

If the pressure at height z is $p(z)$, we have, from (2.18),

$$-\int_{p(z)}^{p(\infty)} dp = \int_z^{\infty} g\rho dz$$

or, since $p(\infty) = 0$,

$$p(z) = \int_z^{\infty} g\rho dz \quad (2.19)$$

That is, the pressure at level z is equal to the weight of the air in the vertical column of unit cross-sectional area lying above that level. If the mass of earth's atmosphere were uniformly distributed over the globe, the pressure at sea level would be 1013 mb, or 1.013×10^5 Pa, which is referred to as *normal atmospheric pressure* and abbreviated as 1 atm.

2.2.1 Geopotential

The *geopotential* Φ at any point in the atmosphere is defined as the work that must be done against the earth's gravitational field in order to raise a mass of 1 kg from sea level to that point. In other words, Φ is the gravitational potential for unit mass. The units of geopotential are J kg^{-1} or $\text{m}^2 \text{s}^{-2}$. The force (in newtons) acting on 1 kg at height z above sea level is numerically equal to g . The work (in joules) in raising 1 kg from z to $z + dz$ is $g dz$; therefore,

$$d\Phi = g dz = -\alpha dp \quad (2.20)$$

The geopotential $\Phi(z)$ at height z is thus given by

$$\Phi(z) = \int_0^z g dz \quad (2.21)$$

where the geopotential $\Phi(0)$ at sea level ($z = 0$) has, by convention, been taken as zero. It should be emphasized that the geopotential at a particular point in the atmosphere depends only on the height of that point and not on the path through which the unit mass is taken in reaching that point. The work done in taking a mass of 1 kg from point A with geopotential Φ_A to point B with geopotential Φ_B is $\Phi_B - \Phi_A$.

We can also define a quantity called the *geopotential height* Z as

$$Z \equiv \frac{\Phi(z)}{g_0} = \frac{1}{g_0} \int_0^z g dz \quad (2.22)$$

where g_0 is the globally averaged acceleration due to gravity at the earth's surface (taken as 9.8 m s^{-2}). Geopotential height is used as the vertical coordinate in most atmospheric applications in which energy plays an important role (for example, the large-scale motions discussed in Chapter 8). It can be seen from Table 2.1 that the values of z and Z are almost the same in the lower atmosphere where $g_0 \approx g$.

Table 2.1

Values of the geometric height (z), geopotential height (Z), and acceleration due to gravity (g) at 40° latitude

z (km)	Z (km)	g (m s^{-2})
0	0	9.802
1	1.000	9.798
10	9.986	9.771
20	19.941	9.741
30	29.864	9.710
60	59.449	9.620
90	88.758	9.531
120	117.795	9.443
160	156.096	9.327
200	193.928	9.214
300	286.520	8.940
400	376.370	8.677
500	463.597	8.427
600	548.314	8.186

In meteorological practice it is not convenient to deal with density ρ , which cannot be measured directly. By making use of (2.2) or (2.16) to eliminate ρ in (2.18), we obtain

$$\frac{dp}{dz} = -\frac{pg}{RT} = -\frac{pg}{R_d T_v}$$

Rearranging the last expression and using (2.20),

$$d\Phi = -RT \frac{dp}{p} = -R_d T_v \frac{dp}{p} \quad (2.23)$$

If we now integrate between pressure levels p_1 and p_2 , with geopotentials Φ_1 and Φ_2 , respectively,

$$\Phi_2 - \Phi_1 = -R_d \int_{p_1}^{p_2} T_v \frac{dp}{p}$$

Dividing both sides of the last equation by g_0 and reversing the limits of integration yields

$$Z_2 - Z_1 = \frac{R_d}{g_0} \int_{p_2}^{p_1} T_v \frac{dp}{p} \quad (2.24)$$

2.2.2 Scale height and the hypsometric equation

For an isothermal (temperature constant with height) and dry atmosphere, (2.24) becomes

$$Z_2 - Z_1 = H \ln(p_1/p_2) \quad (2.25)$$

or

$$p_2 = p_1 \exp\left[-\frac{(Z_2 - Z_1)}{H}\right] \quad (2.26)$$

where

$$H = \frac{RT}{g_0} = \frac{R_d T_v}{g_0} = 29.3 T_v \quad (2.27)$$

H is called the *scale height*. If $Z_2 - Z_1$ is set successively equal to 0, H , $2H$, $3H$, ..., p_2/p_1 is equal to 1, $\exp(-1)$, $\exp(-2)$, $\exp(-3)$, ... That is, the pressure decreases by a factor e ($=2.718$) for each increase H in geopotential height. It should be noted that (2.26) is equivalent to (1.8) which was derived empirically in Section 1.3.1.

Since the atmosphere is well mixed below the turbopause, the pressure and densities of the individual gases decrease with altitude at the same rate and with a scale height corresponding to the apparent molecular weight of the mixture. If we take a value for T_v of 288°K near the earth's surface, the scale height H for air in the atmosphere is found from (2.27) to be $8\frac{1}{2}$ km. Above the turbopause (about 105 km) the vertical distribution of gases is largely controlled by molecular diffusion and a scale height may then be defined for each of the individual gases in air. Since for each gas the scale height is proportional to the gas constant for a unit mass of a gas, which in turn is inversely proportional to the molecular weight of that gas, the pressures (and densities) of heavier gases fall off more rapidly with height above the turbopause than do those of lighter gases.

Problem 2.2 If the ratio of the number density of oxygen atoms to the number density of hydrogen atoms at a geopotential height of 200 km above the earth's surface is 10^5 , calculate the ratio of the number densities of these two constituents at a geopotential height of 1400 km assuming an isothermal atmosphere with a temperature of 2000°K .

Solution At these altitudes the distribution of the individual gases is determined by diffusion and therefore by (2.26). Also, at constant temperature the ratio of the number densities of two gases is equal to the ratio of their pressures. From (2.26),

$$\begin{aligned} \frac{(p_{1400\text{km}})_O}{(p_{1400\text{km}})_H} &= \frac{(p_{200\text{km}})_O \exp[-1200 \text{ km}/H_O(\text{km})]}{(p_{200\text{km}})_H \exp[-1200 \text{ km}/H_H(\text{km})]} \\ &= 10^5 \exp\left[-1200 \text{ km} \left(\frac{1}{H_O} - \frac{1}{H_H}\right)\right] \end{aligned}$$

From the definition of scale height we have, at 2000°K ,

$$H_O = \frac{R^* 2000}{16 \cdot 9.8} \text{ m} \quad \text{and} \quad H_H = \frac{R^* 2000}{1 \cdot 9.8} \text{ m}$$

therefore,

$$\frac{1}{H_O} - \frac{1}{H_H} = 8.84 \times 10^{-6} \text{ m}^{-1}$$

and

$$\frac{(p_{1400\text{km}})_O}{(p_{1400\text{km}})_H} = 10^5 \exp(-10.6) = 2.47$$

Hence, the ratio of the number densities of oxygen to hydrogen atoms at a geopotential height of 1400 km is 2.47.

The temperature of the atmosphere generally varies with height. In this case (2.24) may be integrated if we define a mean virtual temperature \bar{T}_v with respect to $\ln p$ as shown in Fig. 2.2. That is,

$$\bar{T}_v \equiv \frac{\int_{\ln p_2}^{\ln p_1} T_v d(\ln p)}{\int_{\ln p_2}^{\ln p_1} d(\ln p)} = \int_{p_2}^{p_1} T_v \frac{dp}{p} / \ln\left(\frac{p_1}{p_2}\right) \quad (2.28)$$

Then, from (2.24) and (2.28),

$$Z_2 - Z_1 = \frac{R_d \bar{T}_v}{g_0} \ln\left(\frac{p_1}{p_2}\right) = \bar{H} \ln\left(\frac{p_1}{p_2}\right) \quad (2.29)$$

where the scale height \bar{H} is now defined as

$$\bar{H} \equiv \frac{R_d \bar{T}_v}{g_0} = 29.3 \bar{T}_v \quad (2.30)$$

Equation (2.29) is called the *hypsometric equation*.

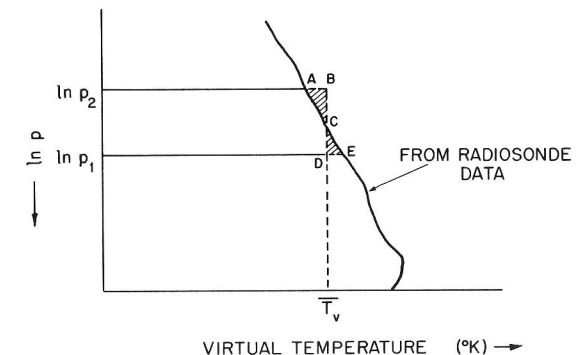


Fig. 2.2 $\ln p$ versus T_v diagram. If area ABC = area CDE, T_v is the mean virtual temperature with respect to $\ln p$ between the pressure levels p_1 and p_2 .

2.2.3 Thickness and heights of constant pressure surfaces

The difference in geopotential height $Z_2 - Z_1$ between any two levels in the atmosphere is called the *thickness* of the intervening layer. It can be seen from (2.29) and (2.30) that the thickness of the layer between any two pressure levels p_2 and p_1 is proportional to the mean virtual temperature of the layer.

We can visualize that as \bar{T}_v increases, the air between the two pressure levels expands so that the layer becomes thicker.

Problem 2.3 Calculate the thickness of the layer between the 1000- and 500-mb pressure surfaces (a) at a point in the tropics where the mean virtual temperature of the layer is 9°C , and (b) at a point in the polar regions where the corresponding mean virtual temperature is -40°C .

Solution From (2.29)

$$\Delta Z = Z_{500\text{mb}} - Z_{1000\text{mb}} = \frac{R_d \bar{T}_v}{g_0} \ln\left(\frac{1000}{500}\right) = 20.3 \bar{T}_v$$

Therefore, for the tropics with $\bar{T}_v = 282^\circ\text{K}$, $\Delta Z = 5725$ m. For polar regions with $\bar{T}_v = 233^\circ\text{K}$, $\Delta Z = 4730$ m.

Thickness may readily be evaluated from radiosonde data which provide measurements of the pressure, temperature, and humidity at various levels in the atmosphere. The virtual temperature T_v at each level may be found from the measurements of temperature and humidity and these values plotted against the pressure on a $\ln p$ versus T_v diagram (see Fig. 2.2). The mean virtual temperature \bar{T}_v for the layer can be computed on an equal area basis (that is, area ABC = area CDE in Fig. 2.2) and the thickness evaluated using (2.29). If p_1 is the pressure at ground level, then the value of Z_2 determined from (2.29) is the geopotential height at which the air pressure is p_2 . Given sounding data from a network of stations it is possible to construct topographical maps of the distribution of geopotential height on selected pressure surfaces (see Chapter 3). Calculations such as this are carried out routinely by weather services throughout the world.

In moving from a given pressure surface to another located above or below it, the change in the geopotential height is geometrically related to the thickness of the intervening layer which, in turn, is directly proportional to the mean virtual temperature of the layer. Thus, if the three-dimensional distribution of virtual temperature is known together with the distribution of geopotential height on one pressure surface, it is possible to infer the distribution of geopotential height of any other pressure surface. The same hypsometric relationship between the three-dimensional temperature field and the shape of pressure surface can be used in a qualitative way to gain some useful insights into the three-dimensional structure of atmospheric disturbances. Let us consider the following examples:

- From the earth's surface up to the tropopause the core of a hurricane is warmer than its surroundings. Consequently, the intensity of the storm (as measured by the depression of the isobaric surfaces) must decrease with height (Fig. 2.3a). Such *warm core lows* always exhibit their greatest intensity near the ground and diminish with increasing height above the ground.

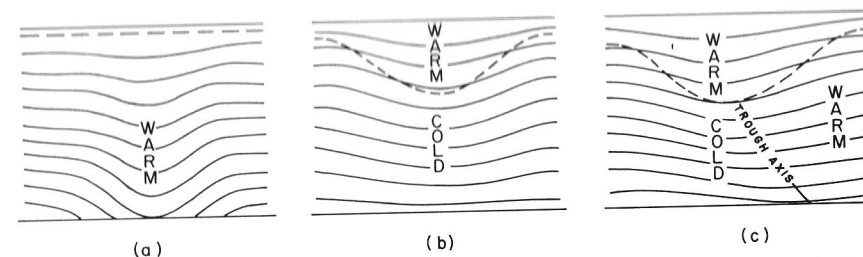


Fig. 2.3 Vertical cross sections through (a) a hurricane, (b) a “cold core” upper tropospheric low in middle latitudes, and (c) a middle-latitude disturbance which tilts westward with increasing height. The solid lines indicate various constant pressure surfaces and the dashed lines represent the tropopause. The sections are drawn such that the thickness between adjacent pressure surfaces is smaller in regions labeled *cold* and larger in regions labeled *warm*.

- Some upper air lows do not extend downward to the ground as indicated in Fig. 2.3b. Such lows are referred to as being *cold core* because they exhibit negative thickness anomalies relative to their surroundings.
- Most middle-latitude disturbances have vertical structure similar to that shown in Fig. 2.3c with the coldest air and the lowest thickness to the west of the surface low. Such systems slope westward with height in agreement with the hypsometric equation.
- The disturbances depicted in Fig. 2.3b and c exhibit amplitude maxima at the tropopause level and damp out rapidly with height in the lower stratosphere. This behavior is consistent with a phase reversal in the temperature field at the tropopause level; for example, warm air in the lower stratosphere is situated over the cold low in the upper troposphere. These concepts are illustrated further in Chapter 3.

2.2.4 Reduction of pressure to sea level

In mountainous regions the difference in surface pressure from one station to another is largely due to differences in elevation. In order to isolate that part of the pressure field which is due to the passage of weather systems, it is necessary to reduce the pressures to a common reference level. For this purpose, sea level is normally used.

Let the subscripts g and 0 refer to conditions at the ground and at sea level ($Z = 0$), respectively. For the layer between the earth's surface and sea level the hypsometric equation (2.29) assumes the form

$$Z_g = \bar{H} \ln \frac{p_0}{p_g}$$

which can be solved to obtain the sea level pressure

$$p_0 = p_g \exp\left(\frac{Z_g}{\bar{H}}\right) = p_g \exp\left(\frac{g_0 Z_g}{R_d \bar{T}_v}\right) \quad (2.31)$$

where (2.30) has been used to obtain the last expression which shows how the sea level pressure depends on the mean virtual temperature between ground and sea level. If Z_g is small, the scale height \bar{H} can be evaluated from the ground temperature. Also, if $Z_g/\bar{H} \ll 1$, the exponential in (2.31) can be approximated by $1 + Z_g/\bar{H}$. Since $\bar{H} \simeq 8$ km for the observed range of ground temperatures on earth, this approximation is satisfactory provided that Z_g is less than a few hundred meters. With this approximation, (2.31) becomes

$$p_0 - p_g \simeq p_g \frac{Z_g}{\bar{H}} = p_g \left(\frac{g_0 Z_g}{R_d \bar{T}_v} \right) \quad (2.32)$$

Since $p_g \simeq 1000$ mb and $\bar{H} \simeq 8$ km, the pressure correction (in millibars) is roughly equal to Z_g (in meters) divided by 8. In other words, near sea level the pressure falls about 1 mb for every 8 m of vertical ascent.

When Z_g is on the order of 1 km or greater, there is difficulty in estimating what the mean virtual temperature of the layer would be in the absence of topography. In practice, a number of empirical corrections are applied to the surface temperature in the estimation of the scale height \bar{H} . These procedures are not entirely satisfactory in eliminating the effects of topography. Therefore, sea level pressure analysis in mountainous regions still leaves much to be desired.

Problem 2.4 Calculate the geopotential height of the 1000-mb pressure surface when the pressure at sea level is 1014 mb. The scale height of the atmosphere may be taken as 8 km.

Solution From the hypsometric equation (2.29),

$$Z_{1000\text{mb}} = \bar{H} \ln \left(\frac{p_0}{1000} \right) = \bar{H} \ln \left(1 + \frac{p_0 - 1000}{1000} \right) \simeq \bar{H} \left(\frac{p_0 - 1000}{1000} \right)$$

where p_0 is the sea level pressure and the relationship $\ln(1+x) \simeq x$ for $x \ll 1$ has been used. Substituting $\bar{H} \simeq 8000$ m into this expression gives

$$Z_{1000\text{mb}} \simeq 8(p_0 - 1000)$$

Therefore, with $p_0 = 1014$ mb, the geopotential height Z_{1000} of the 1000-mb pressure surface is found to be 112 m above sea level. [Note: similar expressions can be derived to relate the height of other pressure surfaces to the distribution of pressure on a nearby constant height level; the constant of proportionality will, in general, depend on the pressure levels.]

Problem 2.5 Derive a relationship for the height of a given pressure surface (p) in terms of the pressure p_0 and temperature T_0 at sea level assuming that the temperature decreases uniformly with height at a rate Γ deg km⁻¹.

Solution Let the height of the pressure surface be z ; then its temperature T is given by

$$T = T_0 - \Gamma z \quad (2.33)$$

Combining the hydrostatic equation (2.18) with the ideal gas equation (2.2) yields

$$\frac{dp}{p} = -\frac{g}{RT} dz \quad (2.34)$$

From (2.33) and (2.34)

$$\frac{dp}{p} = -\frac{g}{R(T_0 - \Gamma z)} dz$$

Integrating this equation between pressure levels p_0 and p and corresponding heights 0 and z and neglecting the variation of g with z , we obtain

$$\int_{p_0}^p \frac{dp}{p} = -\frac{g}{R} \int_0^z \frac{dz}{(T_0 - \Gamma z)}$$

or

$$\ln \frac{p}{p_0} = \frac{g}{R\Gamma} \ln \left(\frac{T_0 - \Gamma z}{T_0} \right)$$

Therefore,

$$z = \frac{T_0}{\Gamma} \left[1 - \left(\frac{p}{p_0} \right)^{R\Gamma/g} \right] \quad (2.35)$$

(This equation forms the basis for the calibration of aircraft altimeters. An altimeter is simply an aneroid barometer which measures the air pressure p . However the scale of the altimeter is expressed as the height z of the aircraft, where z is related to p by (2.35) with values for T_0 , p_0 , and Γ appropriate to the U.S. Standard Atmosphere, namely, $T_0 = 288^\circ\text{K}$, $p_0 = 1013.25$ mb, and $\Gamma = 6.50$ deg km⁻¹.)

2.3 THE FIRST LAW OF THERMODYNAMICS

In addition to the macroscopic kinetic and potential energy that a body as a whole may possess, it also contains *internal energy* due to the kinetic and potential energy of its molecules or atoms. Increases in internal kinetic energy in the form of molecular motions are manifested as increases in the temperature of the body, while changes in the potential energy of the molecules are caused by changes in their relative configurations.

Let us suppose that a body of unit mass takes in a certain quantity of heat energy q (measured in joules), which it can receive by either thermal conduction or radiation. As a result the body may do a certain amount of *external work* w (also measured in joules). The excess of the energy supplied to the body over and above the external work done by the body is $q - w$. Therefore, if there is no change in the macroscopic kinetic and potential energy of the body, it follows from the principle of conservation of energy that the internal energy of the body must increase by $q - w$. That is,

$$q - w = u_2 - u_1 \quad (2.36)$$

where u_1 and u_2 are the internal energies of the body before and after the change. In differential form (2.36) becomes

$$dq - dw = du \quad (2.37)$$

where, dq is the differential increment of heat added to the body, dw the differential element of work done by the body, and du the differential increase in

internal energy of the body.[†] Equations (2.36) and (2.37) are statements of the *First Law of Thermodynamics*. In fact, (2.37) provides a definition of du . It should be noted that the change in internal energy du is a function only of the initial and final states of the body and is therefore independent of the manner by which the body is transferred between these two states.

In order to visualize the work term dw in (2.37) in a simple case, consider a substance, often called the *working substance*, contained in a cylinder of fixed cross-sectional area which is fitted with a movable, frictionless piston (Fig. 2.4). The volume of the material is then proportional to the distance from the base of the cylinder to the face of the piston, and can be represented on the horizontal line of the graph shown in Fig. 2.4. The pressure of the substance in the cylinder can be represented on the vertical line of this graph. Therefore, every state of the substance, corresponding to a given position of the cylinder, is represented by a point of the graph. When the substance is in equilibrium at a state represented by the point P on this graph its pressure is p and its volume V . If the piston moves outwards through an incremental distance dx , while the pressure remains essentially constant at p , the work dW done by the substance in expanding is equal to the force exerted on the piston (this force is equal to pA where A is the cross sectional area of the piston) multiplied by the distance

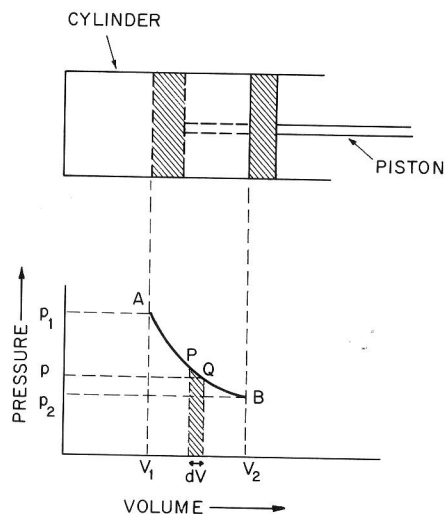


Fig. 2.4 Representation of the state of a substance in a cylinder on a p - V diagram.

[†] It should be noted that neither the heat q nor the work w are functions of state. Therefore, dq and dw are not *perfect differentials*. The internal energy u is a function of state; therefore, du is a perfect differential. If $z = f(x, y)$, dz is a perfect differential provided that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

dx through which the piston moves. That is,

$$dW = pA dx = p dV \quad (2.38)$$

In other words, the work done by the substance when its volume increases by a small amount is equal to the pressure of the substance multiplied by its increase in volume. It should be noted that $dW = p dV$ is equal to the shaded area in the graph shown in Fig. 2.4; that is, it is equal to the area under the curve PQ. When the substance passes from state A with volume V_1 to state B with volume V_2 (Fig. 2.4), during which its pressure p changes, the work W done by the material is equal to the area under the curve AB. That is,

$$W = \int_{V_1}^{V_2} p dV \quad (2.39)$$

Equations (2.38) and (2.39) are quite general and represent the work done by any material due to a change in its volume.

The pressure-volume (or p - V) diagram shown in Fig. 2.4 is an example of a *thermodynamic diagram* in which the physical state of a substance is represented by two thermodynamic variables. Such diagrams are very useful in meteorology; we will discuss other examples later in this chapter.

If we are dealing with a unit mass of material, the volume V is replaced by the specific volume α and the work dw which is done when the specific volume increases by $d\alpha$ is

$$dw = p d\alpha \quad (2.40)$$

Combination of (2.37) and (2.40) yields

$$dq = du + p d\alpha \quad (2.41)$$

which is an alternative statement of the First Law of Thermodynamics.[†] It should be noted that the First Law of Thermodynamics can be applied to any system; however, here we shall generally apply it only to gases.

2.3.1 Joule's law

Following a series of laboratory experiments on air, Joule[‡] concluded in 1848 that when a gas expands without doing external work (for example, by expanding into a chamber which has been evacuated), and without taking

[†] We have assumed here that the only external work done by the body is due to its volume changing. We will see in Section 2.9 that there are other ways in which a body may do external work. We have also assumed that the macroscopic kinetic and potential energy remain constant. However, it can be shown (see, for example, R. G. Fleagle and J. A. Businger, "An Introduction to Atmospheric Physics," Academic Press, New York, 1963, p. 37) that even if the macroscopic kinetic and potential energies of a parcel of air in the atmosphere are changing, the First Law still takes the form of (2.41).

[‡] **James Prescott Joule** (1818–1889) English physicist, one of the great experimentalists of the nineteenth century. Started his scientific work (carried out in laboratories in his home and at his own expense) at age 19. Measured the mechanical equivalent of heat. Recognized the dynamical nature of heat, and developed the principle of conservation of energy. Derived the relationship for the heat produced by an electric current.