that occur in clockwise traversal of the external face of G from v_i to v_{i+1} in this order (please see fig. 3.9). Since $v_{i,1}$, $v_{i,2}$, $\ldots$, $v_{i,r}$ have been placed in between the straight lines uv_i and uv_{i+1} and "outside" [1] the arc xzy, all these type 1B vertices

[1] The term "outside" is mentioned with respect to the vertex u.

are outside $R[v_1, v_2, \infty]$. This is illustrated in fig. 3.12.

Figure 3.13: $R[v_1, v_2, \beta]$ does not contain any vertex of type 2.

We now prove that $R[v_1, v_2, \beta]$ does not contain any vertex of type 2. This proof is illustrated in fig. 3.13. The vertices of type 2 have been placed inside obtuse triangles according to lemma 3.2.5. So, the type 2 vertices that have been placed inside the obtuse triangle $\Delta v_1 v_2 p_2$ are outside $R[v_1, v_2, \beta]$. Furthermore, since $\angle p_1 v_1 p_2 > \frac{\pi}{2}$ by the drawing procedure, hence $\angle p_1 v_1 v_2 > \frac{\pi}{2}$. So, the type 2 vertices that have been placed inside $\Delta w_1 v_1 p_1$ are outside $R[v_1, v_2, \infty]$. Thus, all the type 2 vertices occuring before v_1 in clockwise traversal around the external face of G are outside $R[v_1, v_2, \infty]$. By similar argument, the same can be shown for all the type 2 vertices occuring after v_2 in clockwise traversal of the external face of G. This concludes the proof that $R[v_1, v_2, \beta]$ does not contain any vertex of type 2.

(b) *Proximity constraints for any two vertices of type 1B*

- *β-region of two nonadjacent type 1B vertices contains at least another vertex.*

  Let v_1 and v_2 be two nonadjacent vertices of type 1B. So, there is an intermediate vertex v_i of either type 1A or type 1B. As proved earlier, $\angle v_1 v_i v_2 > \frac{\pi}{2}$. Thus the statement follows.

- *β-region of two adjacent type 1B vertices is empty.*

  Let v_1 and v_2 be two adjacent type 1B vertices. By the same argument used in proving that the β-region of two adjacent type 1A vertices is empty, we can show that $R[v_1, v_2, \beta]$ is devoid of any vertex of F_u and of type 2. Now it remains to be shown that $R[v_1, v_2, \beta]$ does not contain vertices of type 1A and type 1B, other than v_1 and v_2 themselves.

  We have already shown that the vertices of type 1 form a convex polytope with the property that for any three vertices of type 1 in clockwise order, the internal angle is greater than $\frac{\pi}{2}$ and less than π. Thus, $R[v_1, v_2, \infty]$ does not contain any vertex of type 1.

(c) *Proximity constraints between a type 1A vertex and a type 1B vertex*

Let v_1 be a type 1A vertex and v_2 be a type 1B vertex. Using similar arguments as above, it can be shown that if $v_1 \leftrightarrow v_2$ then $R[v_1, v_2, \beta]$ is empty, else $R[v_1, v_2, 1]$ contains the type 1 vertices in between v_1 and v_2.

3. *Proximity constraints for any two vertices of type 2*

- *β-region of two nonadjacent type 2 vertices contains at least another vertex.*

  Please see fig. 3.11 for illustration of the proof. Suppose that e_1 is an edge between two type 1 vertices w_1 and w_2. The type 2 vertices that share internal faces with the vertex w_1 have been placed on an obtuse triangle $\Delta w_1 w_2 p_1$ such that $\angle w_1 w_2 p_1 > \frac{\pi}{2}$. Similarly if e_2 is the next edge with the type 1 vertices w_2 and w_3 as endpoints, then the of type 2 vertices sharing an internal face with the vertex w_2 have been drawn inside an obtuse triangle $\Delta w_2 w_3 p_2$, with $\angle w_2 w_3 p_2 > \frac{\pi}{2}$. The points p_1 and p_2 have been chosen such that $\angle p_1 w_2 p_2 > \frac{\pi}{2}$.

First, we prove that for any point x in $\Delta w_1 w_2 p_1$ and y in $\Delta w_2 w_3 p_2$, $w_2 \in R[x, y, \beta]$. The statement is proved by the fact that $\angle x w_2 y > \frac{\pi}{2}$, since $\angle p_1 w_2 p_2 > \frac{\pi}{2}$ and $\angle w_1 w_2 w_3 > \frac{\pi}{2}$. Next, using similar argument, we can also prove that for any two obtuse triangles T_1 and T_2 based on two edges with type 1 vertices as endpoints, a type 1 vertex $w \in R[x, y, \beta]$, where x and y are inside the triangles T_1 and T_2 respectively. Thus the statement follows for this case.

Next, we prove that for two nonadjacent type 2 vertices inside a single obtuse triangle there exists another vertex in their β-region. This follows immediately from lemma 3.2.5.

- *β-region of two adjacent type 2 vertices is empty.*

  Here we use arguments similar to those presented above. First, the region for drawing type 2 vertices has been chosen such that the β-region of any two type 2 vertices does not contain any vertex of F_u. Let v_1 and v_2 be two adjacent type 2 vertices and v_1 and v_2 be drawn inside an obtuse triangle based on an edge with two type 1 vertices w_1 and w_2 as endpoints, where w_1 is the apex. According to lemma 3.2.5, the β-region of v_1 and v_2 does not contain w_1, w_2 and any type 2 vertex inside the same triangle. Now it suffices to prove that $R[v_1, v_2, \beta]$ does not contain any type 1 or type 2 vertex outside the triangle in which v_1 and v_2 are drawn. The illustration for this proof is shown in fig. 3.14 and we use the same notations as used in the description of drawing the type 2 vertices.

  By definition, the edge $v_1 v_2$ is a fan edge of F_{w_1}. Let v_0 be the type 2 vertex adjacent to w_2. Since $\angle v_0 w_2 p_2 > \angle p_1 w_2 p_2 > \frac{\pi}{2}$, any point inside the triangle $\Delta w_2 w_3 p_2$ is outside $R[v_0, w_2, \infty]$, which is indicated by the parallel line strip in fig. 3.14. Since the fan edges of F_{w_1} form a convex chain having the property that the internal angle of any three vertices on this chain is greater than $\frac{\pi}{2}$ and less than π, for any fan edge $v_1 v_2$ the β-region of v_1 and v_2 does not contain

Figure 3.14: β-region of two adjacent type 2 vertices v_1 and v_2 is empty.

any point of $\Delta w_2 w_3 p_2$.

Now, let us consider the points of $\Delta w_0 w_1 p_0$. For any two points x and y inside $\Delta w_1 w_2 p_1$, $R[x, y, 2]$ does not contain any point of $\Delta w_0 w_1 p_0$. This is because $\angle p_0 w_1 p_1 > \frac{\pi}{2}$ and $\angle w_0 w_1 w_2 > \frac{\pi}{2}$. Thus an arc with center at x or at y and with radius xy does not pass through triangle $\Delta w_0 w_1 p_0$ and the intersection of these two arcs does not overlap with $\Delta w_0 w_1 p_0$.

Thus, $R[v_1, v_2, \beta]$ does not contain any type 1 or type 2 vertex outside the triangle in which v_1 and v_2 are drawn.

4. *Proximity constraints for a vertex of F_u and another of type 1*

- *β-region of a vertex of F_u and another of type 1 that are nonadjacent contains at least another vertex*

  For illustration of the proof please see fig. 3.9.

  Let us first consider the case of a type 1A vertex nonadjacent to a vertex of F_u. By definition, a type 1A vertex has a neighboring vertex in F_u. For example, following the illustration in fig. 3.9, the type 1A vertex v_{i+1} is adjacent to $t_{j'} \in V(F_u)$. The type 1A vertex v_{i+1} is placed in such a way that for any vertex $v \neq t_{j'} \in V(F_u)$, $\angle v_{i+1} t_{j'} v > \frac{\pi}{2}$. Thus $R[v_{i+1}, v, 1]$ contains $t_{j'}$. Same is the case for any type 1A vertex and a nonadjacent vertex of F_u.

Now let us consider the other case; that is the case of type 1B vertex. By definition, a type 1B vertex has no neighbor in F_u. Let $v_{i+1,1}$ be a type 1B vertex which occurs in between two type 1A vertices v_{i+1} and v_{i+2} as shown in fig. 3.9. Here, v_{i+1} is adjacent to $t_{j'} \in V(F_u)$ and v_{i+2} is adjacent to $t_{j''} \in V(F_u)$. Since $v_{i+1,1}$ has been placed "above" the tangent $g_1 g'_1$ to the arc xzy at the point v_{i+1}, $\angle t_{j'} v_{i+1} v_{i+1,1} > \frac{\pi}{2}$ and thus $v_{i+1} \in R[t_{j'}, v_{i+1,1}, 1]$. Furthermore, $v_{i+1,1}$ has been placed to the "left" of the line $t_{j'} d_1$ which is perpendicular to the line $t_{j'} t_{j',1}$. Thus, for any vertex $v \in V(F_u)$ to the "right" of $t_{j'}$, $\angle v t_{j'} v_{i+1,1} > \frac{\pi}{2}$. On the other hand, the same is true for any vertex $v \in V(F_u)$ to the "left" of $t_{j'}$, since $\angle v t_{j'} v_{i+1} > \frac{\pi}{2}$.

Thus, the statement is proved.

- *β-region of a vertex of F_u and another of type 1 that are adjacent is empty*

For illustration of the proof please see fig. 3.15. By definition, only a vertex of type 1A can be adjacent to a vertex of F_u.

From lemma 3.2.5 we find that each of the fan vertices of the fan F_u has been placed on an arc with center at u. Furthermore, the fan vertices form a convex chain of fan edges such that for any three fan vertices w_1, w_2 and w_3 on the chain in clockwise order, $\angle w_1 w_2 w_3 > \frac{\pi}{2}$. Furthermore, We have placed all the type 1A vertices on an arc xzy centered at u.

Let v be a type 1A vertex such that $v \leftrightarrow w_2$. We show that $R[w_2, v, 2]$ is empty. By lemma 3.2.5, all the fan vertices to the right of w_2 have been drawn "inside" (w.r.t. u) the tangent hh' to the β-boundary curve $\mathcal{C}_{u,w_2,2}$ at the point w_2. Furthermore, all the fan vertices to the left of w_2 have also been drawn "inside" (w.r.t. u) hh'. If we consider $R[w_2, v, 2]$, we find that hh' is also tangent to the circle centered at v and having radius vw_2, since the β-

Figure 3.15: Illustration for the proof that the β-region of a vertex of F_u and another of type 1 that are adjacent is empty.

boundary curve $\mathcal{C}_{u,w_2,2}$ coincides with $R[u, w_2, 2]$. Thus for any vertex $v \neq w_2$ and $v \in V(F_u)$, $v \notin R[w_2, v, 2]$. Furthermore all the other vertices of type 1 and type 2 are also outside $R[w_2, v, 2]$. Thus $R[w_2, v, 2]$ is empty, implying that $R[w_2, v, \beta]$ is also empty.

5. *Proximity constraints for a vertex of F_u and another of type 2*

   By definition, the type 2 vertices are not adjacent to any vertex of F_u. Thus it suffices to prove that there is at least another vertex in $R[s, t, 1]$, where $t \in V(F_u)$ and s is a type 2 vertex. The illustration for the proof is shown in fig. 3.16.

Figure 3.16: Illustration for the proof that there is at least another vertex in the β-region of a vertex of F_u and another of type 2.

Let us consider an edge $e_1 = w_1w_2$ on the chain of type 1 vertices. In an obtuse triangle based on w_1w_2 we have placed the corresponding type 2 vertices according to lemma 3.2.5, as illustrated in fig. 3.11. Let s be a type 2 vertex placed inside the mentioned triangle and t be any vertex of F_u. By lemma 3.2.5, $\angle w_1sw_2 < \frac{\pi}{2}$ and by the choice of regions $\mathcal{R}_1$ and $\mathcal{R}_2$, $\angle w_1tw_2 < \alpha < \frac{\pi}{3}$. Now if we consider the quadrilateral tw_1sw_2, then we find that $\angle tw_1s + \angle sw_2t > \frac{7\pi}{6}$. As a result, either $\angle tw_1s$ or $\angle sw_2t$ is greater than $\frac{\pi}{2}$, which implies that either w_1 or w_2 is in $R[s, t, 1]$.

6. *Proximity constraints for a vertex of type 1 and a vertex of type 2*

Using the illustration of fig. 3.11 it has been shown that the type 2 vertices have been placed inside obtuse triangles recursively following lemma 3.2.5. It has also been shown that if x and y are two points inside two distinct triangles then $R[x, y, 1]$ contains at least another point. Furthermore, by lemma 3.2.5, inside each obtuse triangle the vertices have been correctly placed. Thus if a vertex of type 1 and a vertex of type 2 are adjacent then their β-region is empty; and if they are nonadjacent then their β-region contains at least another vertex.

3.3 Application

In this section we illustrate a practical application of the studies we have made so far. The application we provide here is drawn from the domain of wireless sensor networks. One of the most interesting aspects of a wireless sensor network is the contention between two conflicting requirements: one is the conservation of battery power and the other is the dissemination of information. The sensors need to disseminate their information to other sensors and at the same time need to save their battery power. That is why energy efficient topology construction is a very important consideration in the field of wireless sensor networks.

We can model an energy efficient topology in a wireless sensor network by using parameterized proximity graphs, because one of the underlying features in a proximity graph is that the group of adjacent vertices tend to stay close together, whereas the group of nonadjacent vertices tend to stay relatively farther away. In the case of a wireless sensor network, the same principle works, that is sensors that are near to each other should communicate with each other and sensors that are relatively farther away from each other should not communicate with each other since it requires greater power. Next we show how we can place sensors in a plane when a desired topology is given as input.

Suppose that we are given a biconnected 2-outerplane graph G describing the topology of sensors in a wireless sensor network. For illustration please see fig. 3.17. We have to

Figure 3.17: The biconnected 2-outerplane graph G describing the topology of sensors in a wireless sensor network.

place the sensors in a plane such that the placement satisfies the proximity constraints for a given value of β, where $1 < \beta < 2$. This range of β value guarantees that if G is β-drawable then there will be no edge-crossing in the β-drawing of G, which is practically desirable. The first question we have is that is the graph G β-drawable for the given value of β? We have proved that if G satisfies the conditions of theorem 3.2.1 then G is β-drawable. Suppose that G satisfies the conditions of theorem 3.2.1. Next we have to draw G maintaining the proximity constraints. The drawing of G shown in fig. 3.17 is illustrated in fig. 3.18.

Since G satisfies the conditions of theorem 3.2.1, by lemma 3.2.3 we can find an appropriate apex $u \in V(G)$. According to the drawing algorithm given in section 3.2, we can draw the fan F_u in an obtuse triangle. For the example given in fig. 3.17, the vertices

Figure 3.18: Drawing of the biconnected 2-outerplane graph G shown in fig. 3.17.

of F_u are u, a, b, c, d, e, f, g, h and i. The drawing of F_u is placed inside the region R_2 as described in section 3.2. Next the remaining graph is drawn in region R_1 according to the drawing algorithm of theorem 3.2.1 and all the edges between the vertices of the fan and the vertices of the remaining graph are drawn.

Finally, we can place the sensors in the real world according to the obtained drawing of G and scaling the drawing to account for the desired maximum length of an edge.

3.4 Chapter Summary

In this chapter we have concentrated on the problem of β-drawability of a biconnected 2-outerplane graph. We have given a necessary condition and a set of sufficient conditions for β-drawability of a biconnected 2-outerplane graph. We have proved the sufficient conditions by construction. This proof method gives us an $O(n^2)$ drawing algorithm. Although the drawing algorithm has a high time complexity, the focus of our work is on answering the question of β-drawability of a biconnected 2-outerplane graph. In the next chapter we are going to explore "forbidden" biconnected 2-outerplane graphs, which do not admit a β-drawing. We are going to identify a subclass of biconnected 2-outerplane graphs that are forbidden. We are also going to show that if a biconnected 2-outerplane graph does not satisfy the sufficient conditions, then it is not necessarily β-drawable.

Chapter 4

Forbidden 2-Outerplane Graphs

Up to this point, we have found a set of sufficient conditions and proved that if a biconnected 2-outerplane graph G satisfies these conditions then G can be β-drawn, for $1 < \beta < 2$. But what can we say about the biconnected 2-outerplane graphs that do not satisfy these conditions? Are these graphs β-drawable? In this chapter we address this question. We define a class of graphs as *forbidden* if no graph of this class is β-drawable for a specified value of β.

Figure 4.1: Forbidden class of biconnected 2-outerplane graphs due to corollary 3.1.2 (marked with "cross") and β-drawable class of biconnected 2-outerplane graphs (marked with "tick") due to theorem 3.2.1. The remaining class, marked with "?" denotes the class of biconnected 2-outerplane graphs for which we seek an answer to the proximity drawability problem.

4.1 Necessary Condition

In Chapter 3, we have specified a necessary condition for β-drawability of biconnected 2-outerplane graphs. This necessary condition is stated in corollary 3.1.2. The necessary condition is that if G is a biconnected 2-outerplane graph having less than five external vertices then G has no β-drawing, for $1 < \beta < 2$. Thus we have identified a forbidden class of biconnected 2-outerplane graphs, that is the class of biconnected 2-outerplane graphs having less than five external vertices. On the other hand, by theorem 3.2.1 we get a subclass of biconnected 2-outerplane graphs that is β-drawable for $1 < \beta < 2$. Fig. 4.1 illustrates these findings.

4.2 Sufficient Condition

We now show that the biconnected 2-outerplane graphs violating the sufficient conditions specified in theorem 3.2.1 are not always β-drawable for $1 < \beta < 2$. In proving this claim we use lemma 3.1.1, which says that a planar embedded graph, that has a complex 3-cycle or a complex 4-cycle, is not β-drawable.

Theorem 4.2.1 *If $\mathcal{F}$ is the class of biconnected 2-outerplane graphs that do not satisfy at least one of the conditions specified in theorem 3.2.1, then every graph in the class $\mathcal{F}$ is not necessarily β-drawable, for $1 < \beta < 2$.*

Proof. For each of the sufficient conditions, we have to find a member of $\mathcal{F}$ that does not satisfy the condition and that is not β-drawable for $1 < \beta < 2$.

Suppose that a biconnected 2-outerplane graph $G \in \mathcal{F}$ violates condition 1. So, G has either 3 or 4 vertices in the external face. Thus by corollary 3.1.2, G is not β-drawable for $1 < \beta < 2$.

Let us consider the biconnected 2-outerplane graph shown in fig. 4.2. In this graph u is the only external vertex for which we have all the internal vertices included in F_u. We

Figure 4.2: A biconnected 2-outerplane graph that satisfies all the conditions of theorem 3.2.1 except condition 2(a).

find that this graph satisfies all the conditions but condition 2(a), i.e. the fan F_u is not 1-outerplane. This graph is not β-drawable for $\beta \in (1, 2)$ as it contains a complex 4-cycle with the vertex v inside the cycle.

Figure 4.3: A biconnected 2-outerplane graph that satisfies all the conditions of theorem 3.2.1 except condition 2(b), i.e. it does not have any external vertex k such that the fan F_k contains all the internal vertices.

Let us consider the biconnected 2-outerplane graph shown in fig. 4.3. This graph satisfies all of the mentioned conditions but condition 2(b), i.e. there is no external vertex k such that F_k is a fan containing all the internal vertices. As can be readily seen, this graph is not β-drawable for $\beta \in (1, 2)$, since it contains complex 4-cycles.

Figure 4.4: A biconnected 2-outerplane graph that satisfies all the conditions of theorem 3.2.1 except condition 2(c).

Let us now consider the biconnected 2-outerplane graphs shown in fig. 4.4(a) and 4.4(b). These two graphs satisfy all the conditions in the lemma except condition 2(c). For the graph shown in fig. 4.4(a), u and v are the only two external vertices the fan of which contains all the internal vertices (condition 2(b)). Considering F_u as the fan in question, v is outside the fan and has two neighbors in the fan. Same is case if we consider the fan F_v. We find that this graph is not β-drawable for $\beta \in (1, 2)$, since the vertex x is inside a 4-cycle. On the other hand, if we consider the graph in fig. 4.4(b) we find that u is the only external vertex for which the fan F_u contains all the internal vertices. A vertex x in the fan F_u has two neighbors outside the fan. This graph is also not β-drawable for $\beta \in (1, 2)$ as there are two complex 4-cycles.

Thus, it is proved that if a biconnected 2-outerplane graph does not satisfy the specified conditions then the graph is not necessarily β-drawable for $\beta \in (1, 2)$. □

4.3 Chapter Summary

In this chapter we have identified a subclass of biconnected 2-outerplane graphs that are forbidden. This class of forbidden graphs is just the direct result of the necessary condition for β-drawability of a biconnected 2-outerplane graph, that is given in the previous chapter. We have also explored the graphs that do not belong to the forbidden class and also do not belong to the β-drawable graph. We have shown that if a biconnected 2-outerplane graph does not satisfy the sufficient conditions, then it is not necessarily β-drawable. In the next chapter we conclude this thesis with an overview of the results and a brief discussion on the open problems related to this thesis.

Chapter 5

Conclusion

In this thesis we have studied the β-drawability problem for biconnected 2-outerplane graphs. The results of the thesis are summarized as follows:

- We have specified a necessary condition for β-drawability of biconnected 2-outerplane graphs, for $1 < \beta < 2$.
- We have found sufficient conditions for β-drawability of biconnected 2-outerplane graphs, for $1 < \beta < 2$.
- For a biconnected 2-outerplane graph that satisfies the sufficient conditions, we have given an $O(n^2)$ drawing algorithm for β-drawing the graph, for $1 < \beta < 2$.
- The specified necessary condition implies a forbidden class of biconnected 2-outerplane graphs which cannot be β-drawn, for $1 < \beta < 2$. The sufficient conditions imply a subclass of biconnected 2-outerplane graphs that are β-drawable for the same range of β values. We have proved by showing counter-examples that if a biconnected 2-outerplane graph does not belong to the forbidden class and also not to the above mentioned β-drawable class, then this graph is not necessarily β-drawable.

We conclude this thesis with the following open problems:

- Complete characterization of biconnected 2-outerplane graphs is yet to be done. In this problem, one has to provide necessary and sufficient conditions for β-drawability of a biconnected 2-outerplane graph. Interestingly, complete characterizations of trees and outerplanar graphs are still open problems as well [BDLL95, LL96].

- β-drawings of biconnected 2-outerplane graphs have been studied in this thesis and those of trees and outerplanar graphs have been studied in [BDLL95, LL96]. However, all these studies concern little about the area requirement. Penna and Vocca report that the existing algorithms, such as those in [BDLL95, LL96], require exponential area [PV04, PV98]. So, β-drawings of trees, outerplanar graphs and biconnected 2-outerplane graphs in polynomial area is an open problem. Interestingly, nobody still has the answer to the simplest of these problems, i.e. polynomial area β-drawings of trees for $\beta = 1$.

- Characterization of β-drawability of k-outerplane graphs, for any $k \geq 1$, will be a very interesting finding. This is because every planar embedded graph is a k-outerplane graph, for some value of $k \geq 1$. Thus if we are able to find an answer the question of β-drawability problem for k-outerplane graphs, for any $k \geq 1$ and for some value of β, then given a planar embedded graph we can tell if that graph can be β-drawn. However, this will be a greatly challenging problem, because even for the simplest of graphs, such as trees, there are some values of β for which the answer to the β-drawability problem is not known.

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