Physics 81 Set 1 Solutions

1. **Birthdays:** Person #1 has a birthday on a particular day. The probability of this is 1. Person #2 has a $\frac{1}{365}$ chance of matching this birthday. More importantly, they have a $\frac{364}{365}$ chance of *not* matching this birthday. If #1 and #2 don't match, then #3 has a $\frac{363}{365}$ chance of *not* matching. Thus, the chance of #1 <u>and</u> #2 <u>and</u> #3 *not* matching is

$$1 \times \frac{364}{365} \times \frac{363}{365} = 0.992$$

since exclusive probabilities multiply.

Keep adding people to the room until the product is 0.5. You will find that if 23 people are in the room, the chance of *no* shared birthday has dropped below 0.5. Thus, with 23 people, the chance of 1 or more shared birthdays is $\geq 50\%$. Surprisingly few!

2. **Rice:** First, convert from tons to grams:

$$120 \text{milliontons} = 2.4 \times 10^{11} \text{lbs} = 1.0886 \times 10^{14} g$$

Next, since I told you that 1 grain has a mass of $0.0194g$, you can show this corresponds to 5.61×10^{15} grains of rice.

On the first day, 1 grain is delivered. On the second day there are $1 + 2 = 3$ grains. On the third day, there are $2^0 + 2^1 + 2^2 = 7$ grains. Thus, on the Nth day, there are

$$\sum_{m=0}^{N} 2^m$$

grains. If you set this up in a spreadsheet, or on your calculator, you'll find that by the end of day 52, the Prince has delivered 4.5×10^{15} grains, and his stockroom is nearly empty. On the next day, he would need an additional 4.5×10^{15} grains, and he doesn't have them.

So, he lasts 52 days and is bankrupt on the 53rd.

An alternate solution, approximating the delivery of rice as a continuous process

It is possible to use the equation for exponential growth to describe the amount of rice that has been delivered, in direct analogy to the number of lily pads in a pond, after a particular day. The only challenge is to get the right value of λ. To find out the correct value, let's write down the equation for day 2 of the delivery. The generic form of the equation

$$N = N_0 e^{\lambda t}$$

becomes

$$2 = 1 \times e^{\lambda 1}$$

since we started with 1 grain ($N_0 = 1$), we wait for 1 day ($t = 1$), and we end up with 2 grains ($N = 2$).

Using the fact that $ln(e^x) = x$, we can solve for λ :

$$ln(2) = ln(e^{\lambda}) = \lambda = 0.6931$$

Now, we just have to ask how many days elapse before the delivery of 1 grain has grown to 5.61×10^{15} grains. This is equivalent to solving for t in the equation:

$$5.61 \times 10^{15} = 1 \times e^{0.6931t}$$

Simply take the ln of both sides, and you get

$$t = \frac{ln(5.61 \times 10^{15})}{0.6931} = 52.3$$

implying that if the prince delived 1 grain on day 1, he will have emptied his stockroom in the late morning of day 53.

3. **Oil Supply:**

   (a) The easiest way to approach this is to imagine that there are 100 units of oil in the ground, and we presently use 1 unit/year. Obviously, if we kept burning oil at the present rate, we would run out in 100 years.

   Now imagine instead that we in the first year we use 1.05 units of oil. At the end of this year, we have 98.95 units left in the ground. In the second year, our usage increases by 5%, so we burn $1.05 \times 105\% = 1.05 \times 1.5 = 1.05^2 = 1.1025$ units. At the end of the 2nd year, we have $100 - 1.05 - 1.1025 = 100 - 1.05^1 - 1.05^2 = 97.848$ units left.

   All you need to do is keep subtracting terms until you have no oil left in the ground. For a 100-year supply, this will take 36 years.

   (b) For a 1000-year supply, the procedure is exactly the same, but you'll need to subtract the annual usage from 1000 and keep subtracting longer:

   $$1000 - 1.05^1 - 1.05^2 - 1.05^3 \ldots - 1.05^N = 0$$

   You'll find this is true for $N = 80$.

   (c) For a 10,000-year supply, the procedure is exactly the same:

   $$10,000 - 1.05^1 - 1.05^2 - 1.05^3 \ldots - 1.05^N = 0$$

   In this case, $N = 127$.

Notice that attacking a shortage of oil on the supply side is not going to do you much good: A 100-fold increase in supply only buys you an extra 90 years unless you curtail the growth of demand.

An alternative, calculus-based solution: In class you were given the formula for exponential growth:

$$N = N_0 e^{\lambda t}$$

We can use this to describe the growth in our consumption, but we need to be careful. λ is not the growth rate per unit time, but is the instantaneous growth rate. In the language of finance, λ is the interest rate, which, when compounded continuously, gives you an annual yield of 5%. So, first, we need to solve for λ by setting up this equation for 1 year:

$$1.05 = 1.0e^{\lambda 1}$$

Since $ln(e^x) = x$, we can exponentiate to solve for λ:

$$\lambda = ln(1.05) = 0.04879$$

Next, recognize that this expression describes the instantaneous rate of consumption. We want the total amount consumed, which is given by the integral of the instantaneous rate. In particular, we want to find the total time we will have to wait (t^*) until some amount (say, 100 units) of oil has been used. Symbolically:

$$\alpha = \int_0^{t^*} e^{\lambda t} dt$$

where α is the number of units initially in the ground (100, 1000 or 10,000). Doing the integral and solving for t^*, we get

$$\alpha = \frac{e^{\lambda t}}{\lambda} - \frac{1}{\lambda}$$

$$t^* = \frac{ln(\alpha\lambda + 1)}{\lambda}$$

If we put in $\lambda = 0.04879$ and $\alpha = 100$, we get $t^* = 36.3$ years. If $\alpha = 1000$ we get $t^* = 80.1$ years. If $\alpha = 10,000$ we get $t^* = 126.9$ years. All are consistent with our other method.

4. **Proofs of scientific theories:** According to the scientific method, a scientist proposes a theory that explains phenomena observed to date, and makes firm predictions.

   These predictions are then tested in further experiments. If the predictions are *inconsistent* with the experimental results, the theory is incorrect and must be modified. If the predictions are *consistent* with the experimental results, this does not mean the theory is a complete description of the underlying way that nature works. The consistency only tells us that the theory does not *yet* need modification. Future experiments might very well reveal problems with the theory.