Solving Fisher's Market: Eisenberg-Gale Convex Program and KKT Conditions

Mohammad T. Irfan
Bowdoin College
mirfan@bowdoin.edu

This note supplies some additional details for understanding [2, Chapter 5]. In particular, We provide some background on the Karush-Kuhn-Tucker (KKT) conditions that are used there. The KKT conditions are applicable to general non-linear optimization problems, but in the context of solving Fisher's market, these conditions are applied to a particular convex program called the Eisenberg-Gale convex program. Remarkably, this convex program binds together the individual but inter-dependent optimizations of the buyers in Fisher's market into a single optimization program.

1 Karush-Kuhn-Tucker (KKT) Conditions

We first specify a general non-linear constrained optimization in a standard format. Here, $\mathbf{x} = (x_1, \ldots, x_n)$ denotes a vector (or array) of n variables.

$$\begin{aligned}
&\text{Min} \quad f(\mathbf{x}) \\
&\text{Subject to} \\
&\qquad g_i(\mathbf{x}) \leq 0, \quad i = 1, \ldots, m
\end{aligned}$$

Below is the Lagrange function for the above mathematical program, where $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_m)$ are dual variables (also known as Lagrange multipliers) corresponding to the constraints $g_i(\mathbf{x}) \leq 0$ for $i = 1, \ldots, m$.

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_i \lambda_i g_i(\mathbf{x}).$$

The KKT conditions are necessary conditions for optimality, meaning that these conditions are true for sure at any optimal solution for the above mathematical program. These necessary conditions become sufficient under various conditions. One such condition is Slater's condition, which is applicable when the program is convex and feasible.

If $\mathbf{x}^*$ is an optimal solution, then there exists $\boldsymbol{\lambda}^*$ such that the following four KKT conditions hold.

- Stationary condition:
  $\nabla L(\mathbf{x}^*, \boldsymbol{\lambda}^*) = 0$.
  That is, $\frac{\partial f(\mathbf{x}^*)}{\partial x_k} + \sum_i \lambda_i^* \frac{\partial g_i(\mathbf{x}^*)}{\partial x_k} = 0$, for any x_k.
- Primal feasibility condition:
  $g_i(\mathbf{x}^*) \leq 0$, for $i = 1, \ldots, m$.
- Dual feasibility condition:
  $\lambda_i^* \geq 0$, for $i = 1, \ldots, m$.
- Complementary slackness condition:
  $\lambda_i^* g_i(\mathbf{x}^*) = 0$, for $i = 1, \ldots, m$.

2 The Eisenberg-Gale Convex Program (1959)

In [2, Chapter 5], the Eisenberg-Gale convex program [1] is presented in the context of Fisher's market. In this market model, there are n goods (indexed by j) and n' buyers (indexed by i). There is 1 divisible unit of each good. Each buyer i has a budget e_i and has a utility $u_{i,j}$ per unit of good j. Buyer i's utility from an allocation $\boldsymbol{x}_i = (x_{i,1}, \ldots, x_{i,n})$ is $u_i(\boldsymbol{x}_i) = \sum_j x_{i,j} u_{i,j}$.

Under the constraint that each buyer uses their budget to maximize own utility, the model seeks to find *equilibrium allocations* $\boldsymbol{x}^*$, which are allocations for market clearance (i.e., no surplus of goods or money). This invites the question *equilibrium prices* of goods. The beauty of duality theory is that equilibrium prices arise as duals of equilibrium allocations.

Usually, it is hard to derive one single optimization routine when there are many individual optimizations (e.g., each buyer maximizing own utility) that are tied together by some global constraints (e.g., a supply of 1 unit per good). However, in this case, computer scientists remarkably connected this market model with the Eisenberg-Gale convex program. The objective function is to maximize the budget-weighted geometric mean of the buyers' utilities: $(\prod_i u_i^{e_i})^{\frac{1}{\sum_i e_i}}$. Since $\frac{1}{\sum_i e_i}$ is a constant, it can be discarded from maximization. Then, taking log gives us $\sum_i e_i \log u_i$, where u_i is used as a shorthand notation for $u_i(\boldsymbol{x}_i) = \sum_j x_{i,j} u_{i,j}$. The Eisenberg-Gale convex program for Fisher's market is given below.

$$
\begin{aligned}
&\text{Max} \quad \sum_i e_i \log \left(\sum_j u_{i,j} x_{i,j} \right) \\
&\text{Subject to} \\
&\qquad \sum_i x_{i,j} \leq 1, \quad \forall j \\
&\qquad x_{i,j} \geq 0, \quad \forall i, j
\end{aligned}
$$

We reshape this program to conform to the standard format given in Section 1.

$$
\begin{aligned}
&\text{Min} \quad -\sum_i e_i \log\left(\sum_j u_{i,j}x_{i,j}\right) \\
&\text{Subject to} \\
&\qquad \sum_i x_{i,j} - 1 \leq 0, \quad \forall j \\
&\qquad - x_{i,j} \leq 0, \quad \forall i, j
\end{aligned}
$$

Below is the Lagrange function, where λ_j and $\mu_{i,j}$ are dual variables corresponding to the first and second group of constraints, respectively. In the context of Fisher's market, λ_j is nothing but the price of good j.

$$
L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = -\sum_i e_i \log\left(\sum_j u_{i,j}x_{i,j}\right) + \sum_j \lambda_j \left(\sum_i x_{i,j} - 1\right) + \sum_i \sum_j \mu_{i,j}(-x_{i,j}).
$$

If $\mathbf{x}^*$ is an optimal solution, then there exist $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ such that the following KKT conditions hold.

Stationary condition:
For any $x_{i,j}$,

$$
\begin{aligned}
&\frac{\partial L}{\partial x_{i,j}} = 0. \\
&-\frac{e_i u_{i,j}}{\sum_j u_{i,j}x^*_{i,j}} + \lambda^*_j - \mu^*_{i,j} = 0. \\
&\frac{e_i u_{i,j}}{\sum_j u_{i,j}x^*_{i,j}} = \lambda^*_j - \mu^*_{i,j}.
\end{aligned}
$$

As we will show in dual feasibility below, $\mu^*_{i,j} \geq 0$. Therefore, in general, the following holds.

$$
\begin{aligned}
\frac{e_i u_{i,j}}{\sum_j u_{i,j}x^*_{i,j}} &\leq \lambda^*_j. \\
\frac{u_{i,j}}{\lambda^*_j} &\leq \frac{\sum_j u_{i,j}x^*_{i,j}}{e_i}. \qquad (1)
\end{aligned}
$$

Moreover, if $\mu^*_{i,j} = 0$, the above inequality becomes an equality as follows.

$$
\frac{u_{i,j}}{\lambda^*_j} = \frac{\sum_j u_{i,j}x^*_{i,j}}{e_i}. \qquad (2)
$$

Primal feasibility:

$$\sum_i x^*_{i,j} \leq 1, \quad \forall j$$
$$x^*_{i,j} \geq 0, \quad \forall i, j$$

Dual feasibility:

$$\lambda^*_j \geq 0, \quad \forall j$$
$$\mu^*_{i,j} \geq 0, \quad \forall i, j$$

Complementary slackness:

1. For the first group of constraints:

$$\lambda^*_j \left(\sum_i x^*_{i,j} - 1 \right) = 0, \quad \forall j$$

   Therefore, if $\lambda^*_j > 0$, then $\sum_i x^*_{i,j} = 1$, for any j. In the context of Fisher's market, this means that <u>at an optimal solution, if the price of a good j is positive, then that good is sold out.</u>

2. For the second group of constraints:

$$\mu^*_{i,j} x^*_{i,j} = 0, \quad \forall i, j$$

   Therefore, if $x^*_{i,j} > 0$, then $\mu^*_{i,j} = 0$. In the context of Fisher's market, whenever a buyer i buys a good j (i.e., $x^*_{i,j} > 0$), it must be the case that $\mu^*_{i,j} = 0$, and in this case, Equation (2) gives us the following.

$$\frac{u_{i,j}}{\lambda^*_j} = \frac{\sum_j u_{i,j} x^*_{i,j}}{e_i}.$$

   Using Inequality (1), this means that <u>whenever a buyer i buys a good j at an optimal solution, the buyer is inevitably maximizing the bang per buck.</u>

The above two underlined statements will be used next to show that the Eisenberg-Gale convex program computes an equilibrium point of Fisher's market.

3 Proof: Eisenberg-Gale Solves Fisher's Market

We present a more elaborate version of the proof given in Theorem 5.1 of [2]. We show that if each good has some interested buyer (i.e., a buyer who gets positive utility from the good), then an optimal solution of the Eisenberg-Gale convex program is an equilibrium point of Fisher's market. The proof consists of two parts. Put together, these two parts show that the market clears at an optimal solution of the Eisenberg-Gale convex program, thereby giving us a constructive proof of equilibrium existence as well.

Part 1: All goods are sold out.

The first complementary slackness condition above shows that at an optimal solution, the price of good j, $\lambda_j^* > 0$, then good j is sold out. Here, we show that for all goods j, $\lambda_j^* > 0$.

Suppose that $\lambda_j^* = 0$ for some good j. Inequality (1) gives us the following for any i.

$$\begin{aligned} u_{i,j} &\le \lambda_j^* \times \frac{\sum_j u_{i,j} x_{i,j}^*}{e_i} = 0, \forall i \\ u_{i,j} &\le 0, \forall i. \end{aligned}$$

That is, no buyer is interested in good j, which is a contradiction.

Part 2: No buyer has unspent money.

Consider any buyer i. Since buyers maximize their utilities, there must be a good j such that $x_{i,j}^* > 0$.[1] The second complementary slackness condition gives us the following.

$$\begin{aligned} &\frac{u_{i,j}}{\lambda_j^*} = \frac{\sum_j u_{i,j} x_{i,j}^*}{e_i} \\ \implies &u_{i,j} e_i = u_i \lambda_i^* \\ \implies &e_i \sum_j u_{i,j} x_{i,j}^* = u_i \sum_j \lambda_i^* x_{i,j}^* \quad \text{[multiplying both sides with } x_{i,j}^* \text{ and summing over all } j\text{]} \\ \implies &e_i u_i = u_i \sum_j \lambda_i^* x_{i,j}^* \\ \implies &e_i = \sum_j \lambda_i^* x_{i,j}^* \quad \text{[dividing by } u_i\text{, since } u_i > 0\text{].} \end{aligned}$$

Therefore, all buyers i have spent all of their budget.

References

[1] E. Eisenberg and D. Gale. Consensus of subjective probabilities: The pari-mutuel method. *The Annals of Mathematical Statistics*, 30(1):165–168, 1959.

[2] N. Nisan, T. Roughgarden, E. Tardos, and V. Vazirani. *Algorithmic Game Theory*. Cambridge University Press, 2007.

[1] Otherwise, buyer i's utility for all goods is 0. As such, buyer i can be taken out of the market.