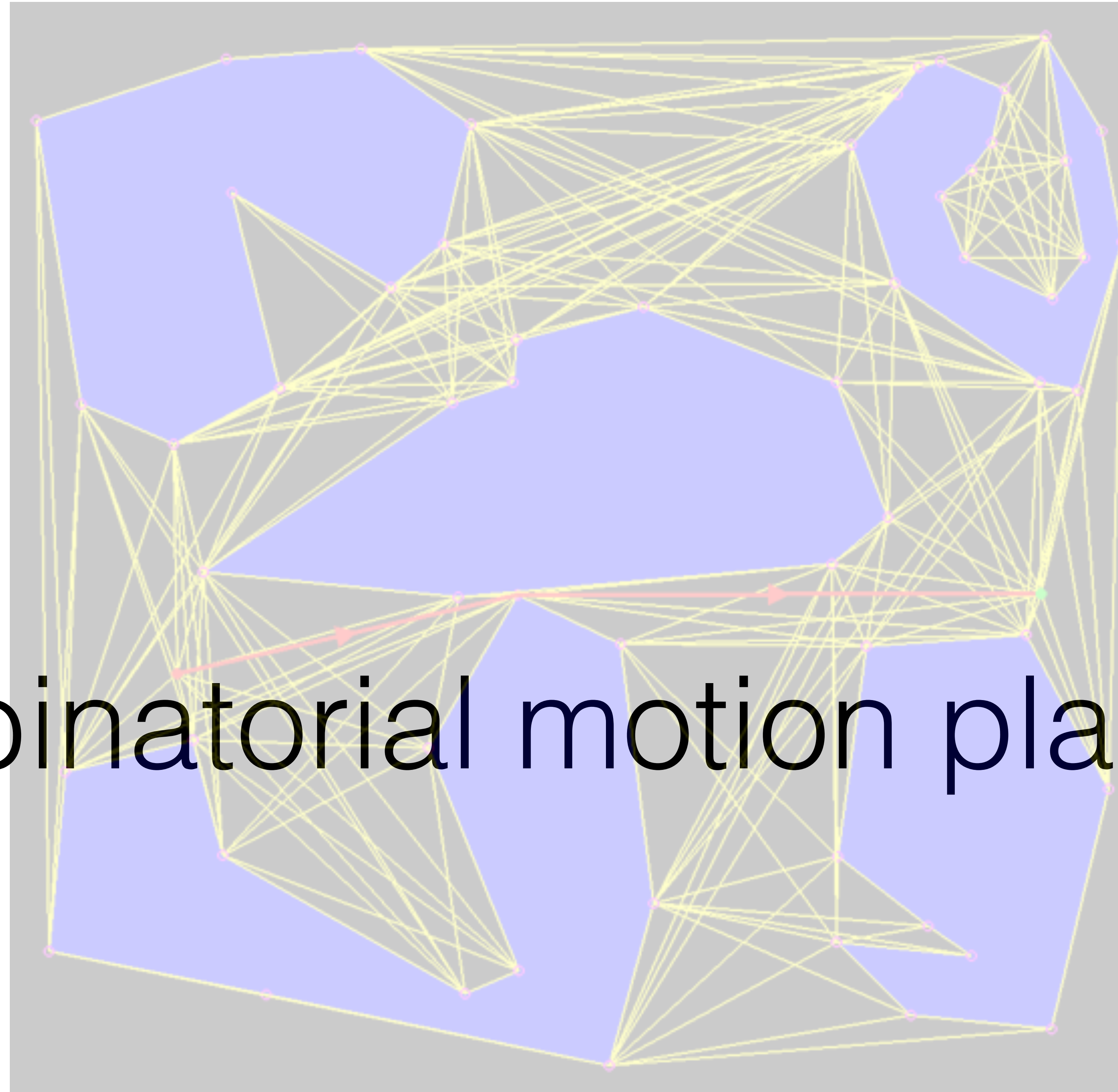


Combinatorial motion planning

Computational Geometry
csci3250
Laura Toma
Bowdoin College

Motion Planning

Input:

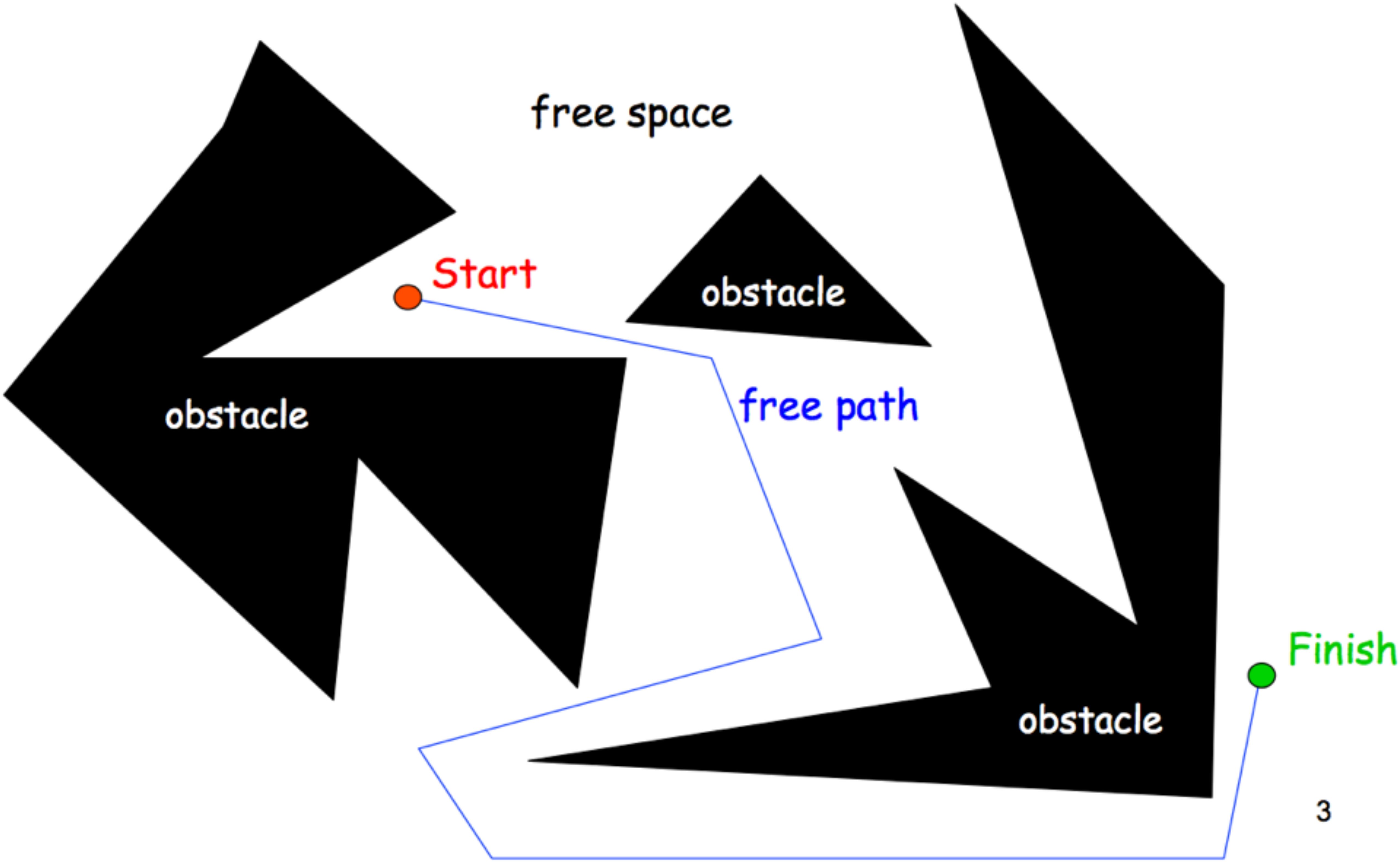
- a robot R
- start and end position
- a set of obstacles $S = \{O_1, O_2, \ldots\}$

Find a path from start to end (that optimizes some objective function).

Parameters:

- geometry of obstacles (polygons, disks, convex, non-convex, etc)
- geometry of robot (point, polygon, disc)
- robot movement (dof)
- objective function to minimize (euclidian distance, nb turns, etc)
- 2d, 3d
- static vs dynamic environment
- exact vs approximate algorithm
- known vs unknown map

Problem



screenshot from: ai.stanford.edu/~latombe/cs26n/2012/slides/**point-robot**-bug.ppt

Motion Planning

Ideally we want a planner to be complete and optimal.

- A planner is complete:
  - it always finds a path when a path exists
- A planner is optimal:
  - it finds an optimal path (wrt objective function)

Approaches

- **Combinatorial (exact)**
  - Compute an exact representation of free space
  - Roadmap: a graph that represents the free space
  - Find a path using the roadmap

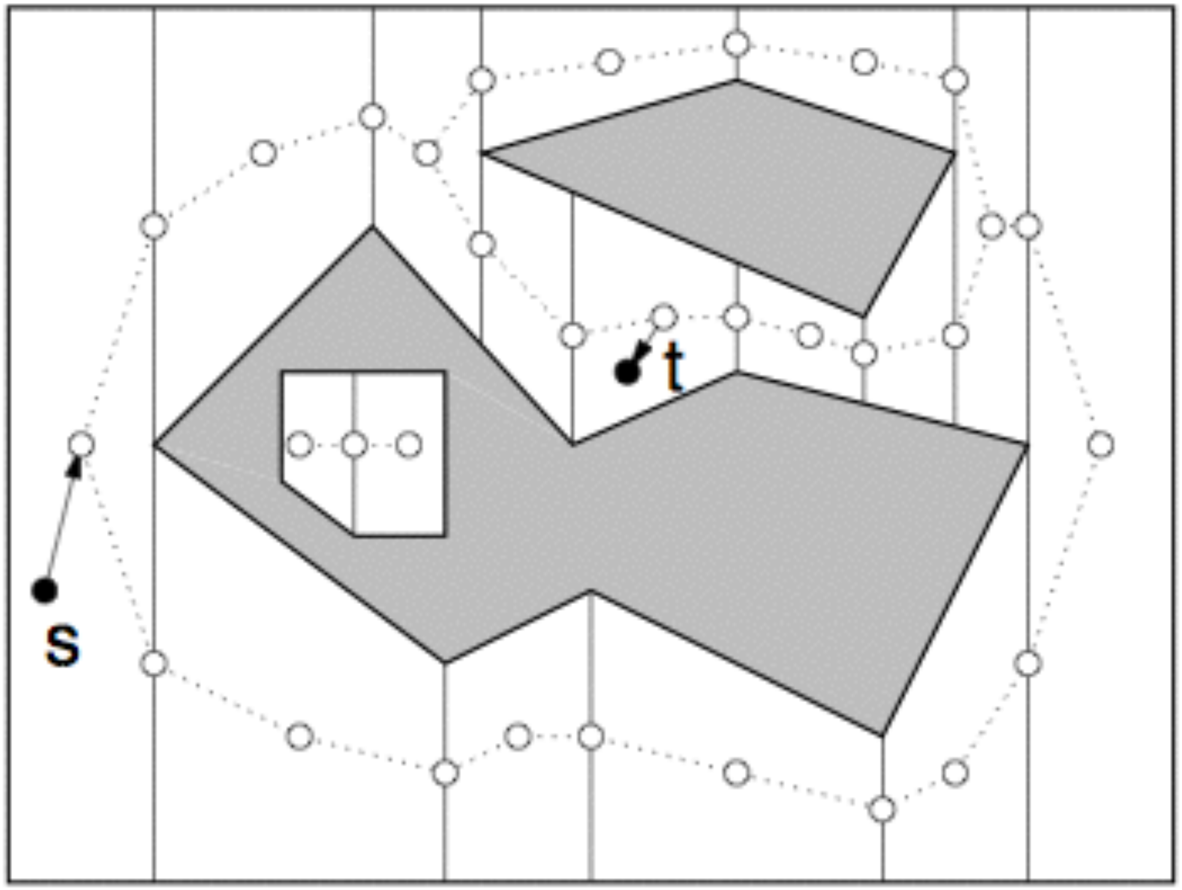
this week

- **Approximate**
  - roadmaps, sampling-based, search based, space decomposition, probabilistic, potential functions

next week

Point robot in 2D

- General idea
  - Compute free space
  - Compute a representation of free space
  - Build a graph of free space
  - Search graph to find path <---------- Reduce motion planning to graph search

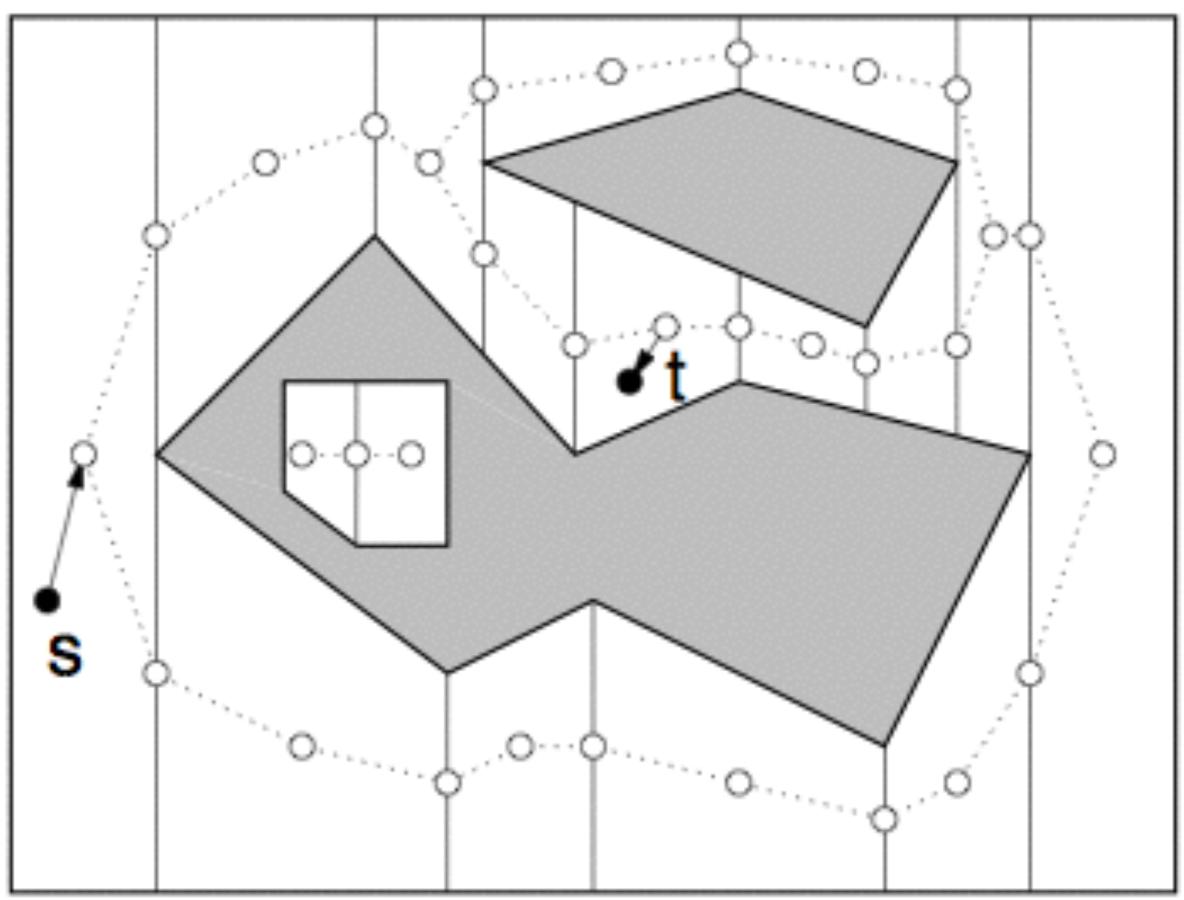


Point robot in 2D

n = complexity of obstacles (total number of edges)

Result:

Let R be a point robot moving among a set of polygonal obstacles in 2D with n edges in total. We can pre-process S in O(n lg n) expected time such that, between any start and goal position, a collision-free path for R can be computed in O(n) time, if it exists.

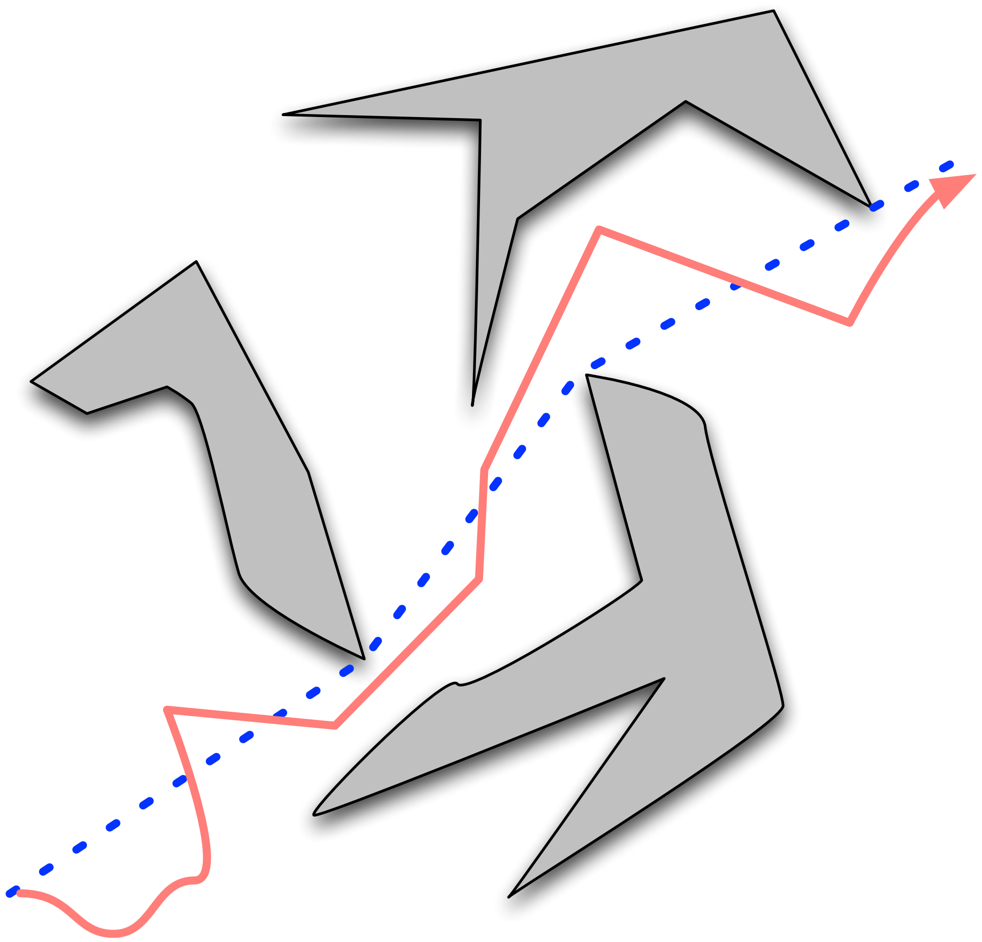


complete

Claim: not optimal

Show that the trapezoid map is not optimal by giving a scene where it dos not give the optimal (shortest) path

Point robot in 2D

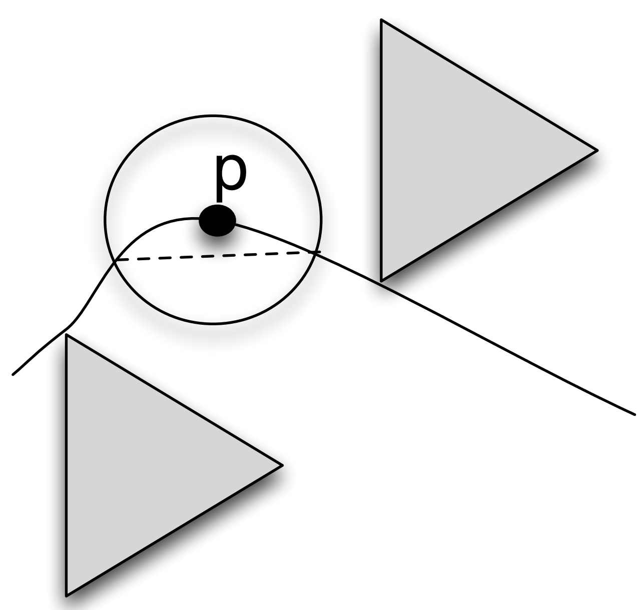


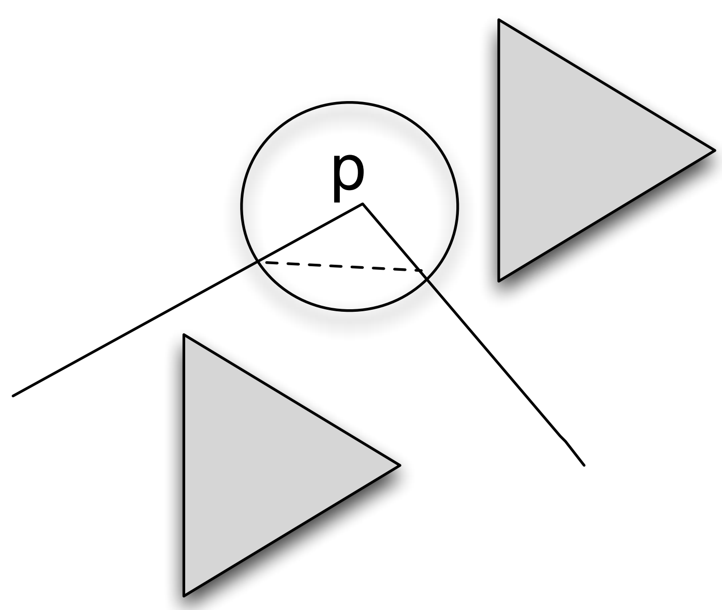
What if we wanted the shortest path?

Shortest paths for point robot in 2D

Claim: Any shortest path among a set S of disjoint polygonal obstacles

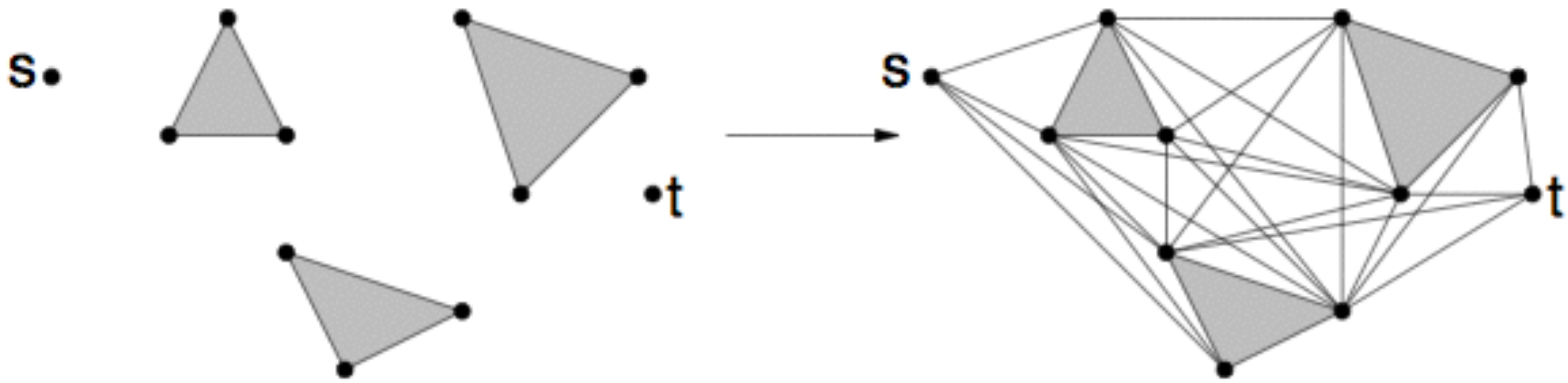
- is a polygonal path (that is, not curved)
- its vertices are the vertices of S.



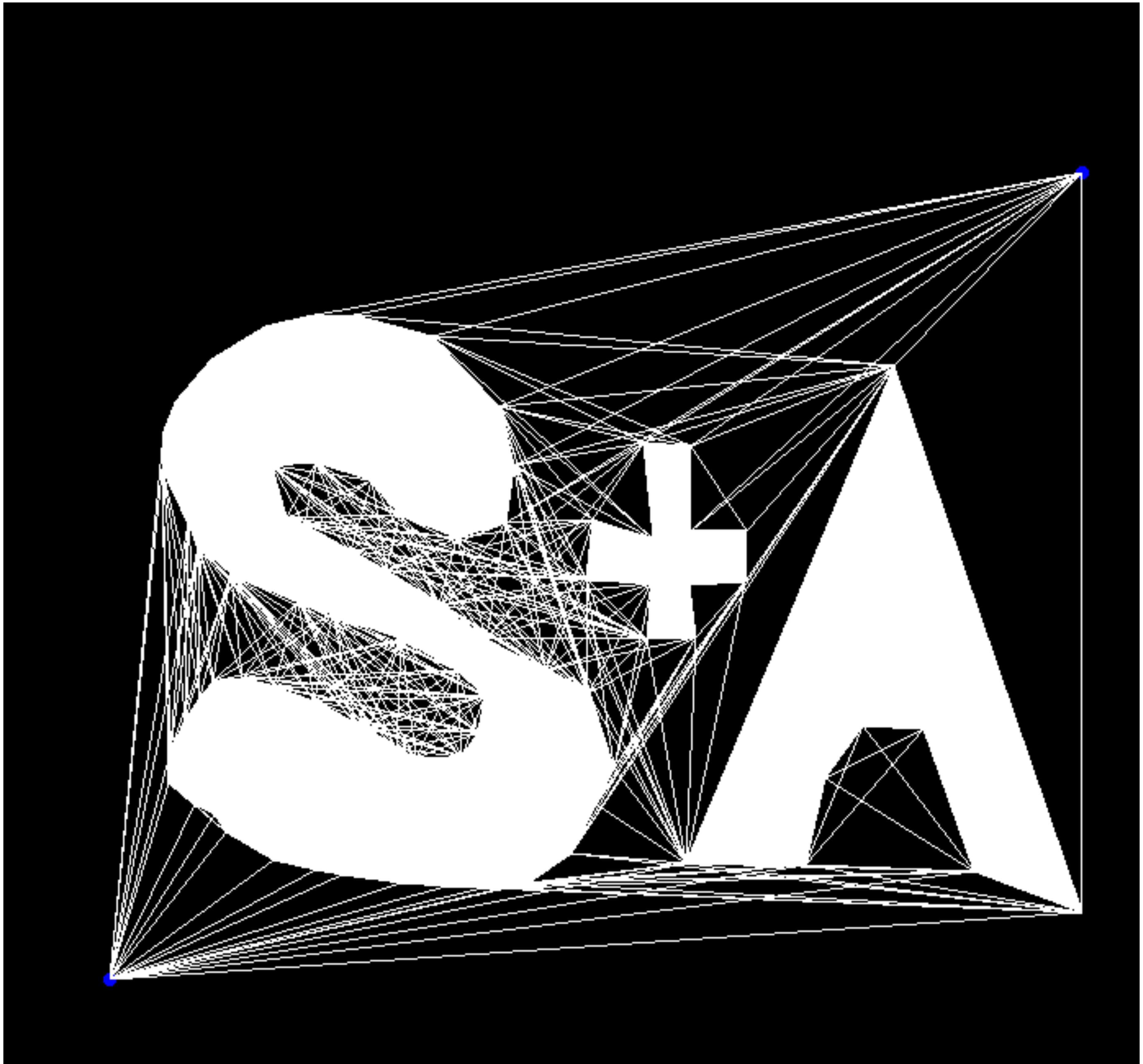


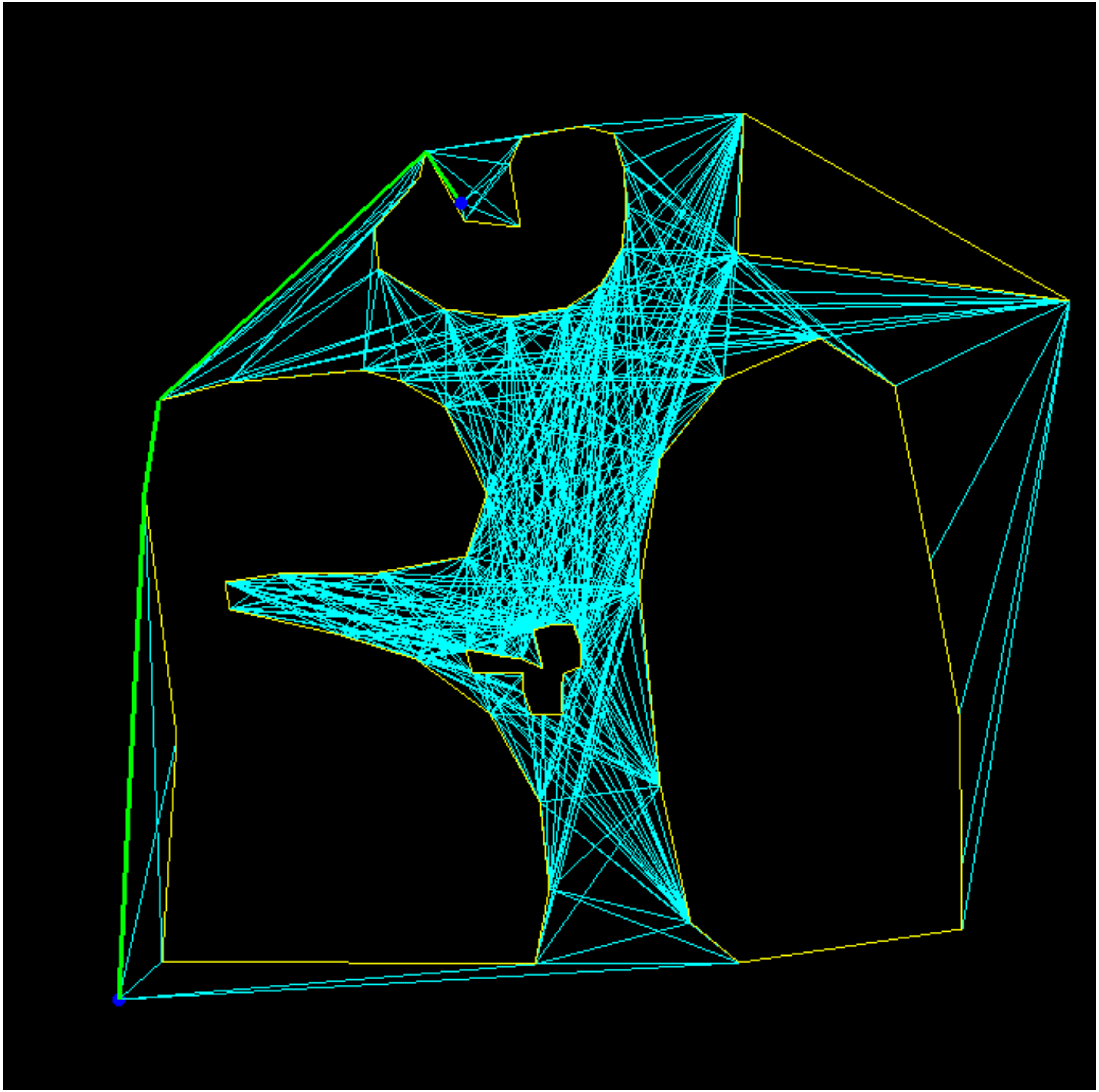
Shortest paths for point robot in 2D

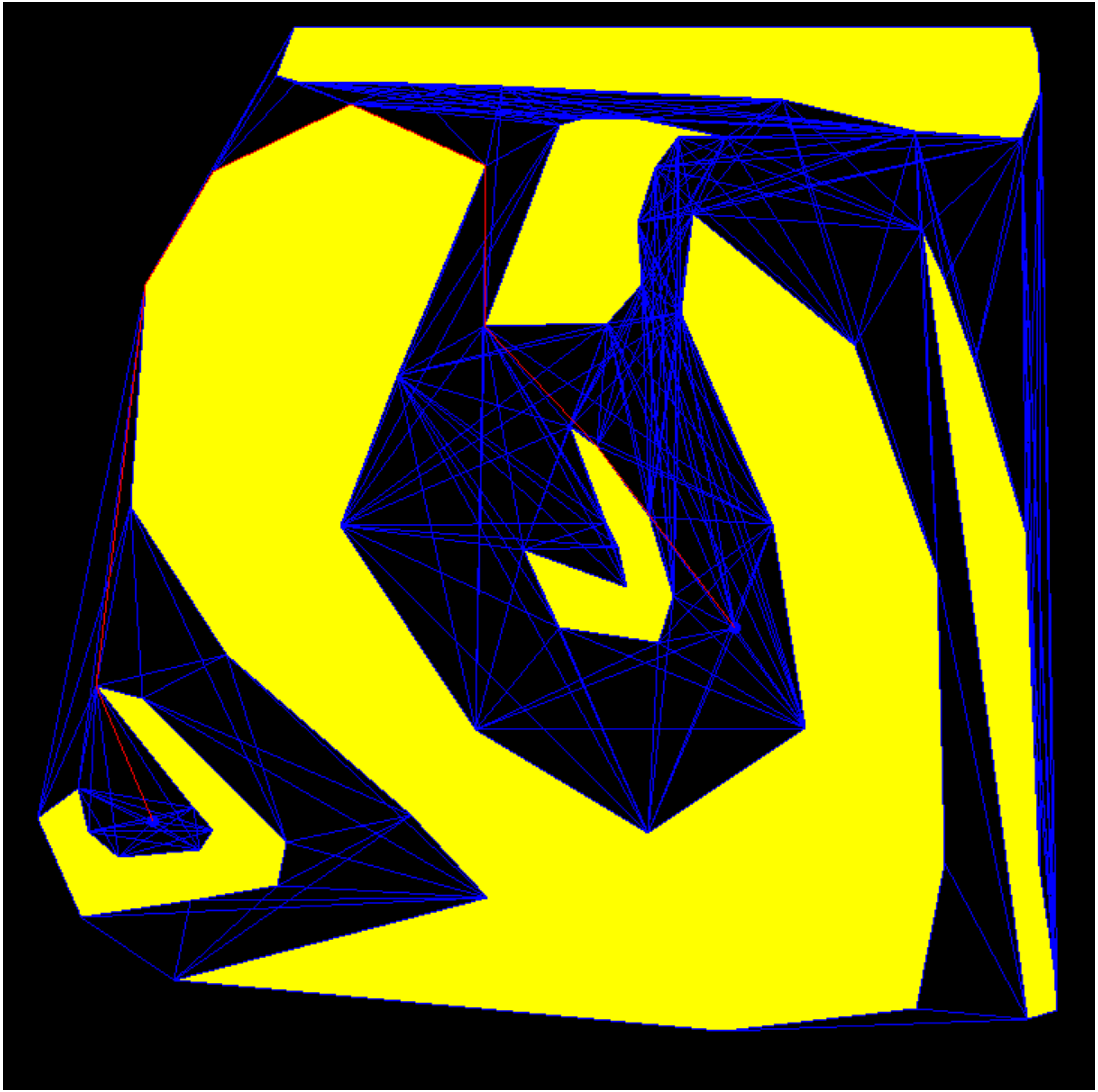
- Idea: Build the visibility graph (VG)
    - all possible ways to travel between the vertices of the obstacles

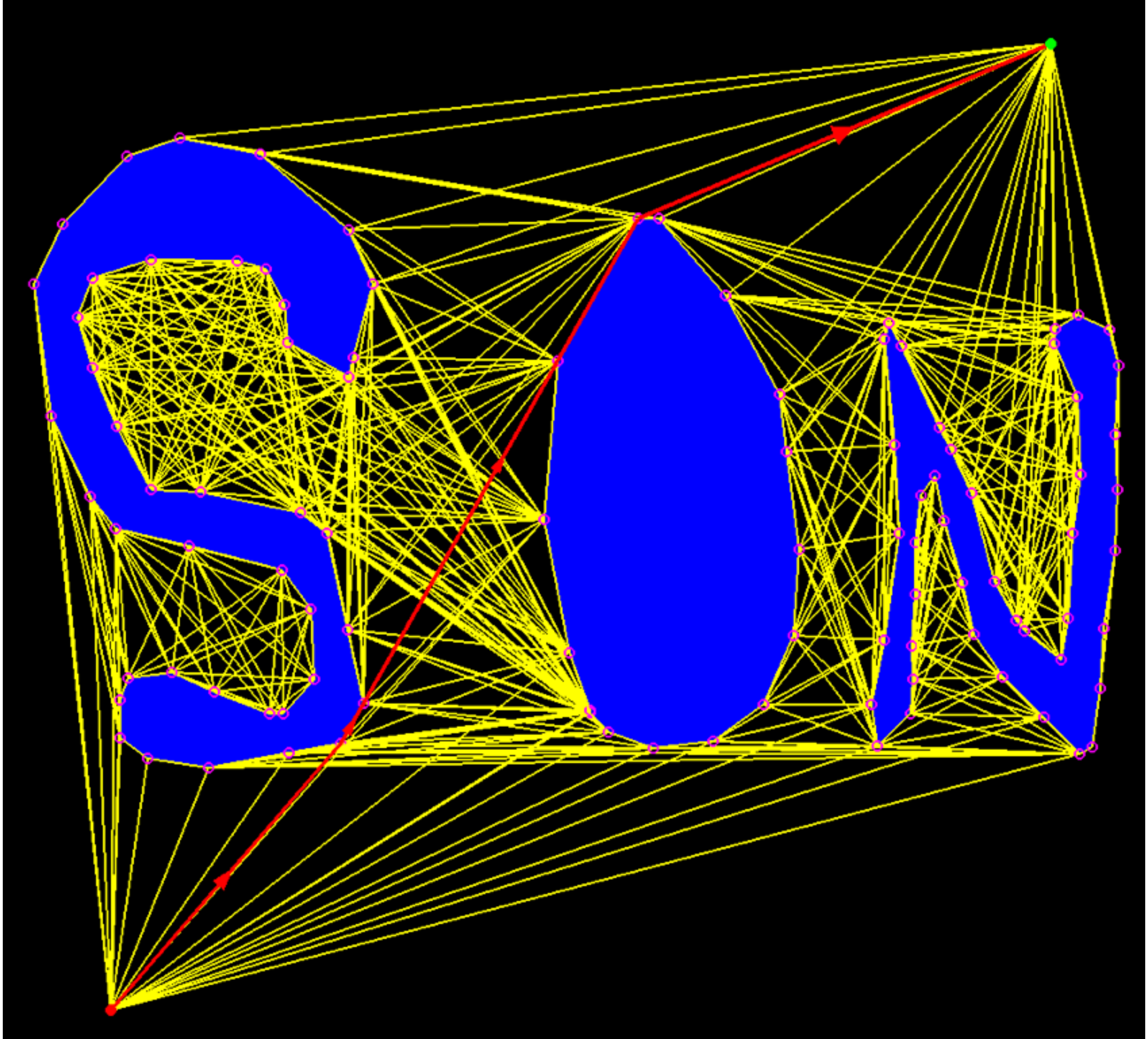


- Claim: any shortest path must be a path in the VG


S+A







n = complexity of obstacles
(total number of edges)

Shortest paths for point robot in 2D

Algorithm

- Compute visibility graph
  - V = {set of vertices of obstacles + p_{start} + p_{end}}
- SSSP (Dijkstra) in VG

- How big is VG and how long does it take to compute it?

n = complexity of obstacles (total number of edges)

Shortest paths for point robot in 2D

Algorithm

- Compute visibility graph
  - V = {set of vertices of obstacles + p_{start} + p_{end}}
- SSSP (Dijkstra) in VG

- Complexity of VG
  - $V_{VG} = O(n)$, $E_{VG}=O(n^2)$ <------- can have quadratic size
- Computing VG
  - naive: for each edge, check if intersects any obstacle. $O(n^3)$
  - improved: $O(n \lg n)$ per vertex, $O(n^2 \lg n)$ total
- Dijkstra on VG: $O(E_{VG} \lg n) = O(n^2 \lg n)$

n = complexity of obstacles (total number of edges)

Shortest paths for point robot in 2D

Algorithm

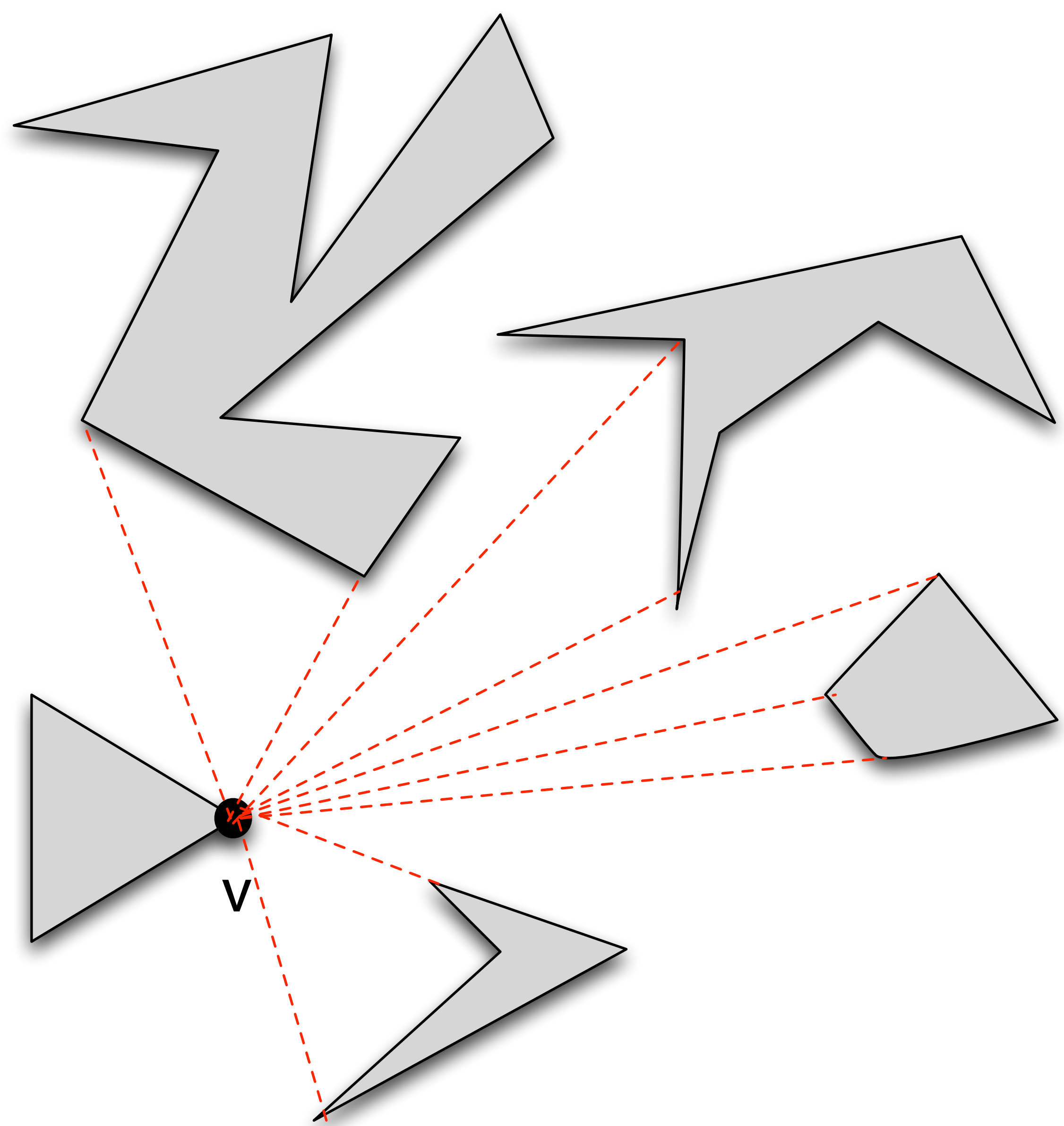
- Compute visibility graph <------- $O(n^2 \lg n)$
- SSSP (Dijkstra) in VG <------- $O(E_{VG} \lg n)$

Theorem:

A shortest path of two points among a set of polygonal obstacles with n edges in total can be computed in $O(E_{VG} \lg n) = O(n^2 \lg n)$ time.

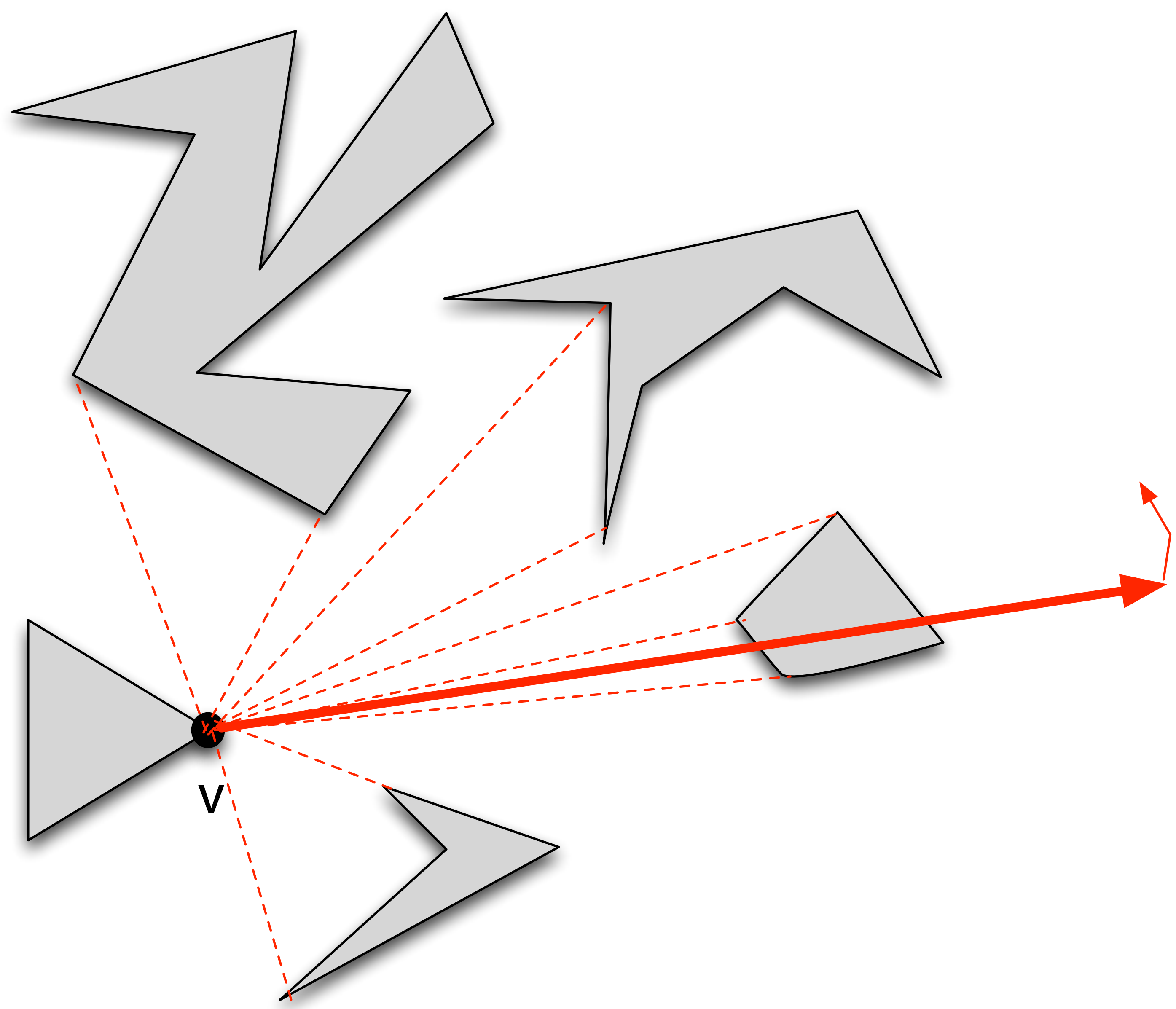
Improved computation of VG

- Idea
    - for every vertex v: compute all vertices visible from v in O(n lg n)



Improved computation of VG

- Idea
  - for every vertex v: compute all vertices visible from v in O(n lg n)



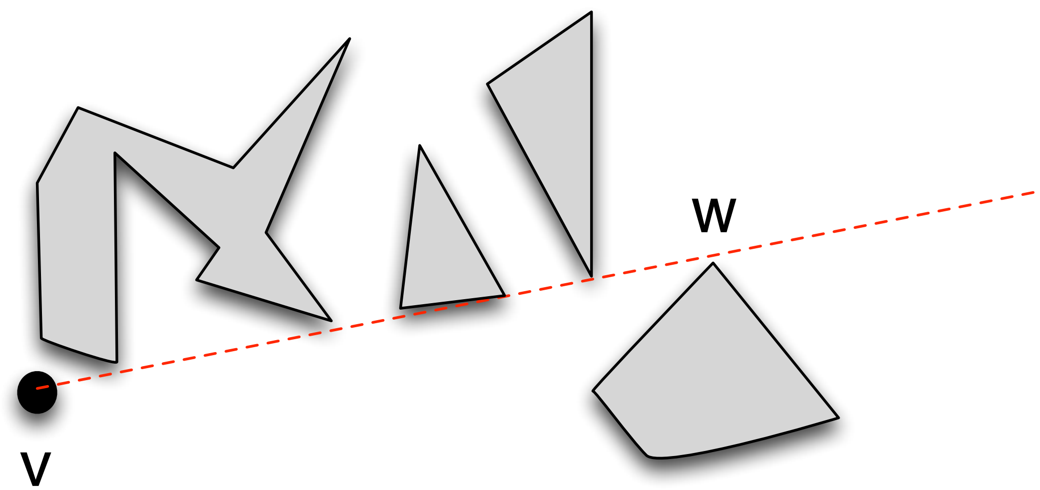
Improved computation of VG

- Idea
    - for every vertex v: compute all vertices visible from v in O(n lg n)

active structure stores all edges intersected by sweep line,
ordered by distance from v

Improved computation of VG

- Idea
  - for every vertex v: compute all vertices visible from v in O(n lg n)



w visible if vw does not intersect the interior of any obstacle

Motion planning with VG

- Optimal and complete
- VG needs to be computed only once, so we can think of it as pre-processing
- VG may be large ==> path planning doomed to quadratic complexity

Motion planning with VG (2D)

- Optimal and complete
- VG needs to be computed only once, so we can think of it as pre-processing
- VG may be large ==> path planning doomed to quadratic complexity

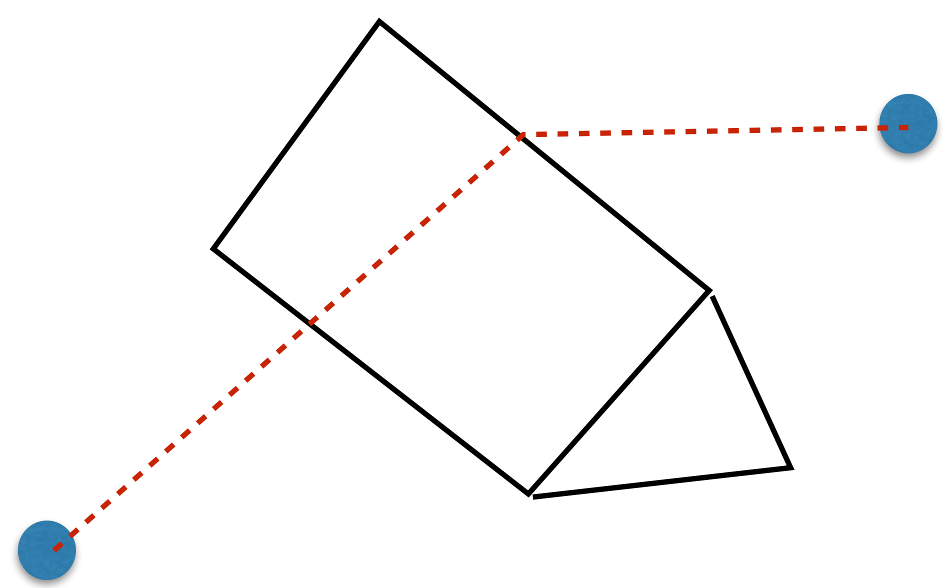
History:

- Quadratic barrier broken by Joe Mitchell: SP of a point robot moving in 2D can be computed in $O(n^{5/3 + eps})$
- Hershberger and Suri [1993]: SP of a point robot moving in 2D can be computed in O(n lg n) ("continuous Dijkstra" approach)
- Special cases can be solved faster:
    - e.g. SP inside a simple polygon w/o holes: O(n) time

VG in 3D

- VG does not generalize to 3D
- Shortest paths in 3D much harder
  - no easy way to discretize the problem: inflection points of SP are not restricted to vertices of S, can be inside edges
  - 3D shortest paths among polyhedral obstacles is NP-complete

  (Complete and optimal planning in 3D is hopeless)

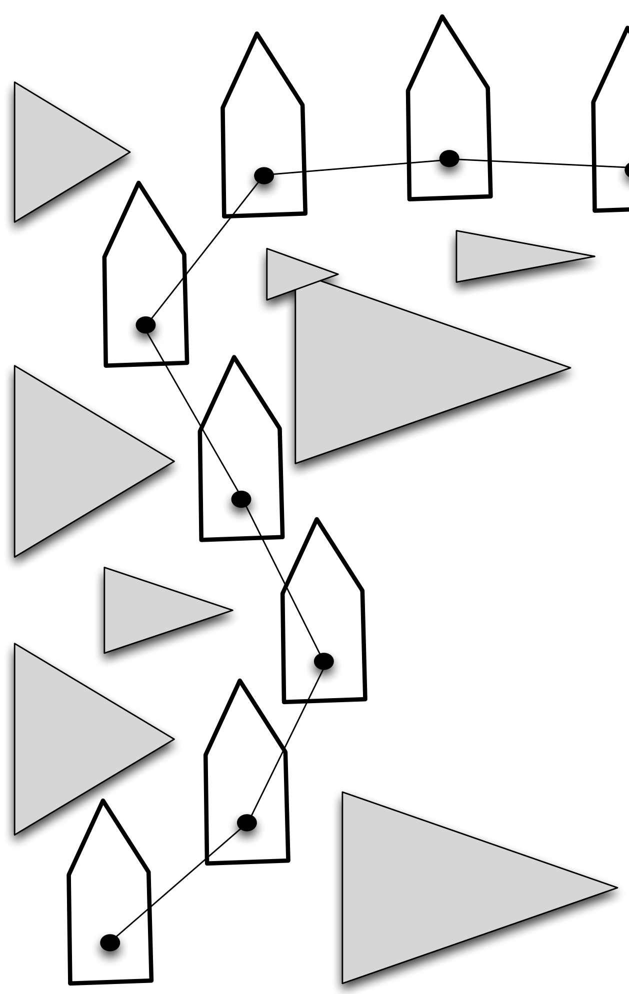


- 2D
  - point robot moving inside an arbitrary polygon
  - point robot moving among (arbitrary) polygons
  - disk robot moving among (arbitrary) polygons
  - convex robot moving among (arbitrary) polygons
  - non-convex robot moving among (arbitrary) polygons
  - robot with arms ….

harder

Convex polygon moving in 2D

- How can the robot move?
  - Translation only
  - Translation + rotation

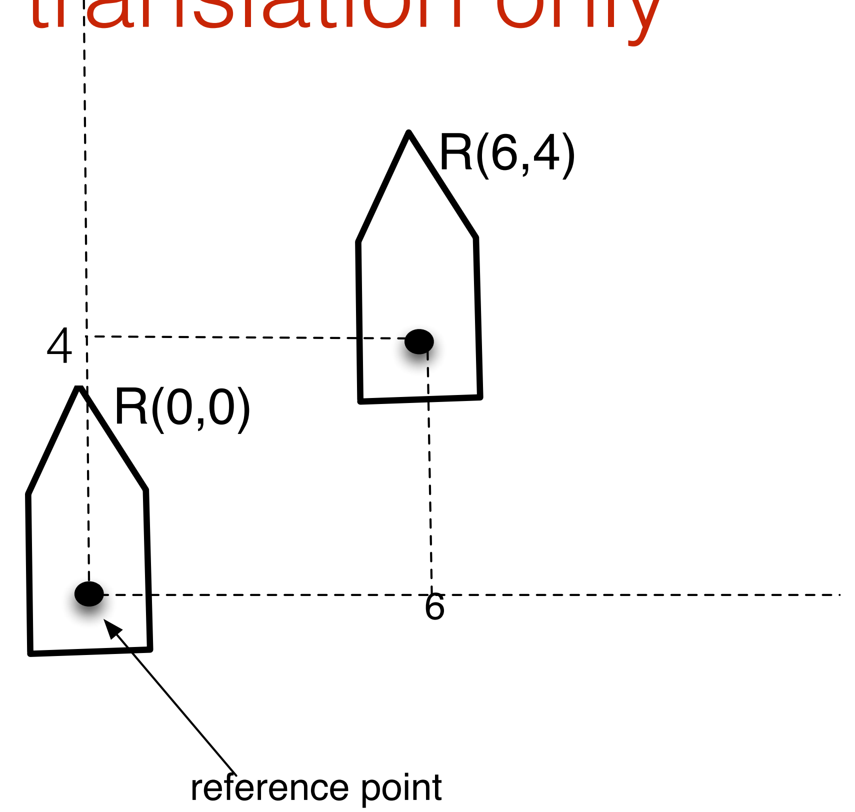


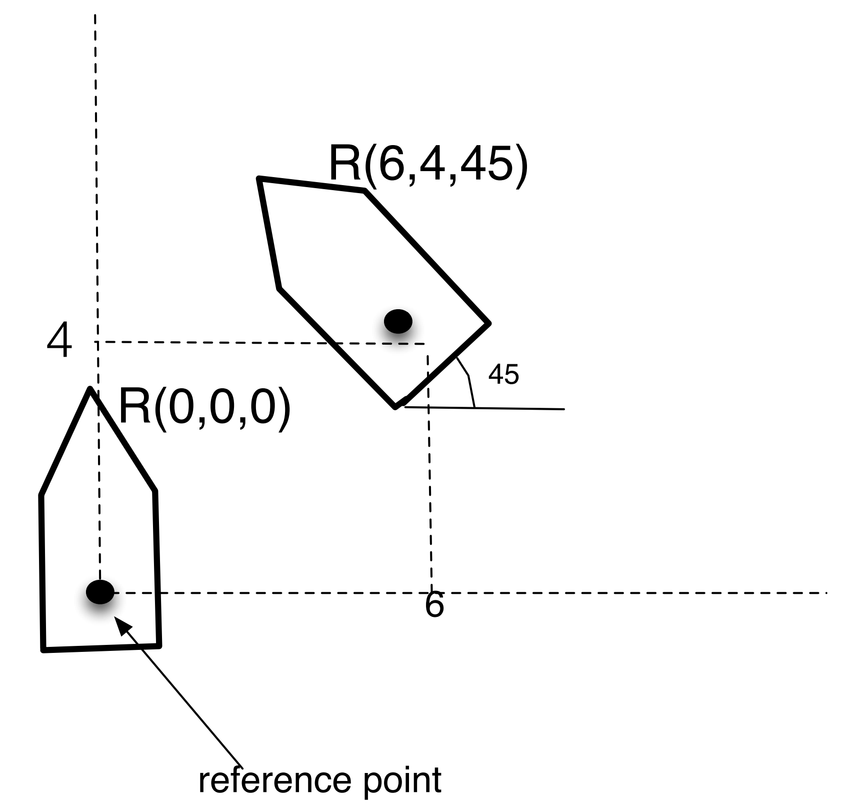
screenshot from internet

Work/physical space

- Space where robot moves around

translation only





translation + rotation

Work/physical space

- Space where robot moves around

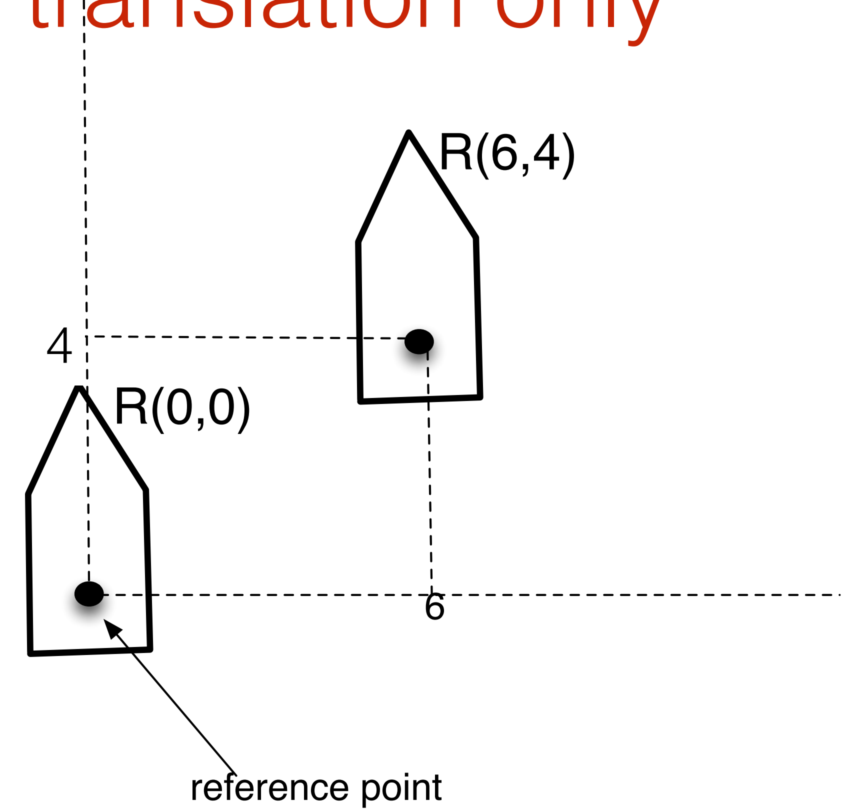
A placement of robot is specified by the degrees of freedom (dof) of the robot

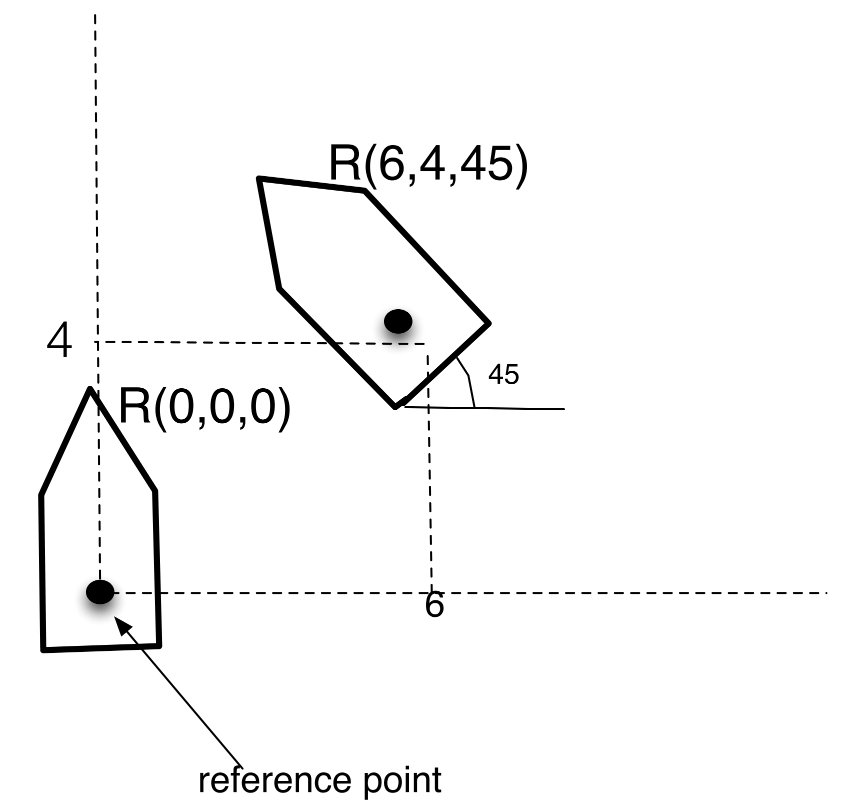
- Example:

  R(x,y)

  R(x,y,θ)

translation only





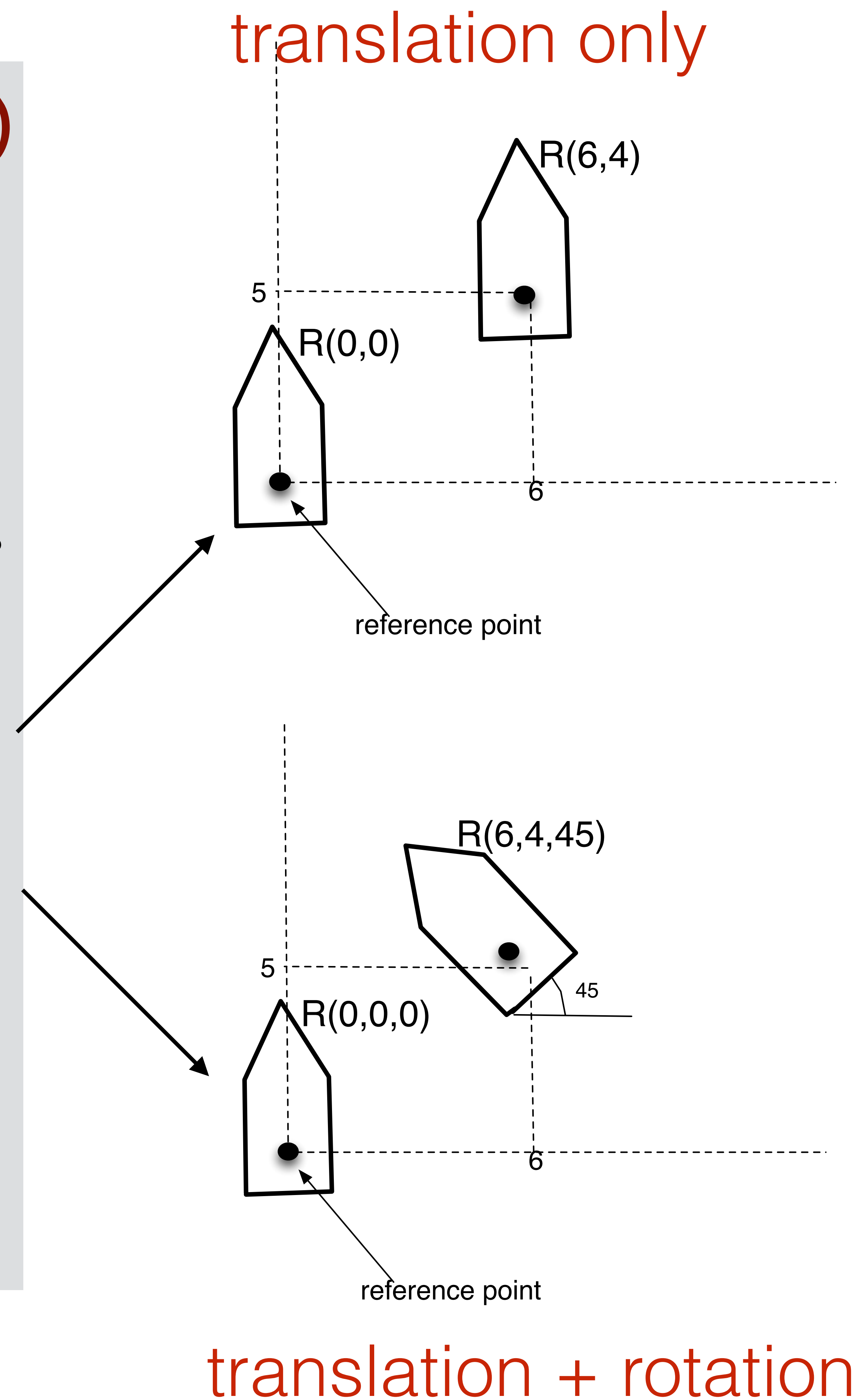
translation + rotation

Configuration space (C-space)

- A point in C-space corresponds to placement of the robot in physical space
- C-space: The parametric space of the robot = space of all possible placements of the robot
- **Examples:**

    2D, translation only <-> **R(x,y)**

    2D, transl.+ rot. <-> **R(x,y, theta)**



Physical Space and C-space

robot	physical space	C-space
polygon, (translation only)	2D	2D

Physical Space and C-space

robot	physical space	C-space
polygon, (translation only)	2D	2D
polygon, (translation + rotations)	2D	3D

Physical Space and C-space

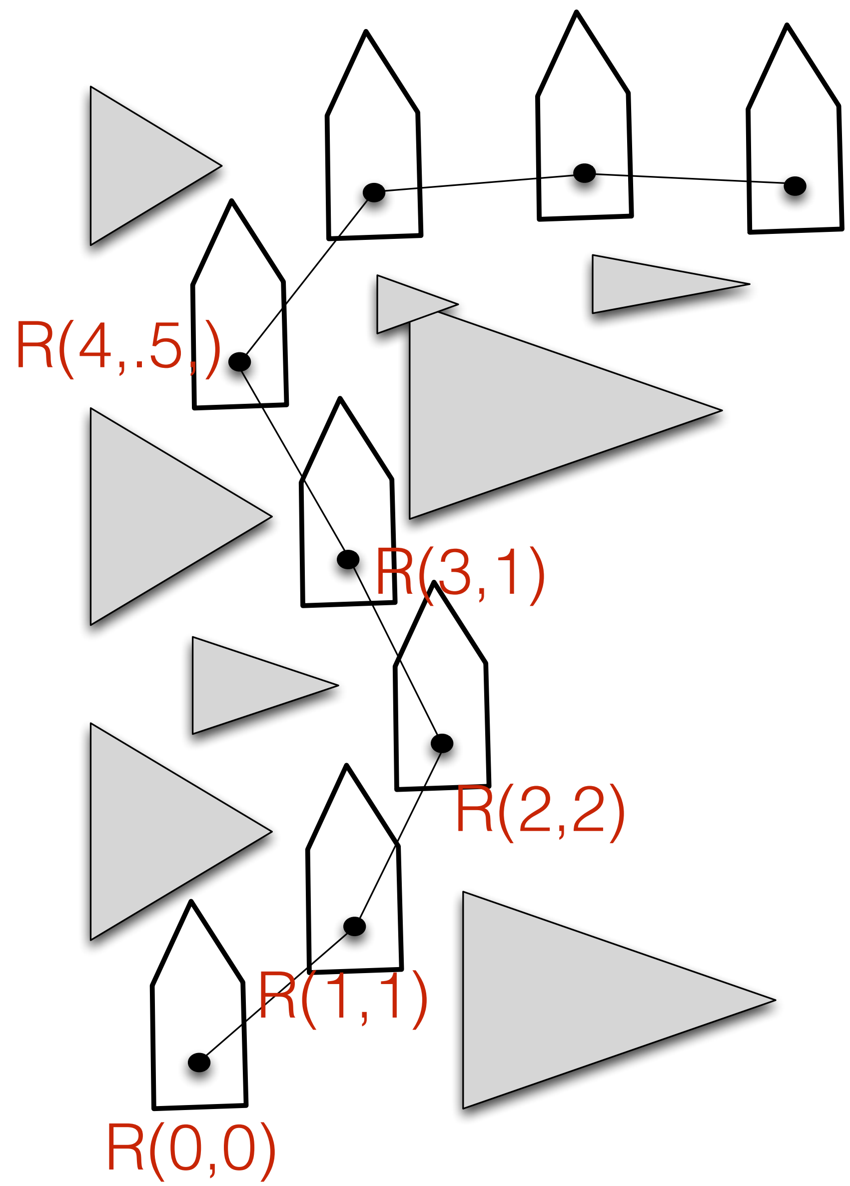
robot	physical space	C-space
polygon, (translation only)	2D	2D
polygon, (translation + rotations)	2D	3D
polygon (translation, rotations)	3D	6D

Physical Space and C-space

robot	physical space	C-space
polygon, (translation only)	2D	2D
polygon, (translation + rotations)	2D	3D
polygon (translation, rotations)	3D	6D
Robot arm with joints	3D	#DOF

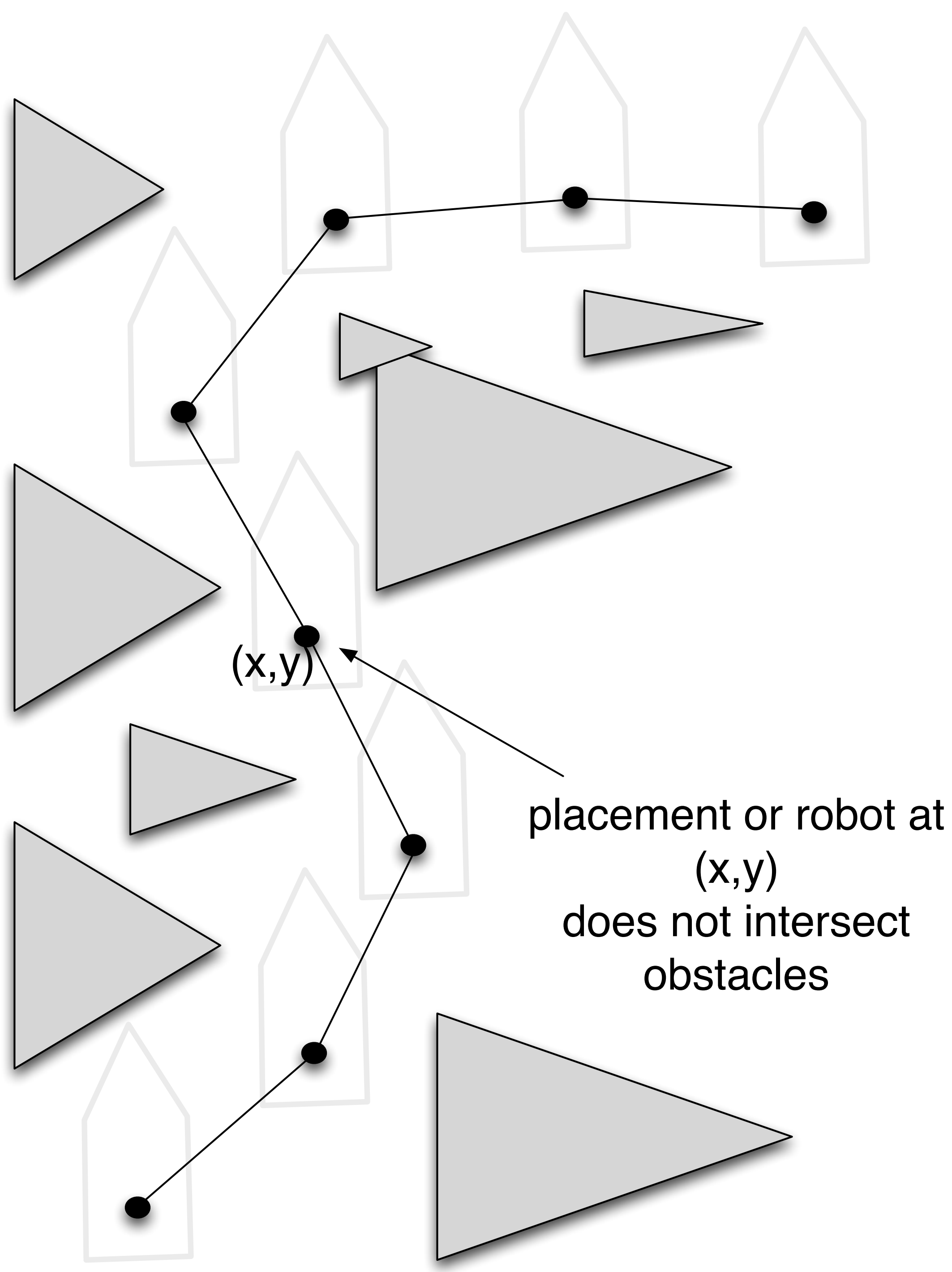
Motion planning

- Any path for R corresponds to a path for R in C-space
- Motion planning => motion planning in C-space



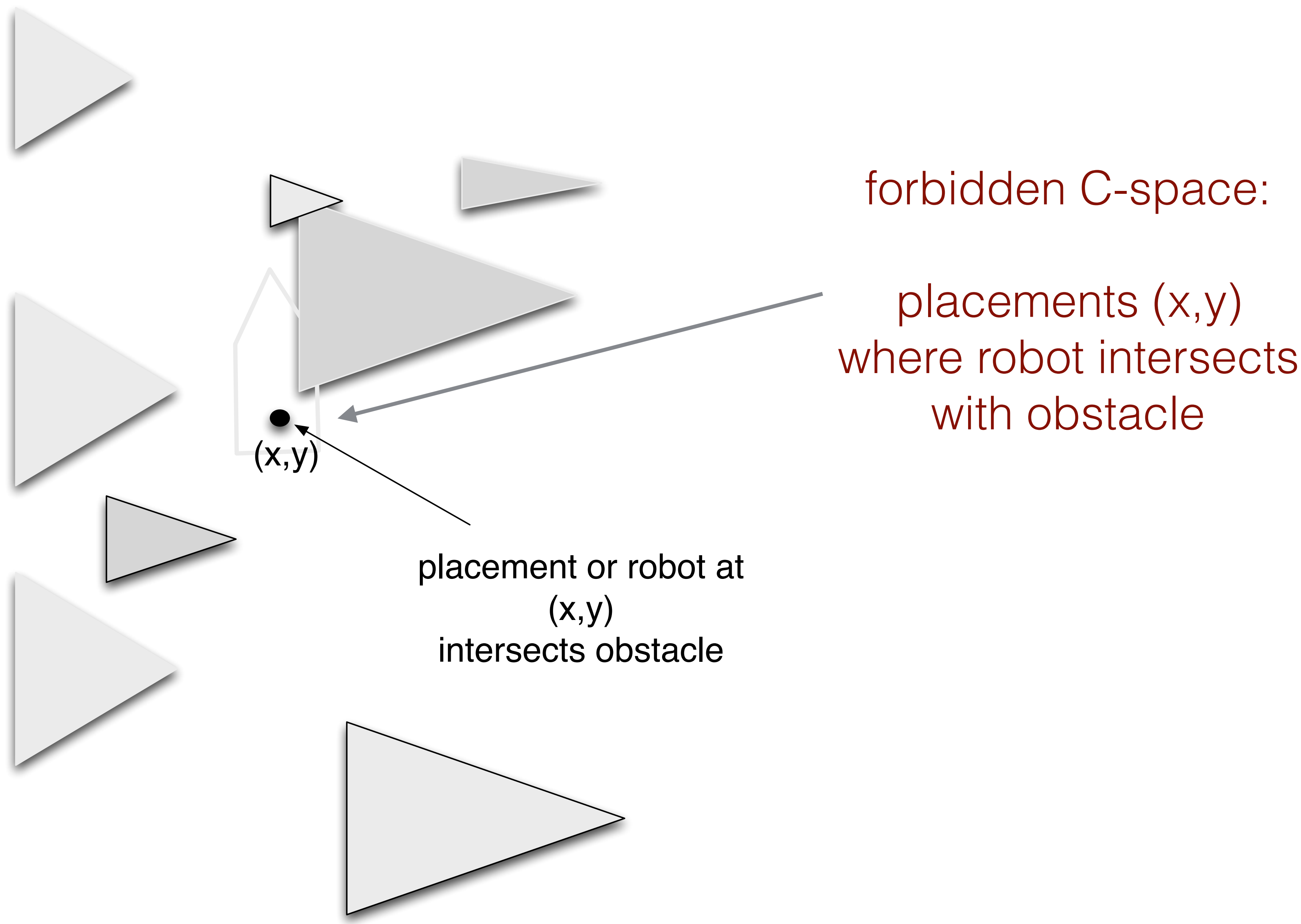
Motion planning in C-space

- Free C-space



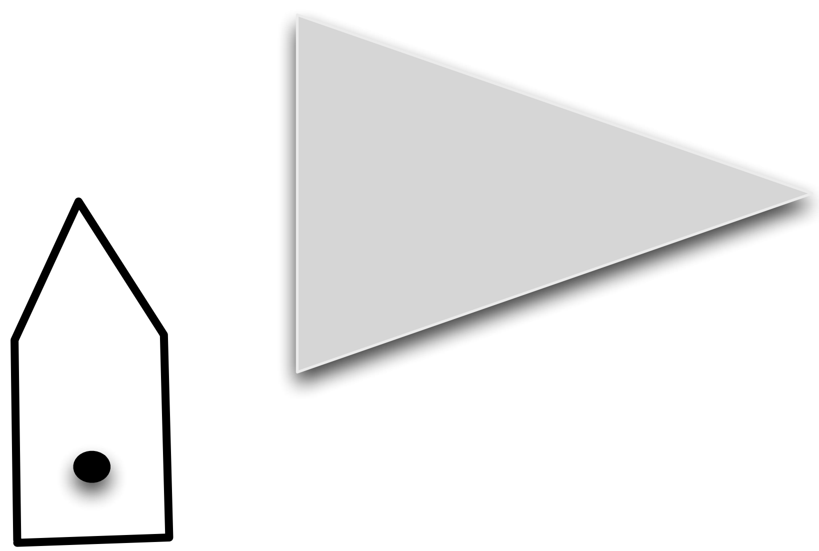
Motion planning in C-space

- Forbidden C-space



C-obstacles

- Extended obstacle, or C-obstacle
  - Given obstacle O, robot R(x,y): what placements cause intersection with O?



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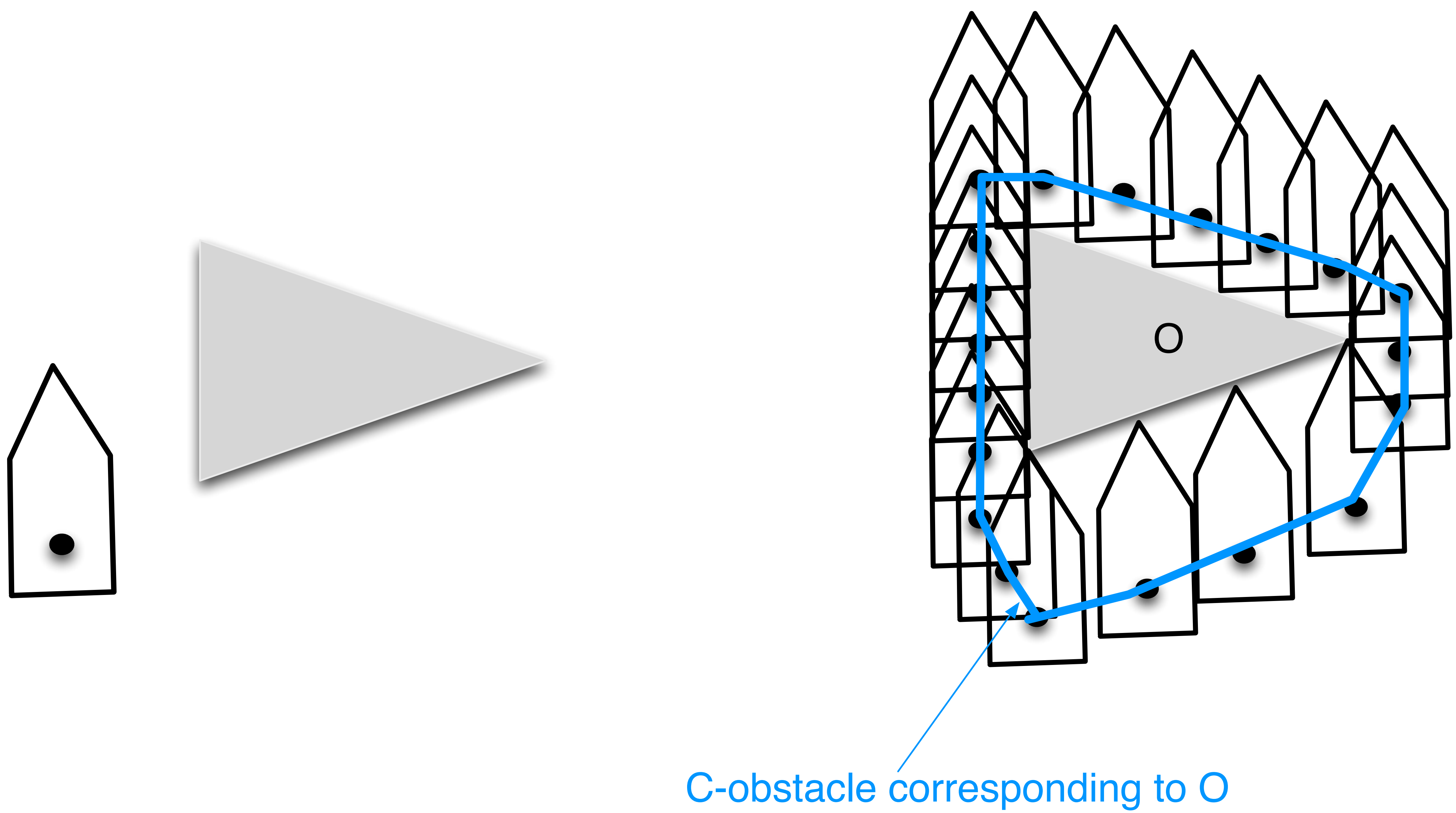

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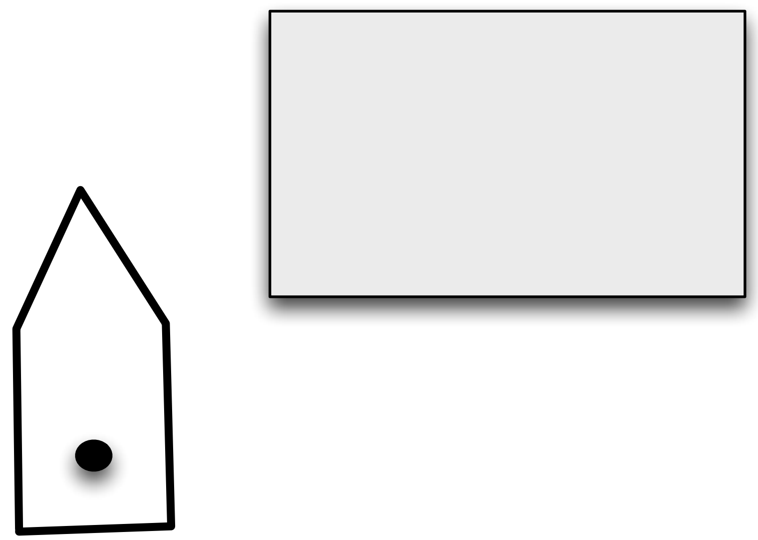

C-obstacles

- Extended obstacle, or C-obstacle
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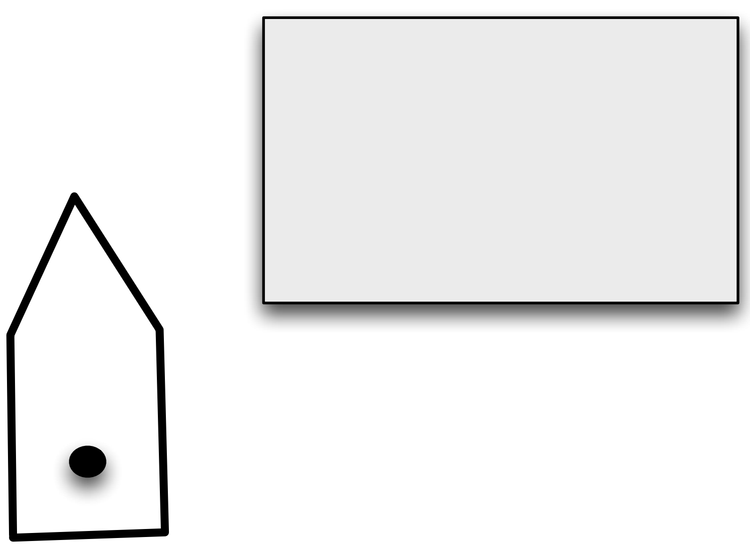
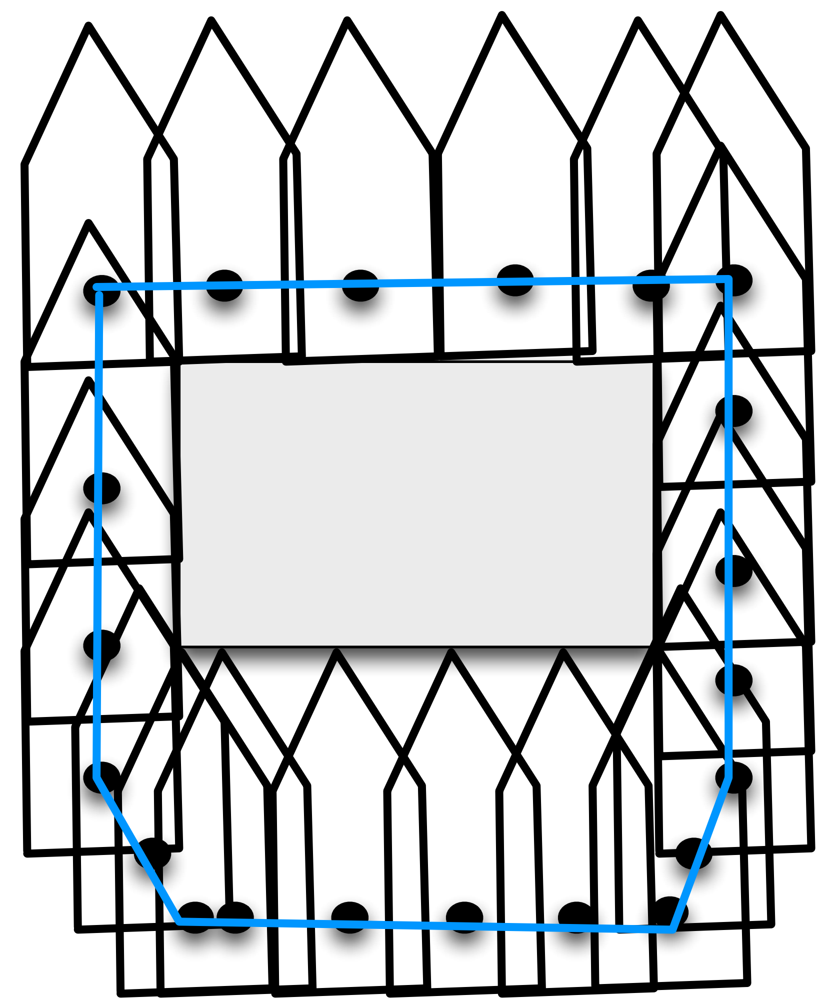
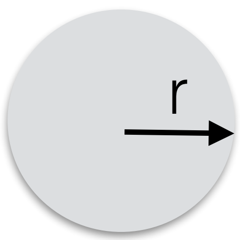
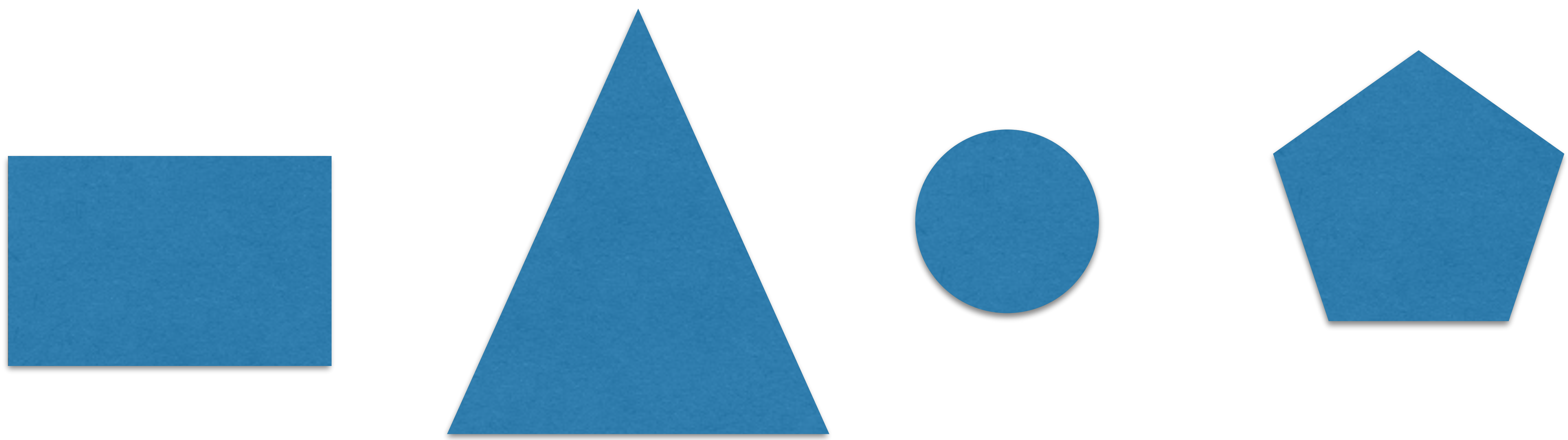




Exercise



robot



Show the corresponding C-obstacles for a disc robot.

Exercise

Show the corresponding C-obstacle.

Polygonal robot translating in 2D

Algorithm

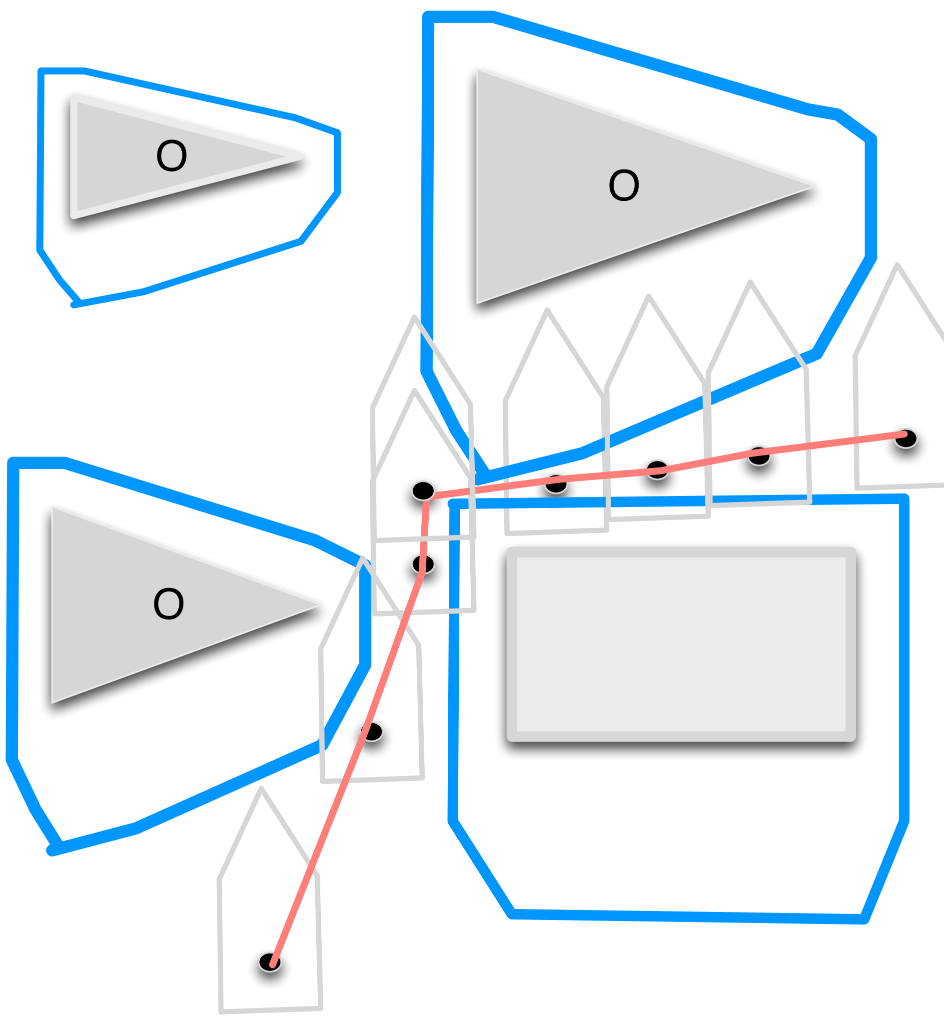
- For each obstacle O, compute the corresponding C-obstacle
- Compute the union of C-obstacles
- Compute its complement. That's the free C-space

//now the problem is reduced to a point

//robot moving in free C-space

- Compute a trapezoidal map of free C-space
- Compute a roadmap

How fast can we do this?



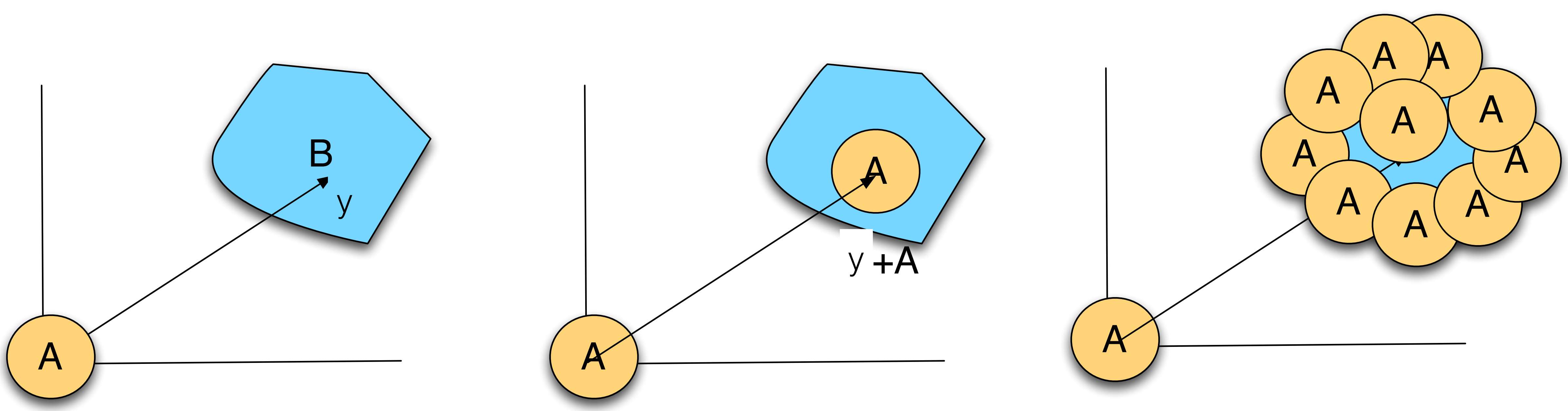
How do we compute C-obstacles?

Minkowski sum

- Let A, B two sets of points in the plane
- Define $A \oplus B = \{ x + y \mid x \text{ in } A, y \text{ in } B\}$ Minkowski sum

  vector sum x, y vectors

- Interpretation: consider set A to be centered at the origin. Then $A \oplus B$ represents many copies of A, translated by y, for all y in B; i.e. place a copy of A centered at each point of B.

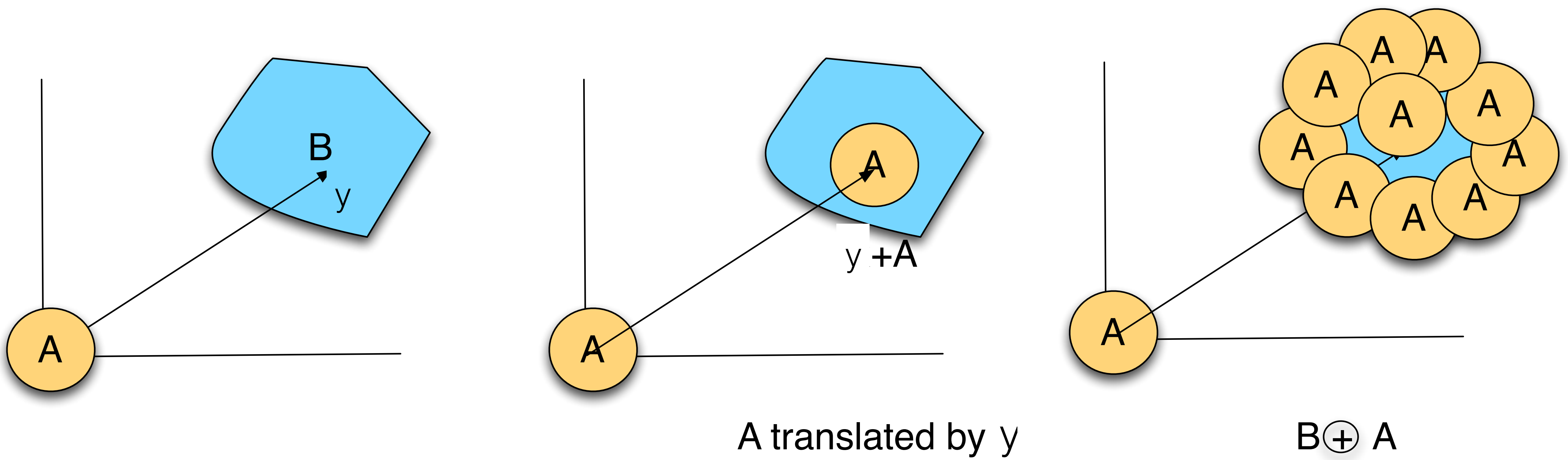


A translated by y

$B \oplus A$

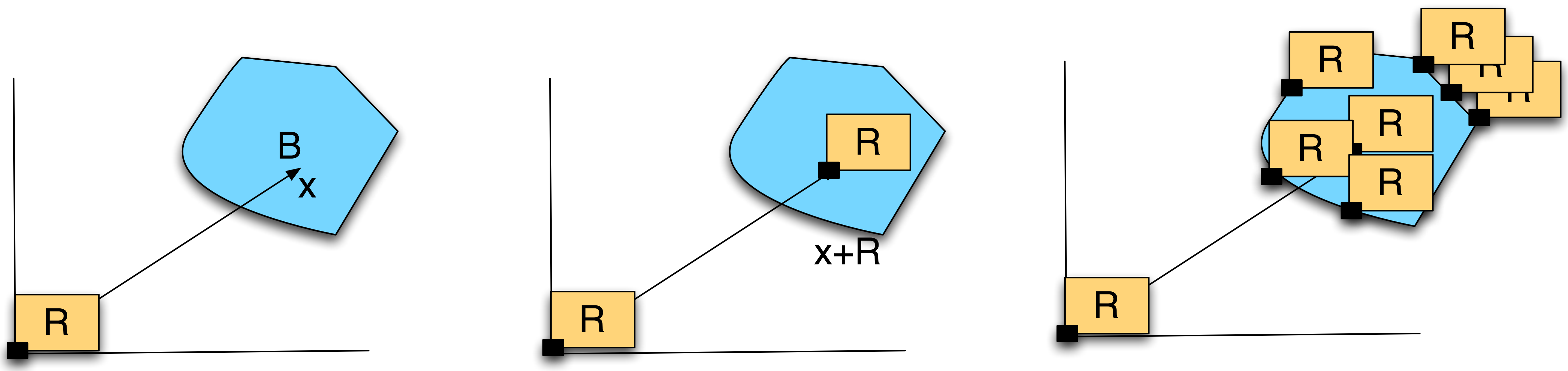
Minkowski sum

- A $\oplus$ B: Slide A so that the center of A traces the edges of B



C-obstacles as Minkowski sums

- Consider a robot R with the center in the lower left corner

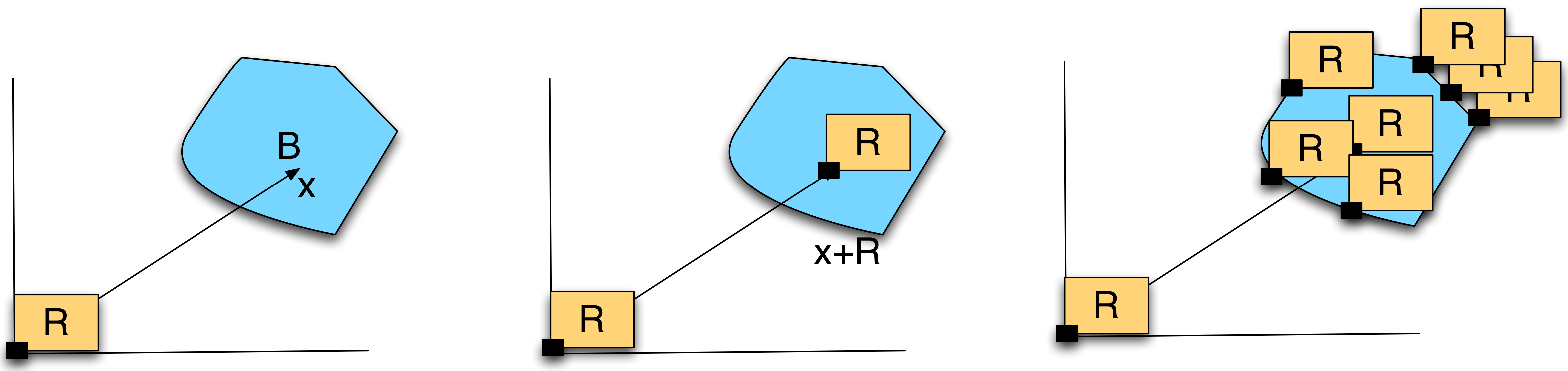


R translated by x

$B \oplus R$

C-obstacles as Minkowski sums

- Consider a robot R with the center in the lower left corner

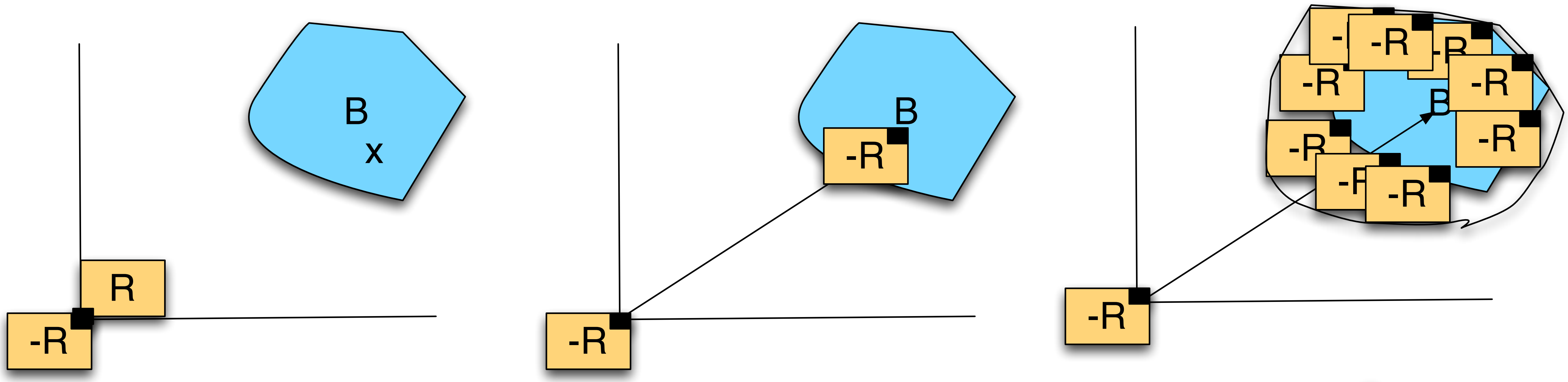


R translated by x

$B \oplus R$

$B \oplus R$ is not quite the C-obstacle of B

C-obstacles as Minkowski sums



-R translated by x

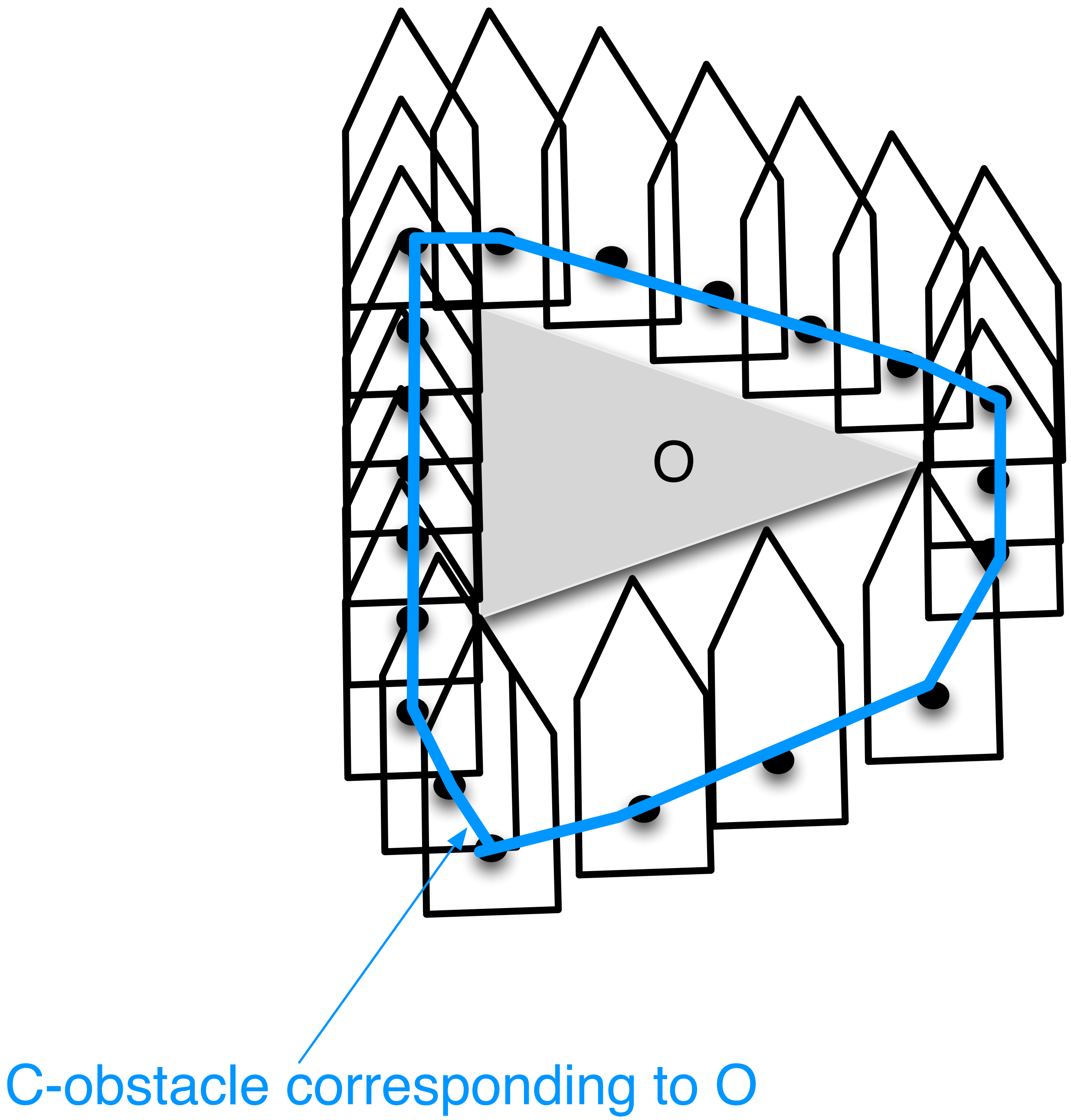
$B \oplus -R$

-R: R reflected by origin

The C-obstacle of B is $B \oplus (- R(0,0))$.

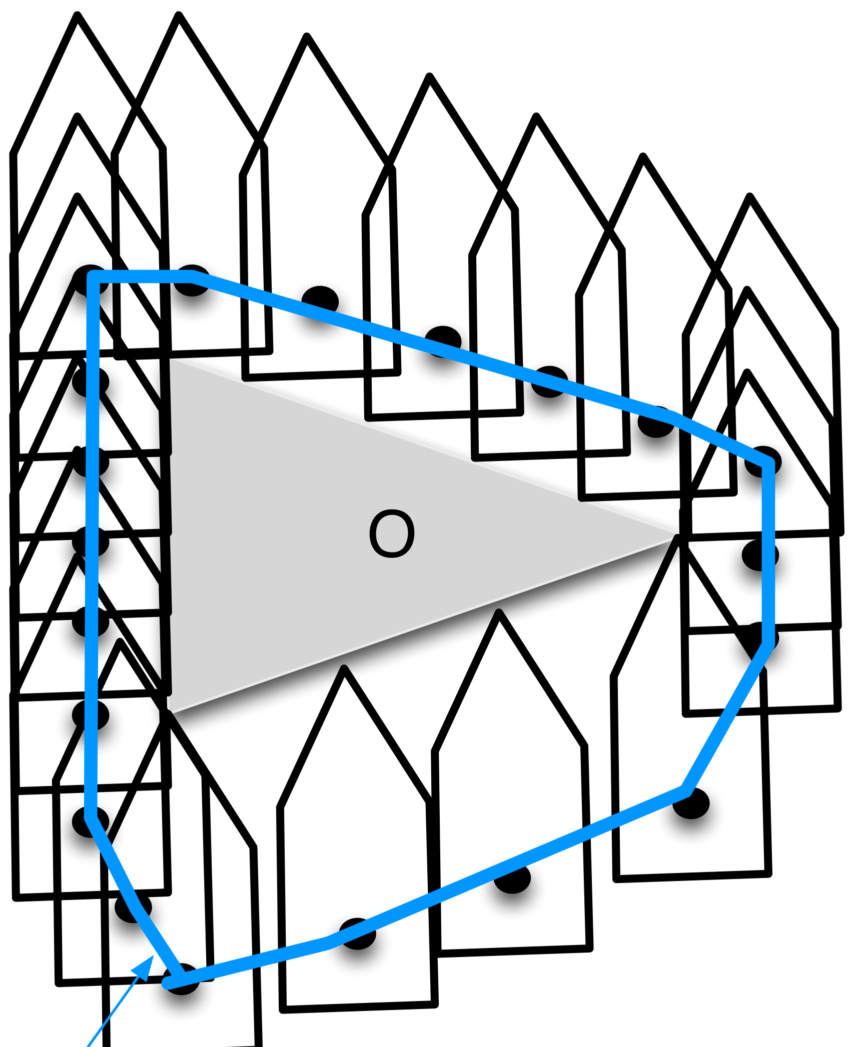
Slide so that R touches the obstacle

Find O + (-R)

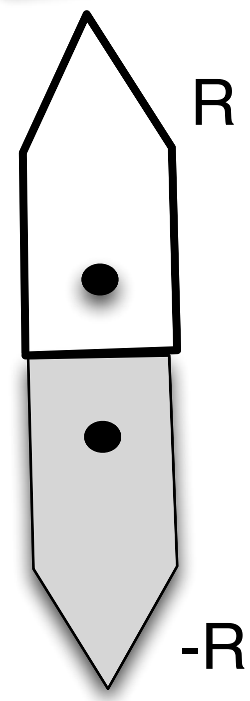


Slide so that R touches the obstacle

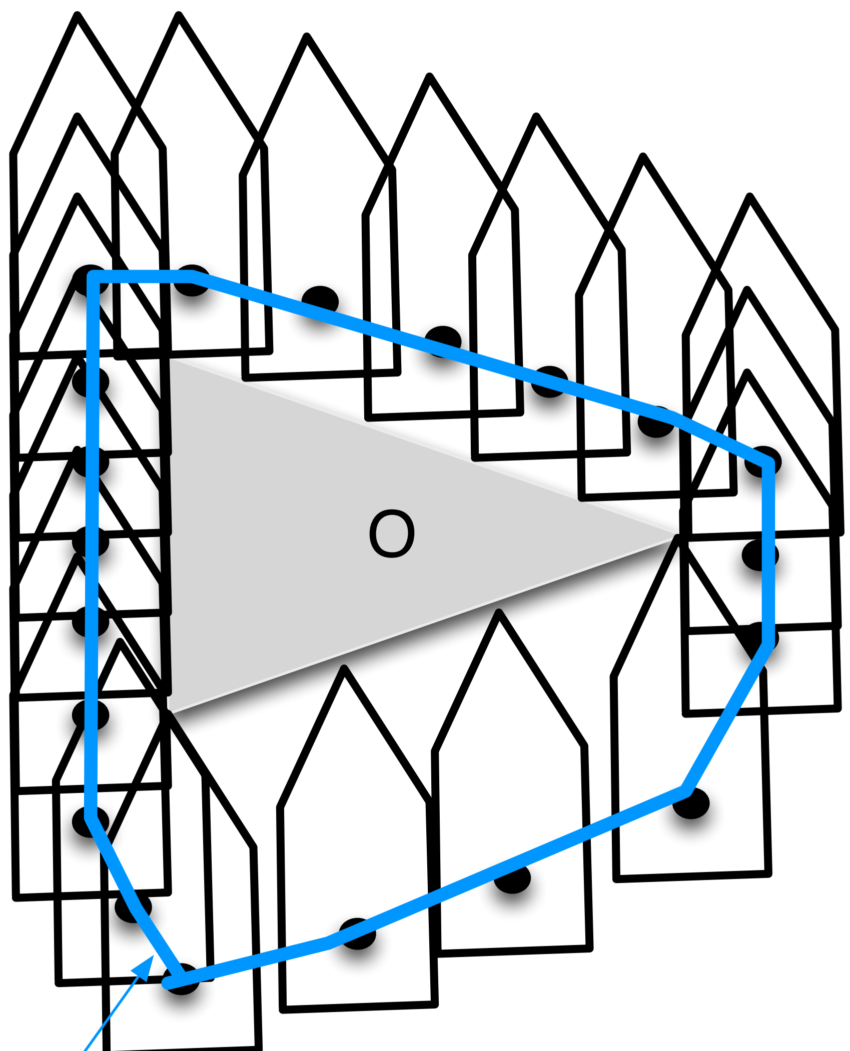
Slide so that centerpoint of -R traces the edges of obstacle



C-obstacle corresponding to O

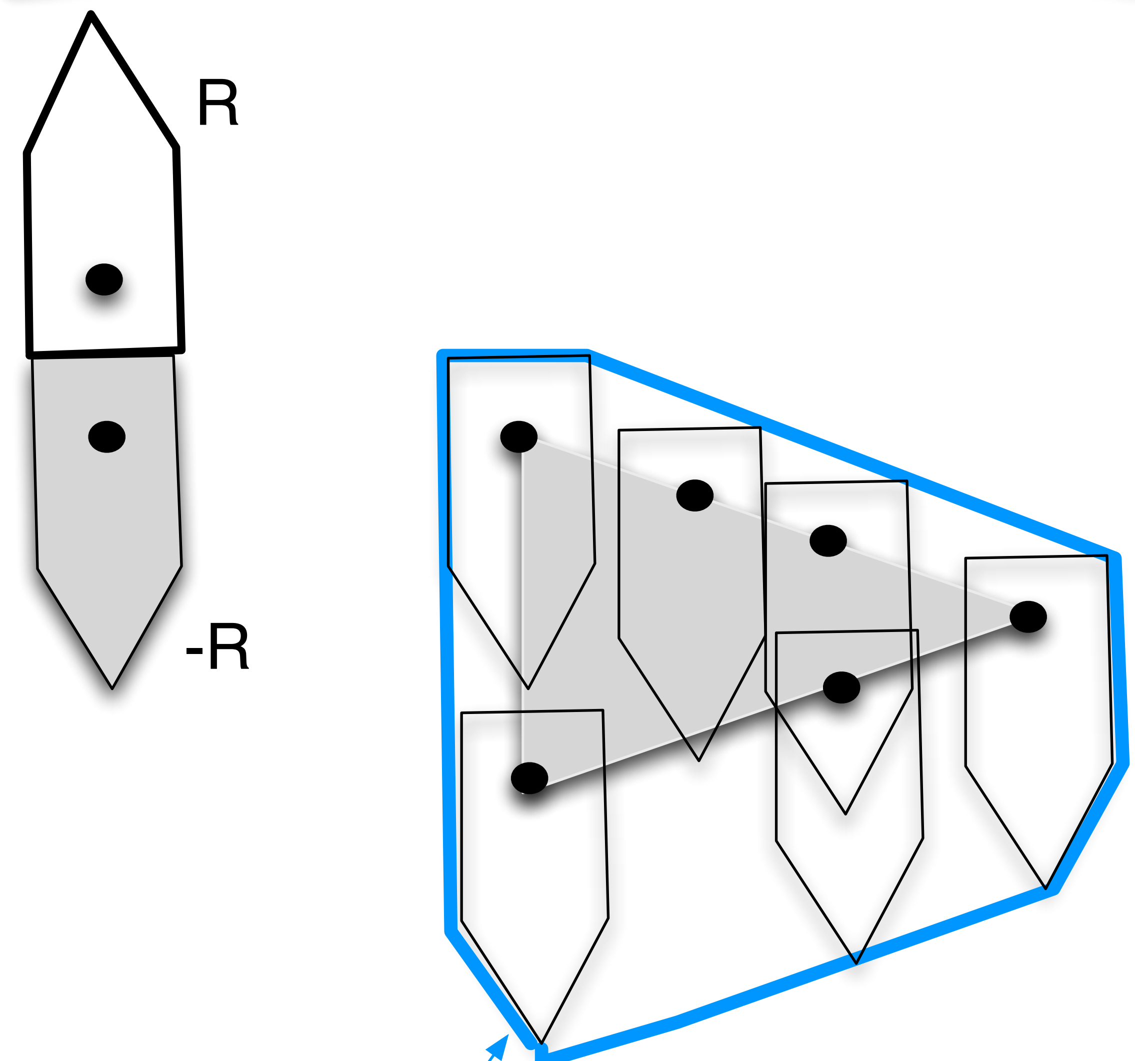


Slide so that R touches the obstacle

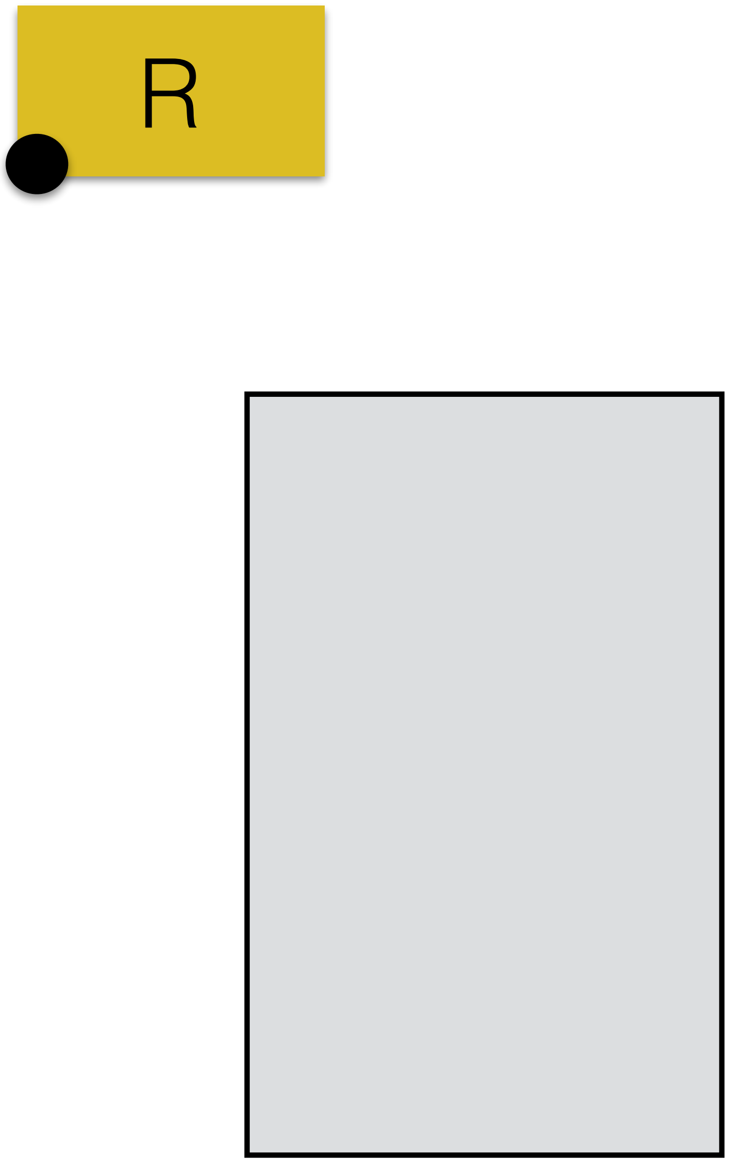


C-obstacle corresponding to O

Slide so that centerpoint of -R traces the edges of obstacle



C-obstacle corresponding to O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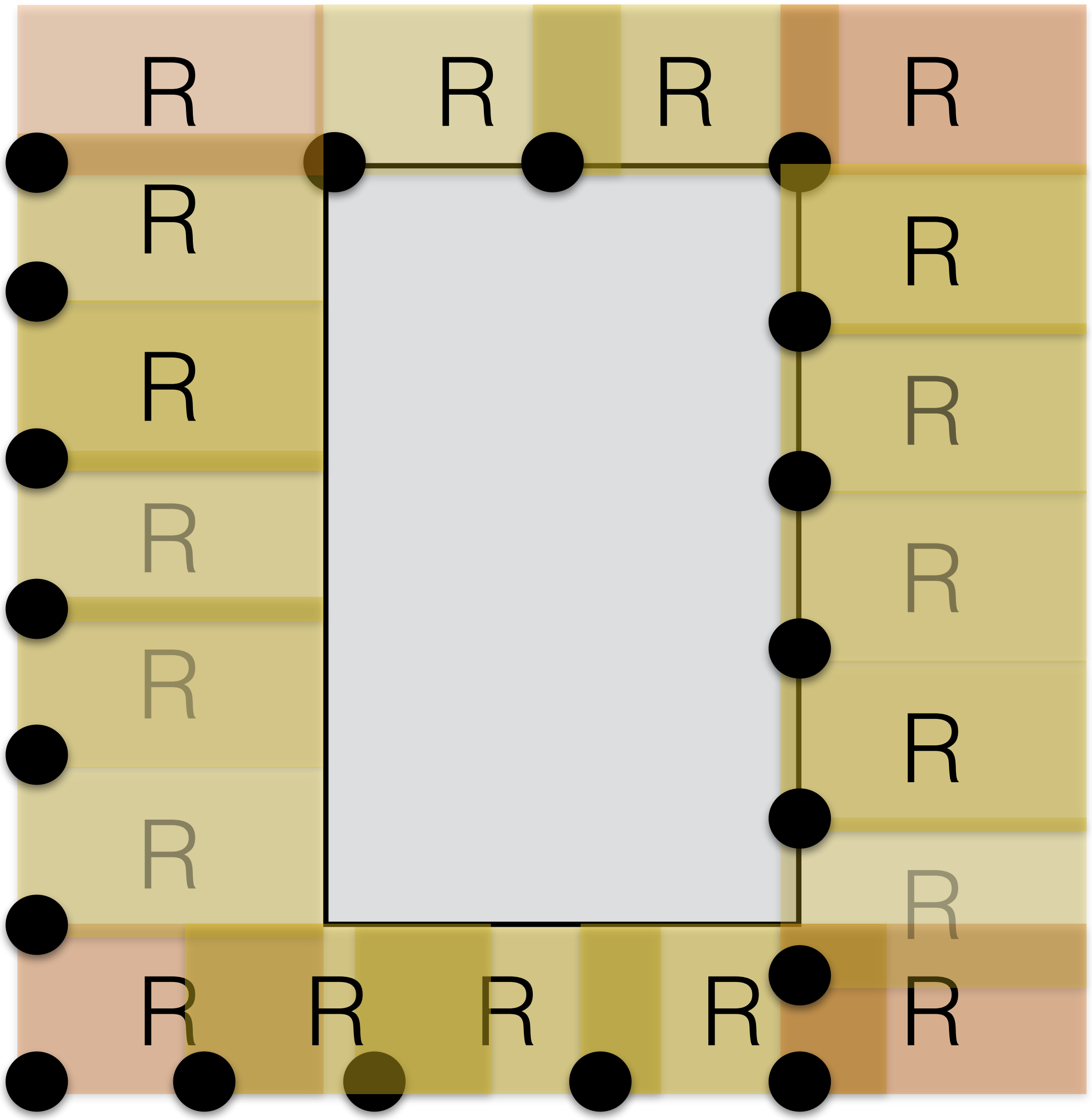
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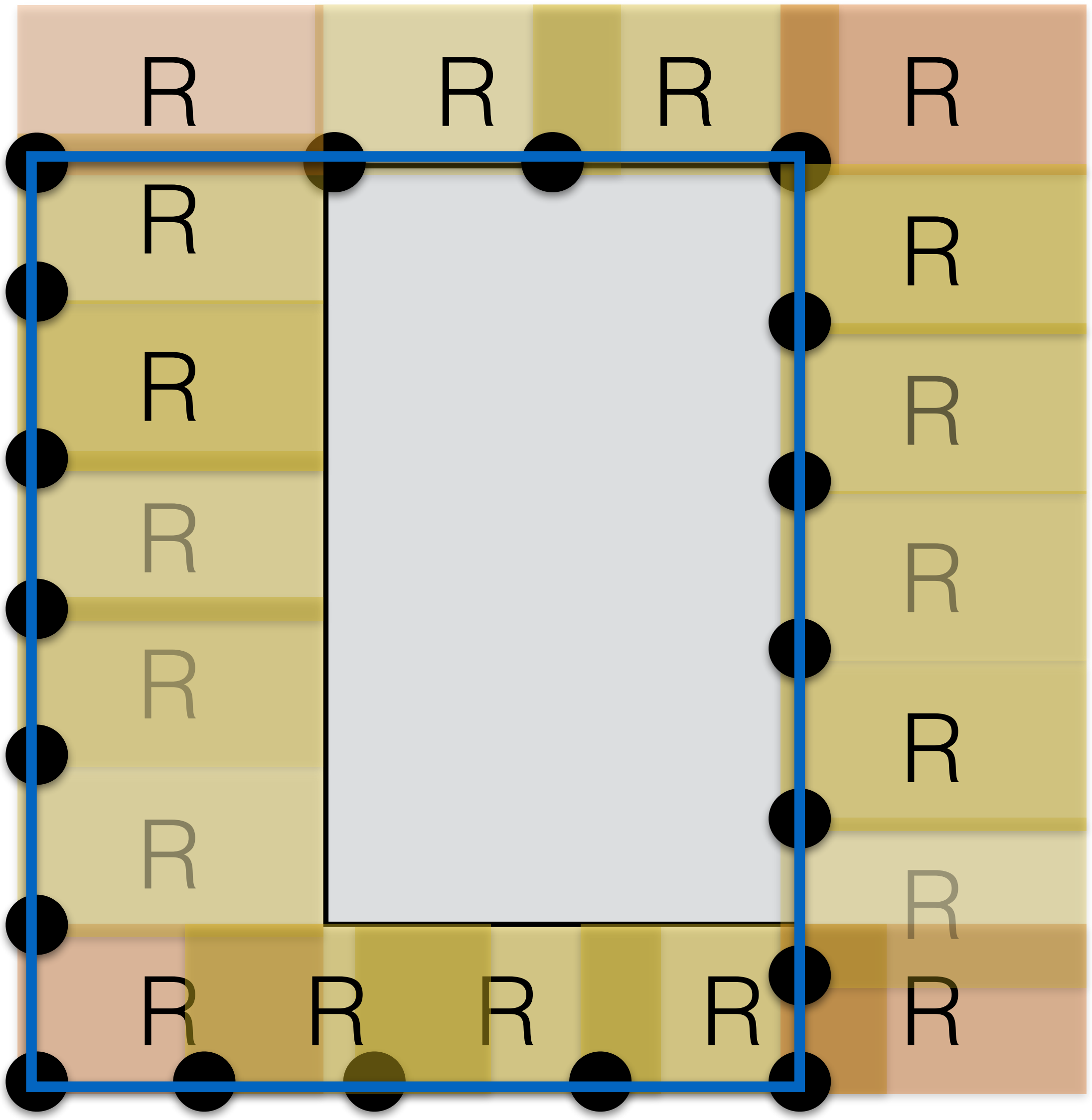
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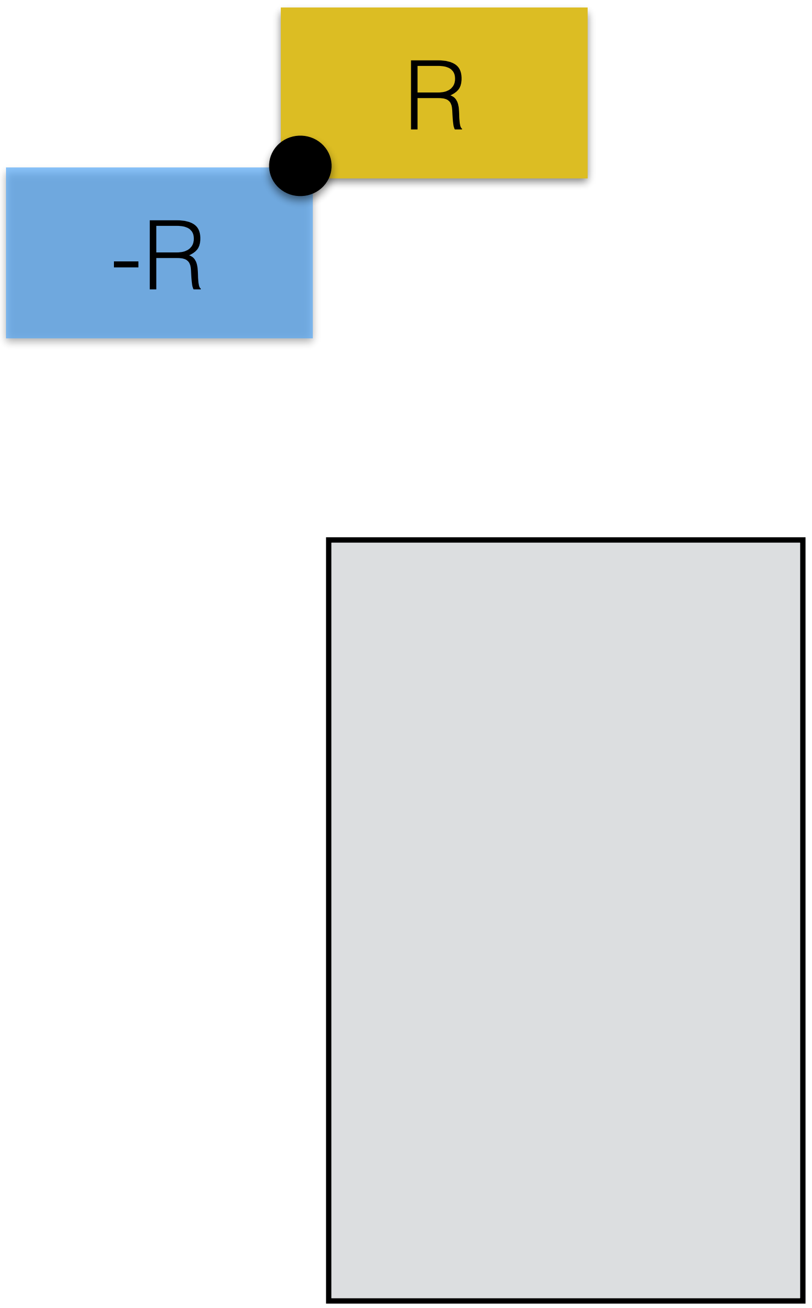
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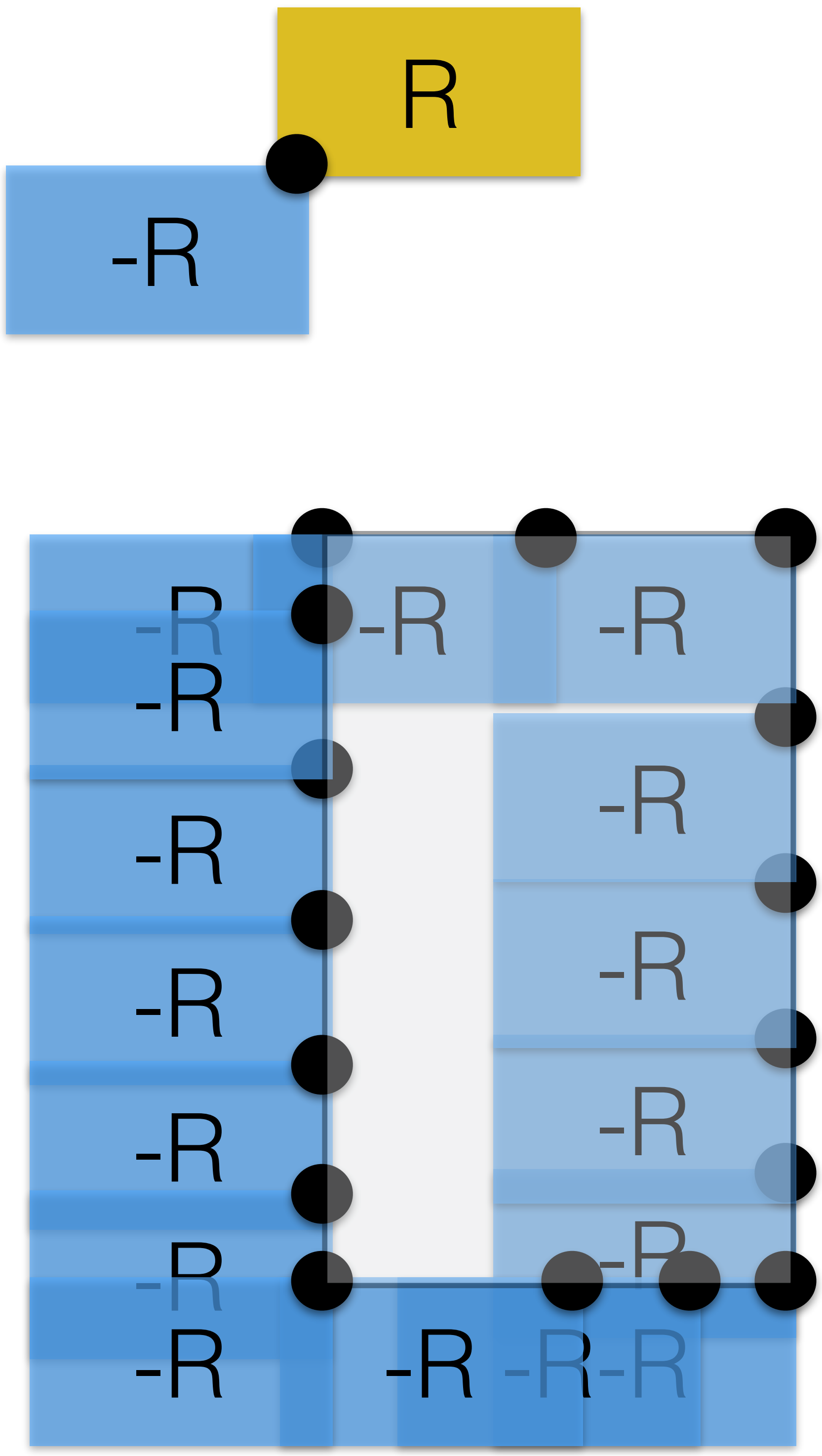
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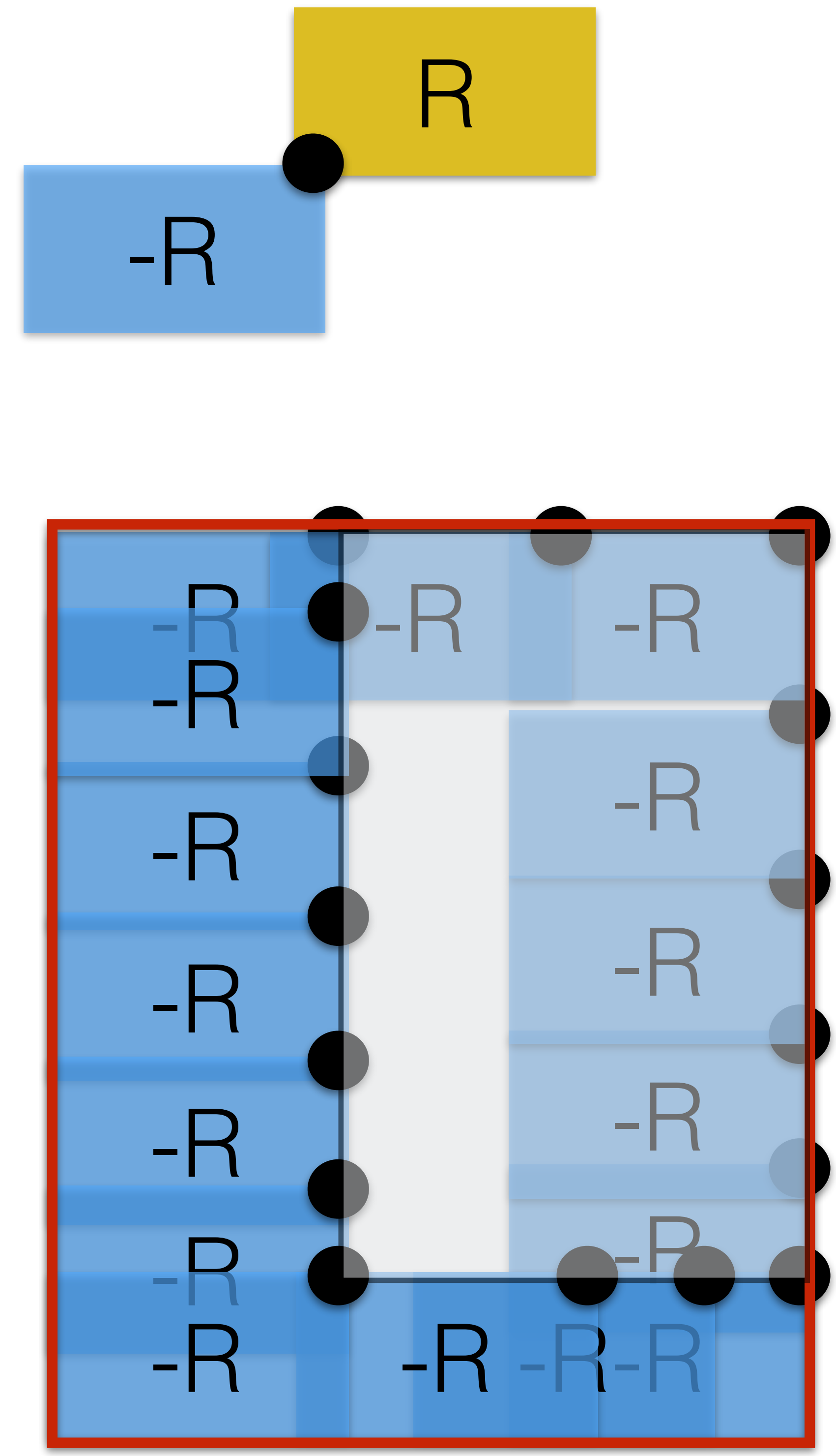
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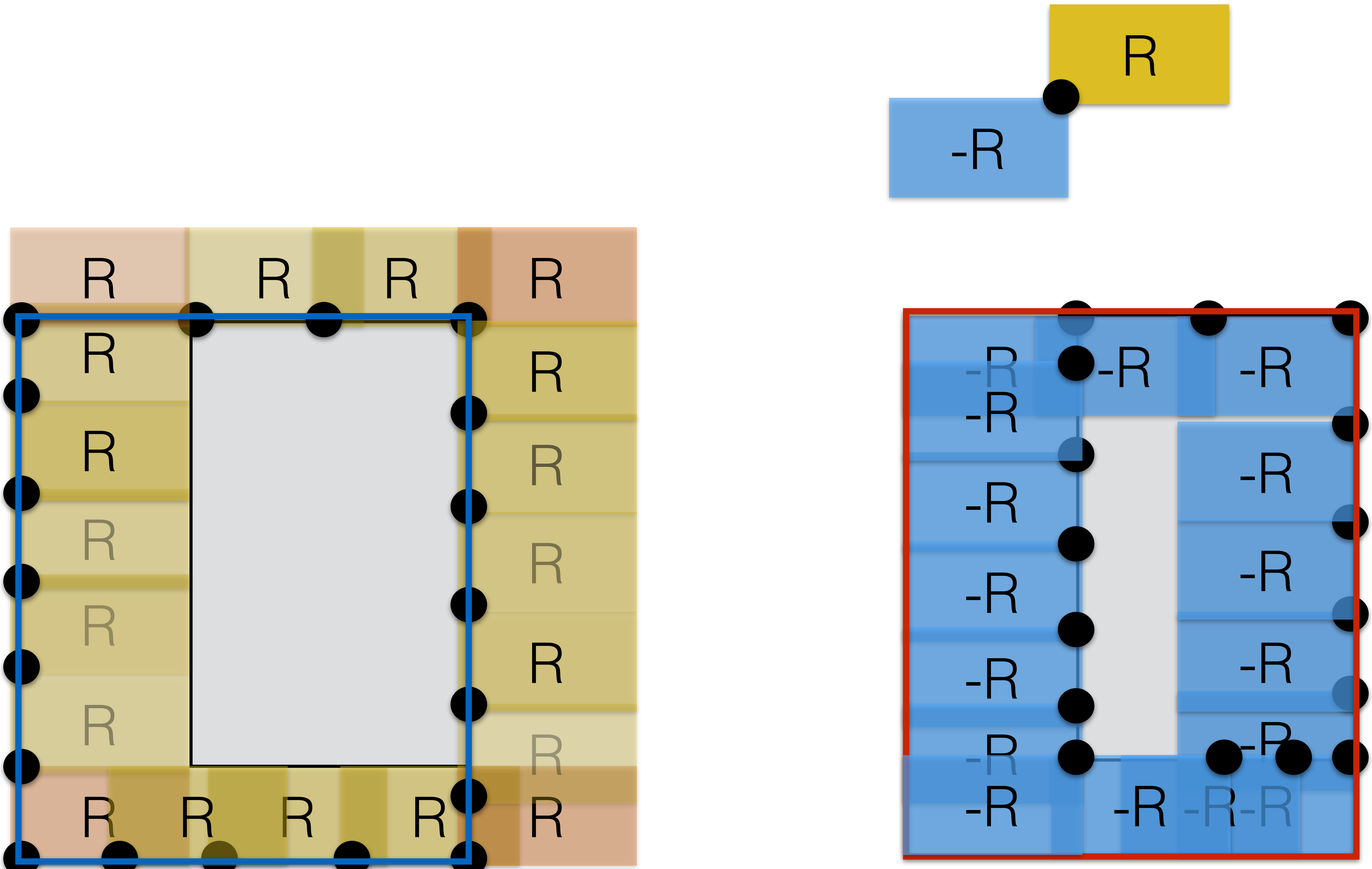

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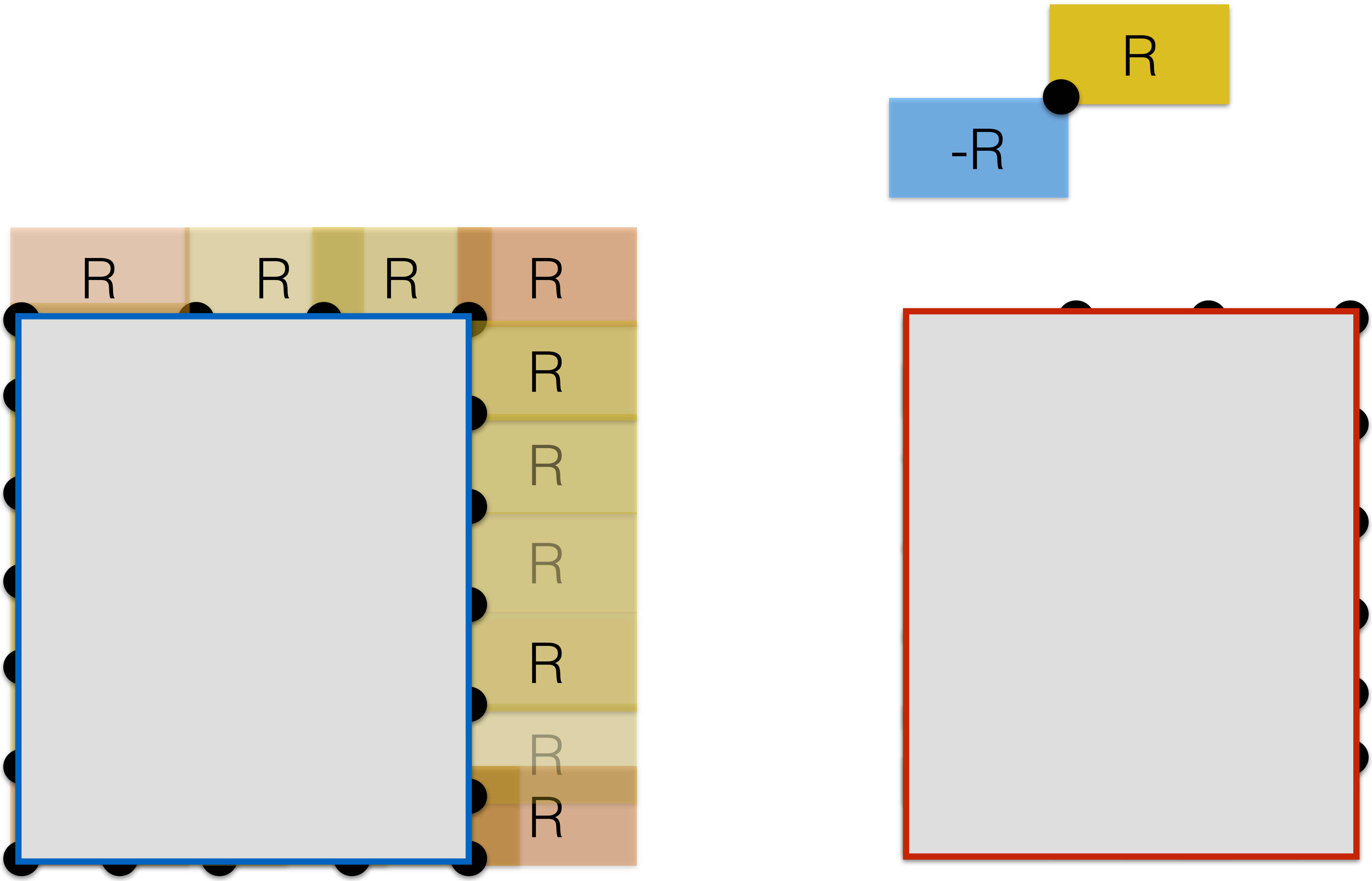
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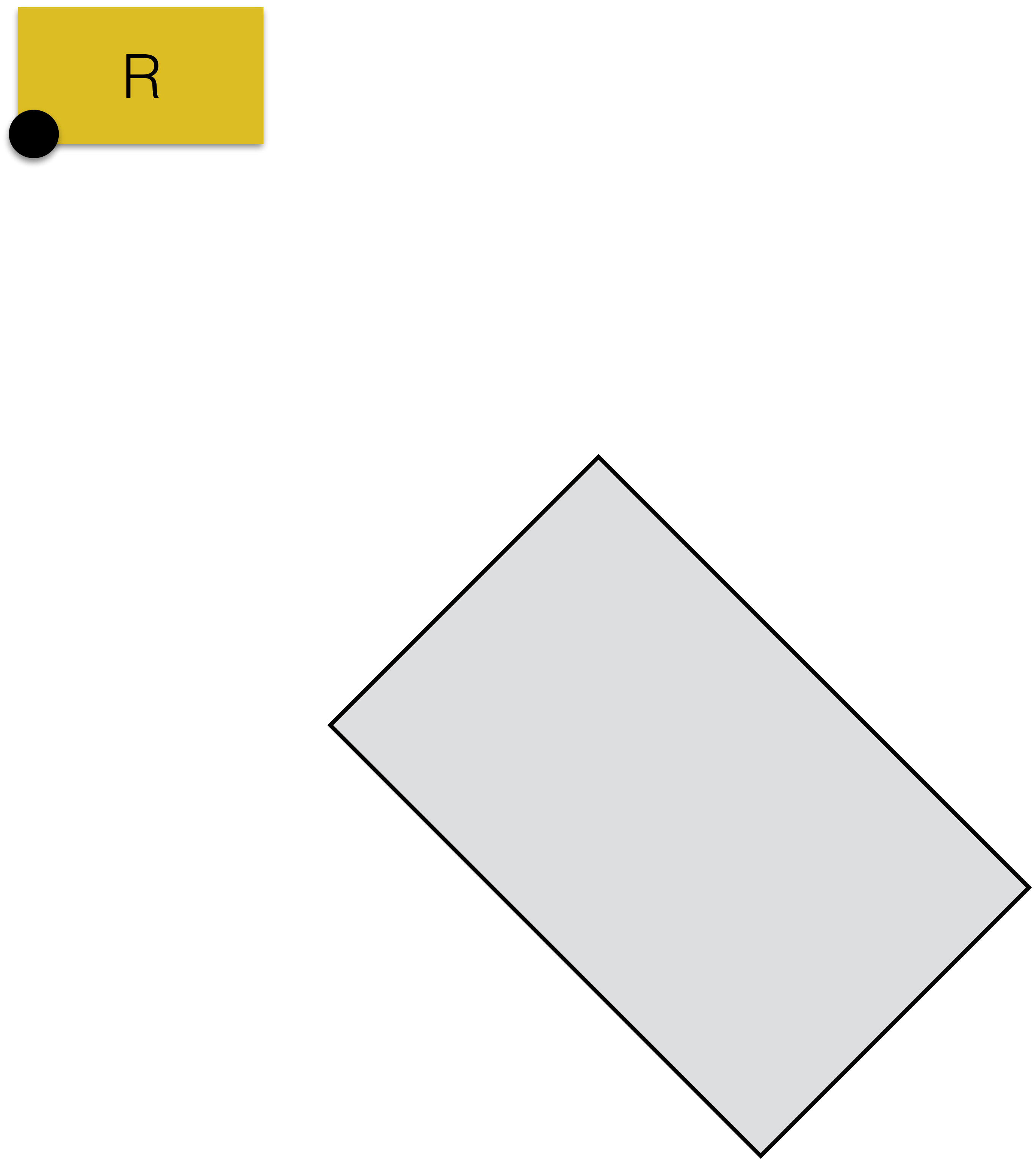
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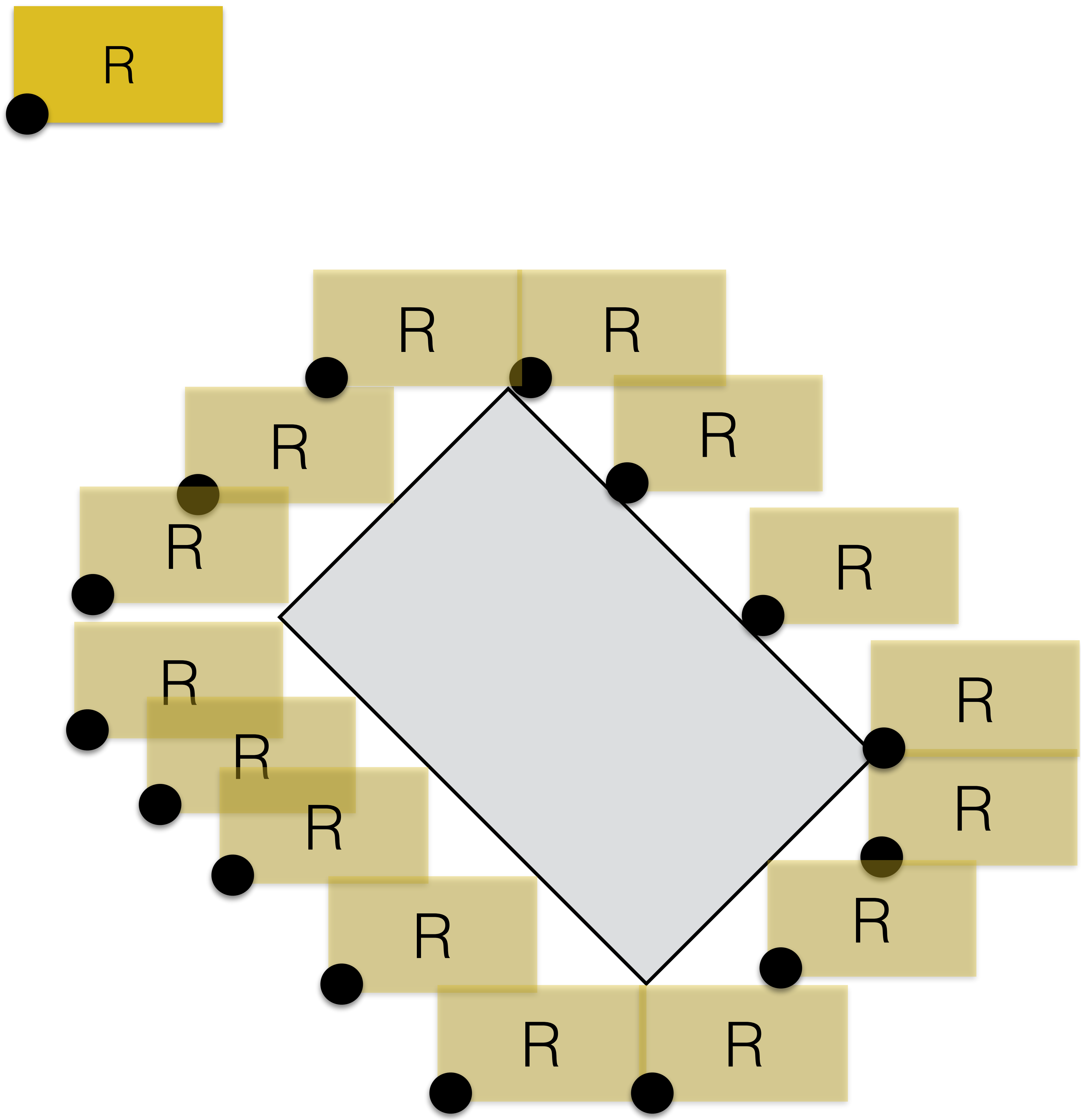
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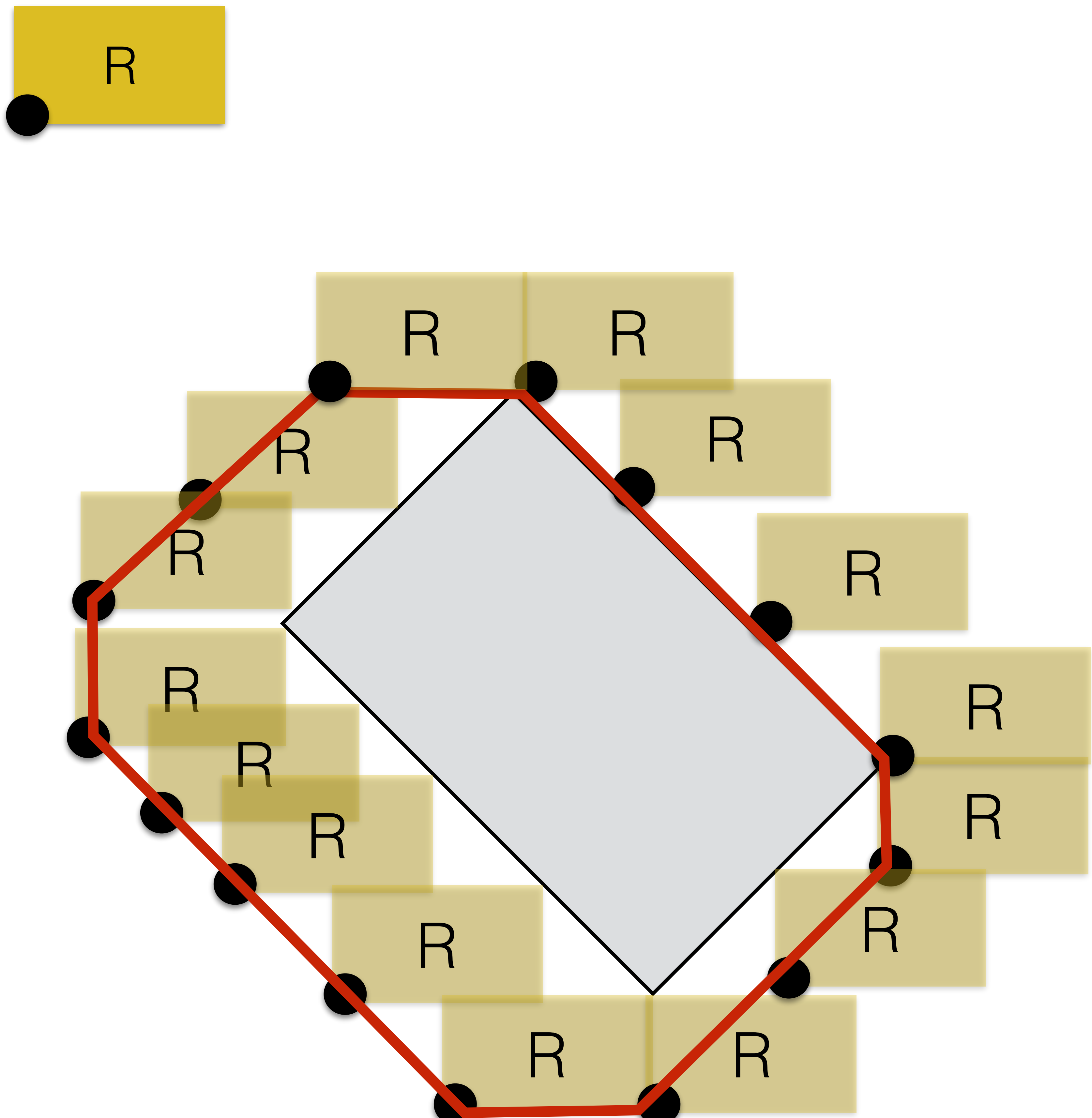
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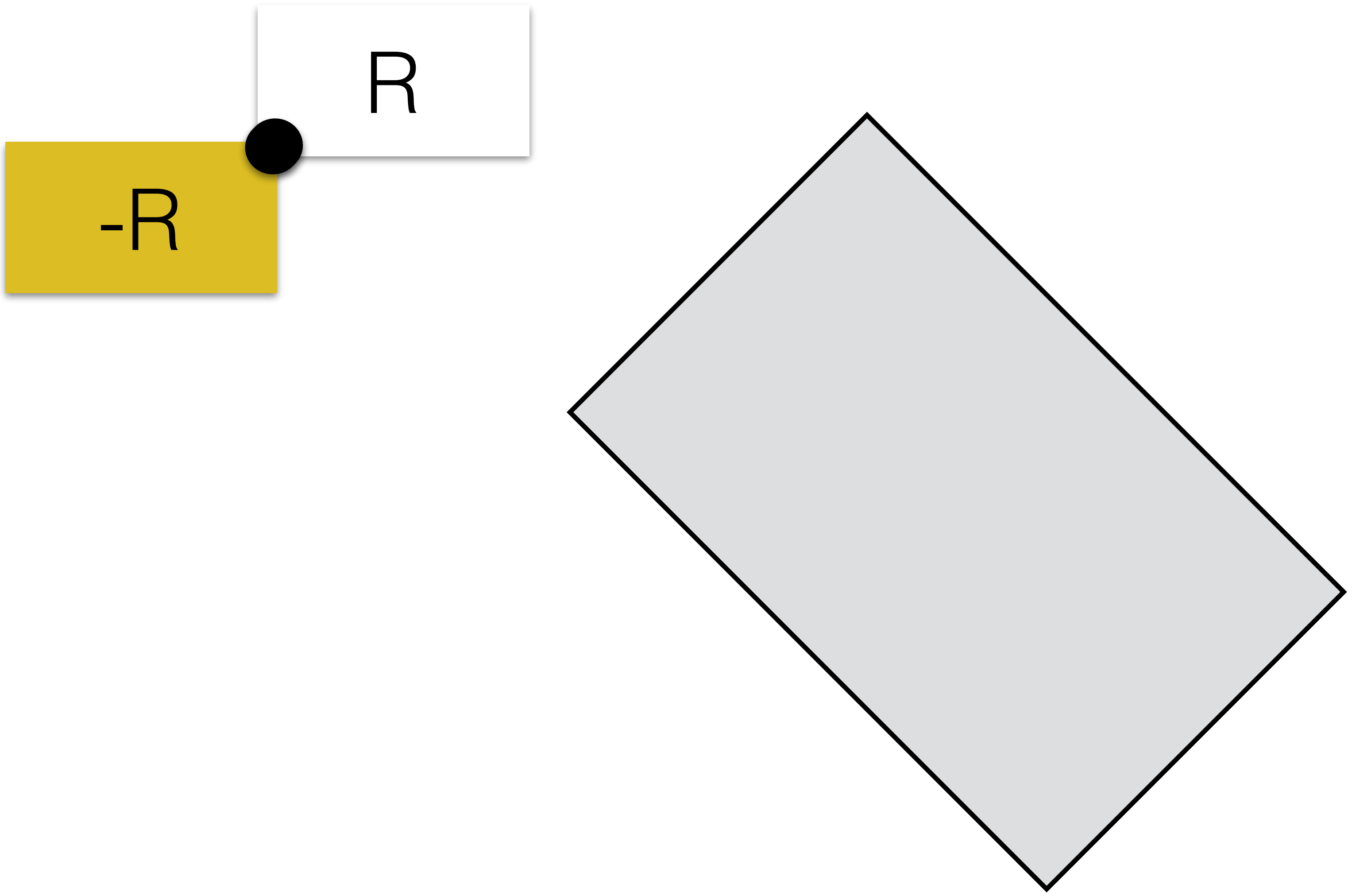
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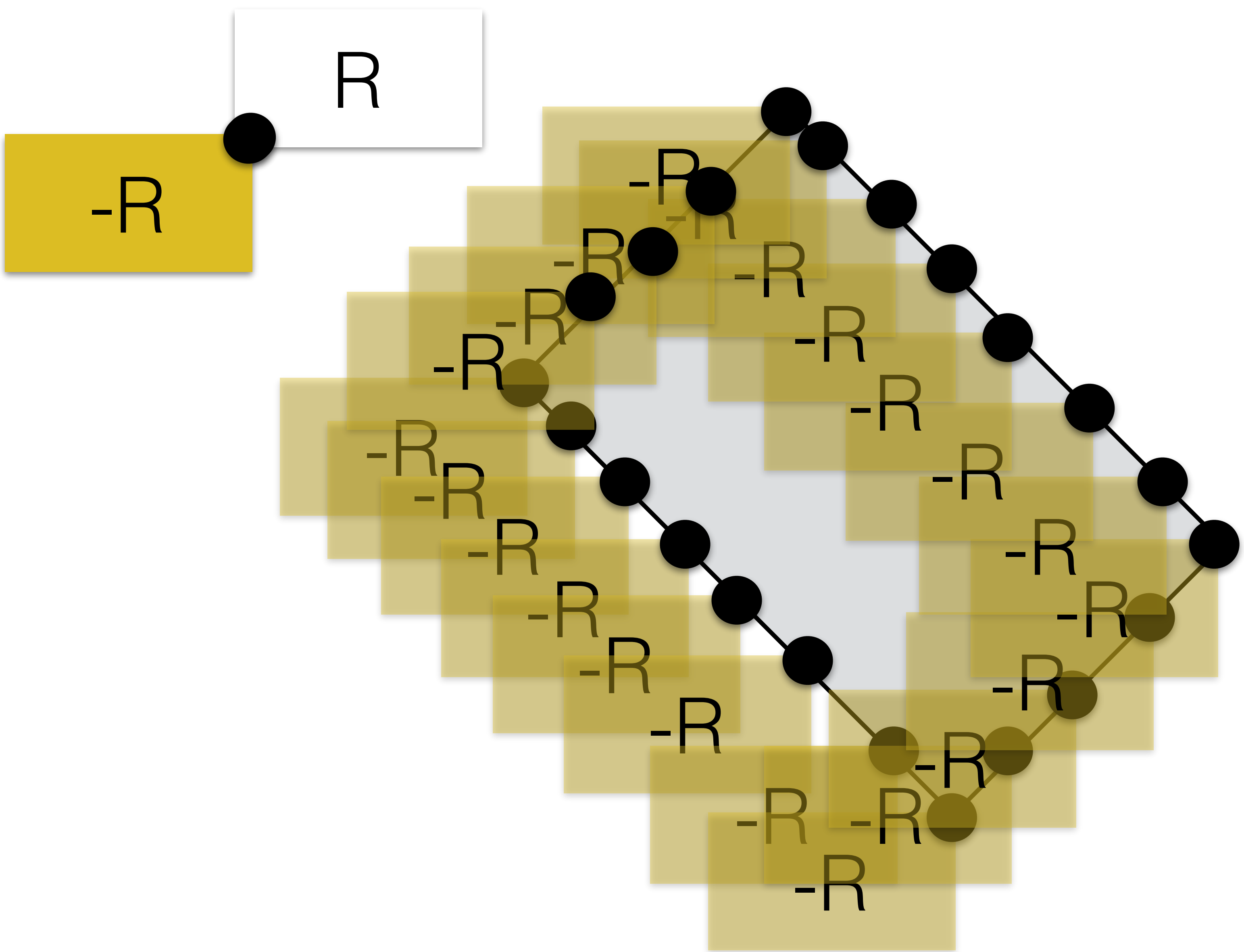
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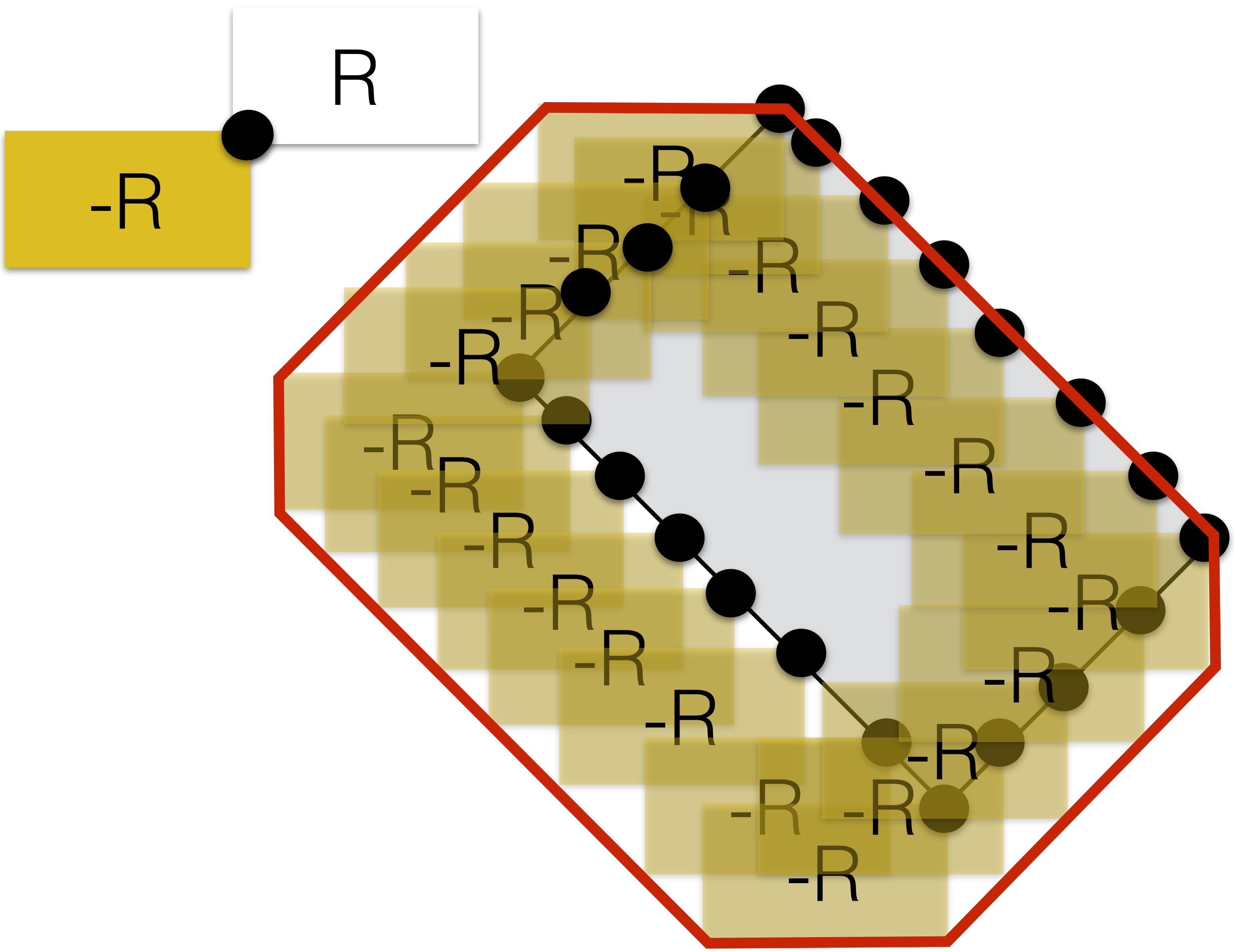
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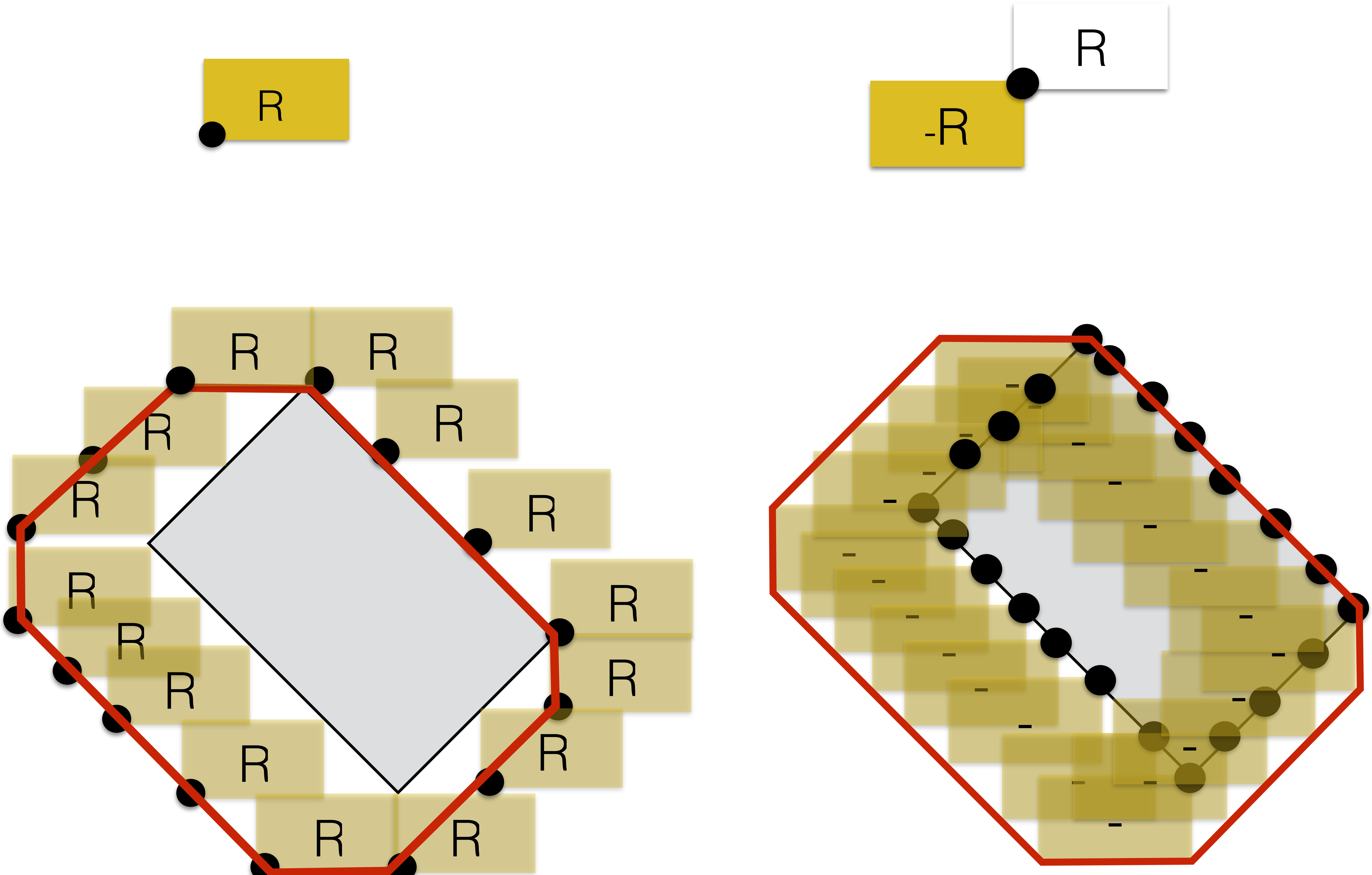
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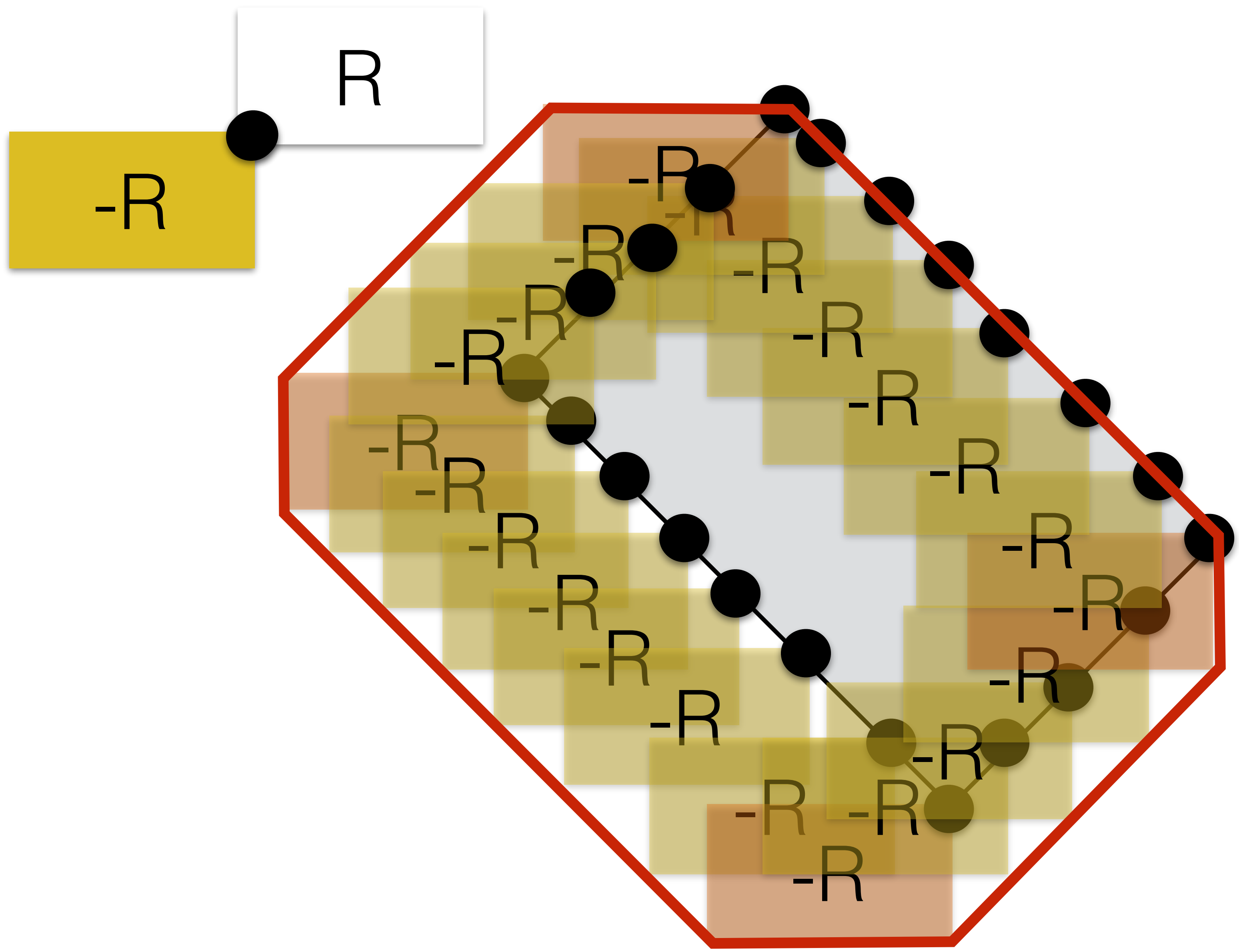
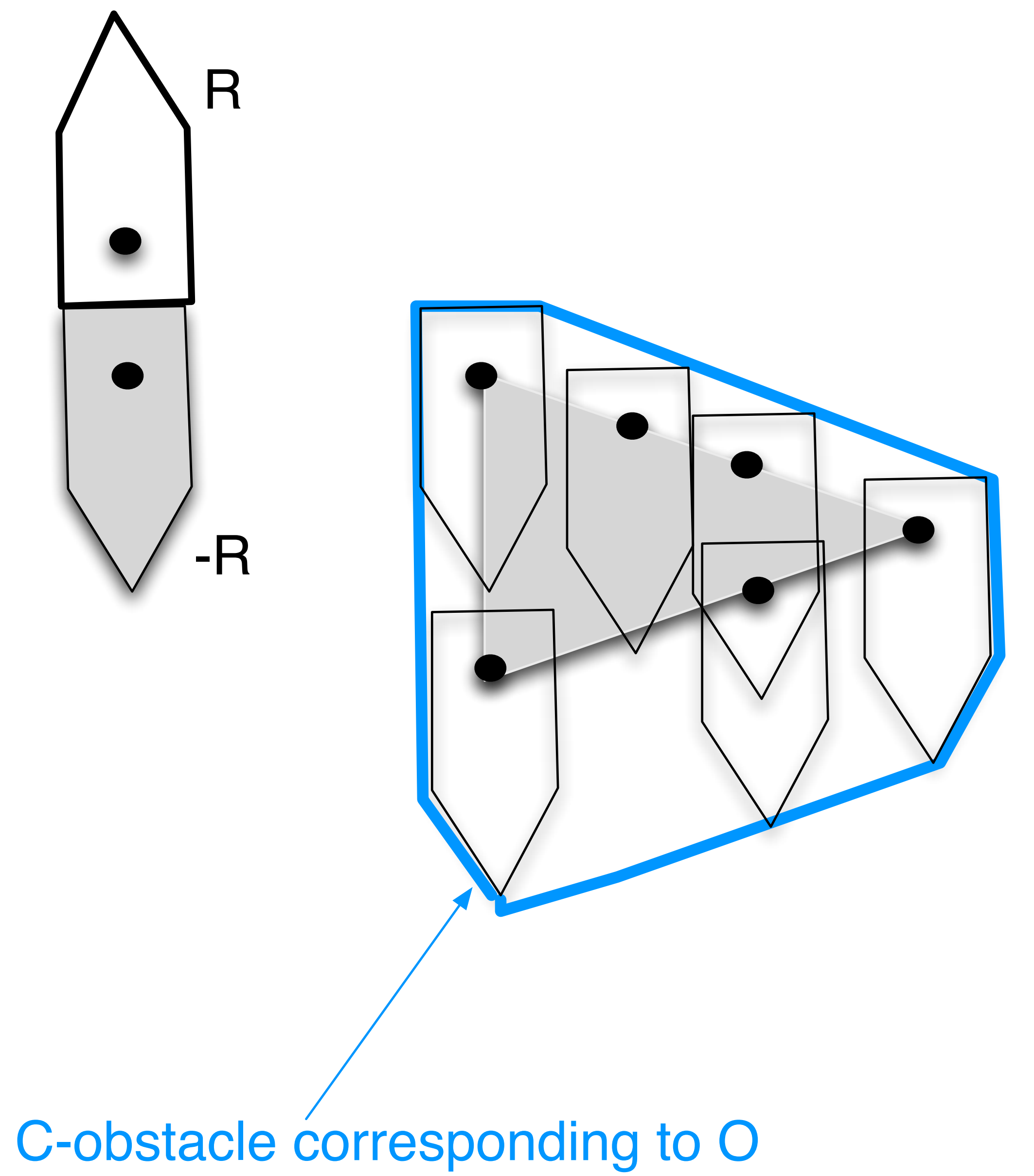
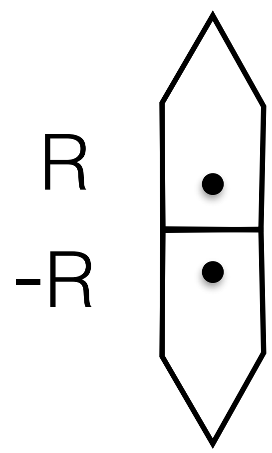
How do we compute Minkowski sums?

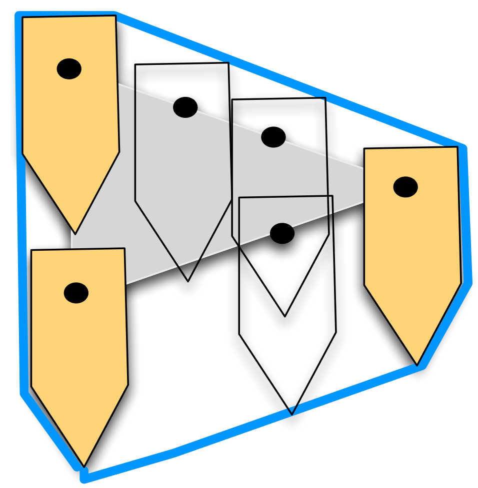


How do we compute Minkowski sums?



Computing Minkowski sums





CASE 1: Convex robot with convex polygon

Observations:

- Each edge in R, O will cause an edge in R+O
- R+O has m+n edges (unless there are parallel edges)
- To compute: Place -R at all vertices of O and compute convex hull
- Possible to compute in O(m+n) time by walking along the boundaries of R and O

Computing Minkowski sums

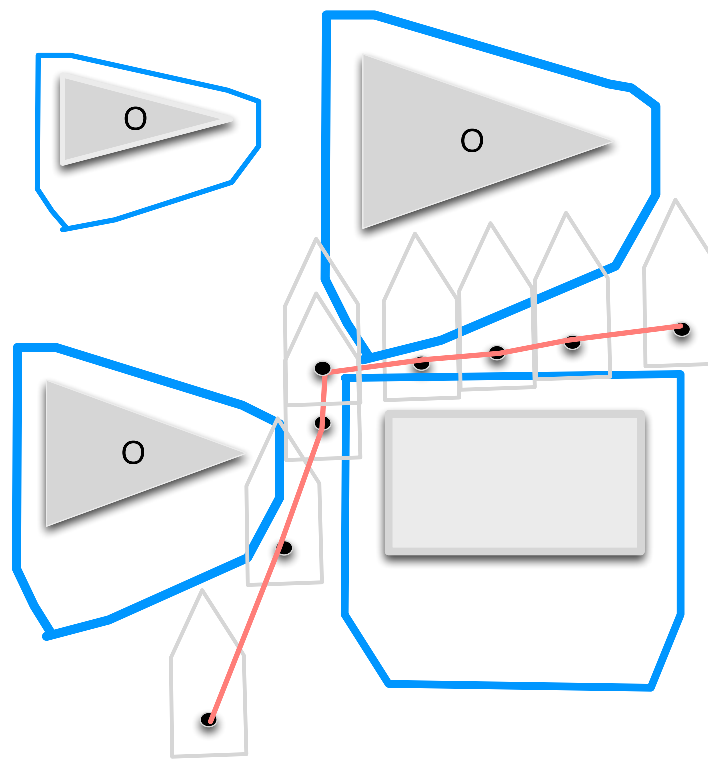
2D

- convex + convex polygons
    - The Minkowski sum of two convex polygons with n, and m edges respectively, is a convex polygon with n+m edges and can be computed in O(n+m) time.
- convex + non-convex polygons
    - Triangulate and compute Minkowski sums for each pair [convex polygon, triangle], and take their union
    - Size of Minkowski sum: O(m+3) for each triangle => O(mn)
- non-convex + non-convex polygons:
    - size of Minkowski sum: $O(n^2m^2)$

3D

- it gets worse . . .

Polygonal robot translating in 2D



Algorithm

- For each obstacle O, compute the corresponding C-obstacle
- Compute the union of C-obstacles
- Compute its complement. That's the free C-space

//now the problem is reduced to point

//robot moving in free C-space

- Compute a trapezoidal map of free C-space
- Compute a roadmap

For a **convex** robot of **O(1) size**

- Free C-space can be computed in $O(n \lg^2 n)$ time.

==> With $O(n \lg^2 n)$ time pre-processing, a collision-free path can be found for any start and end in $O(n)$ time.

Complete, non optimal.

Polygonal robot in 2D with rotations

- Physical space is 2D
- A placement is specifies by 3 parameters: R(x,y, theta) ==> C-space is 3D.

Polygonal robot in 2D with rotations

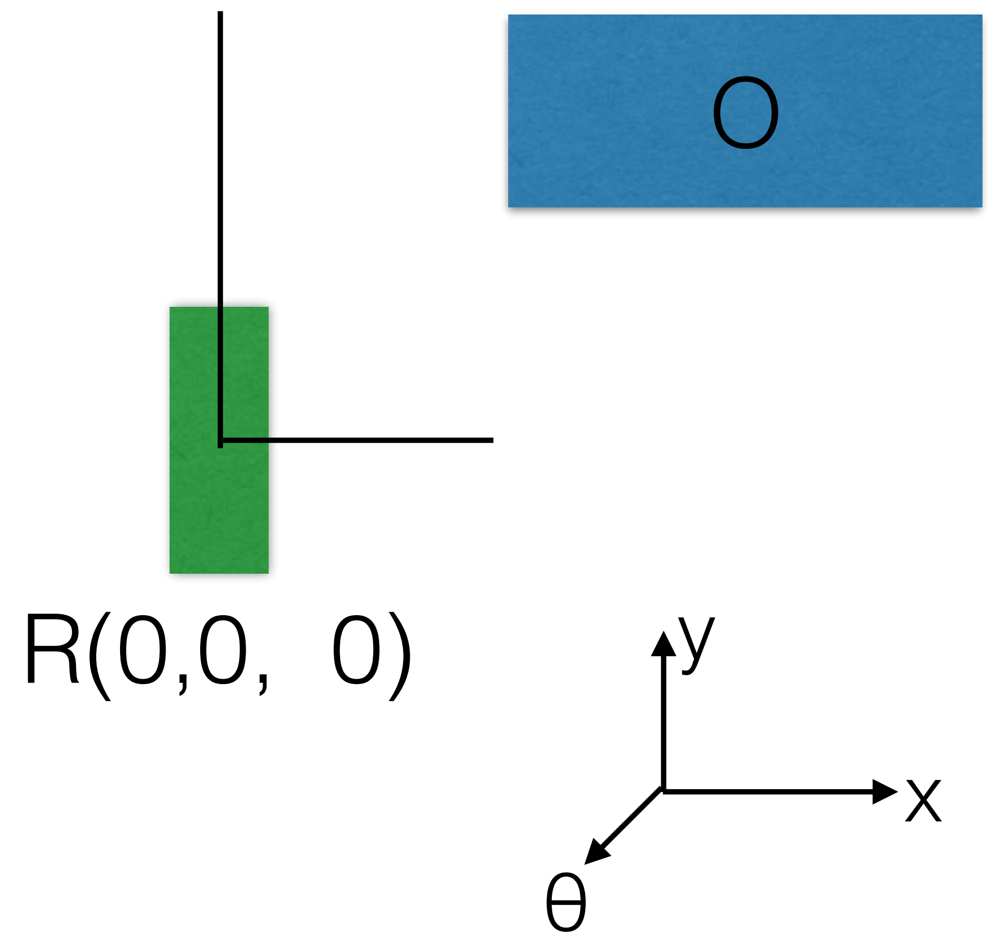
- We'd like to extend the same approach:

Reduce to point robot moving among C-obstacles in C-space.

- Compute C-obstacles
- Compute free space as complement of union of C-obstacles
- Decompose free space into simple cells
- Construct a roadmap
- BFS on roadmap

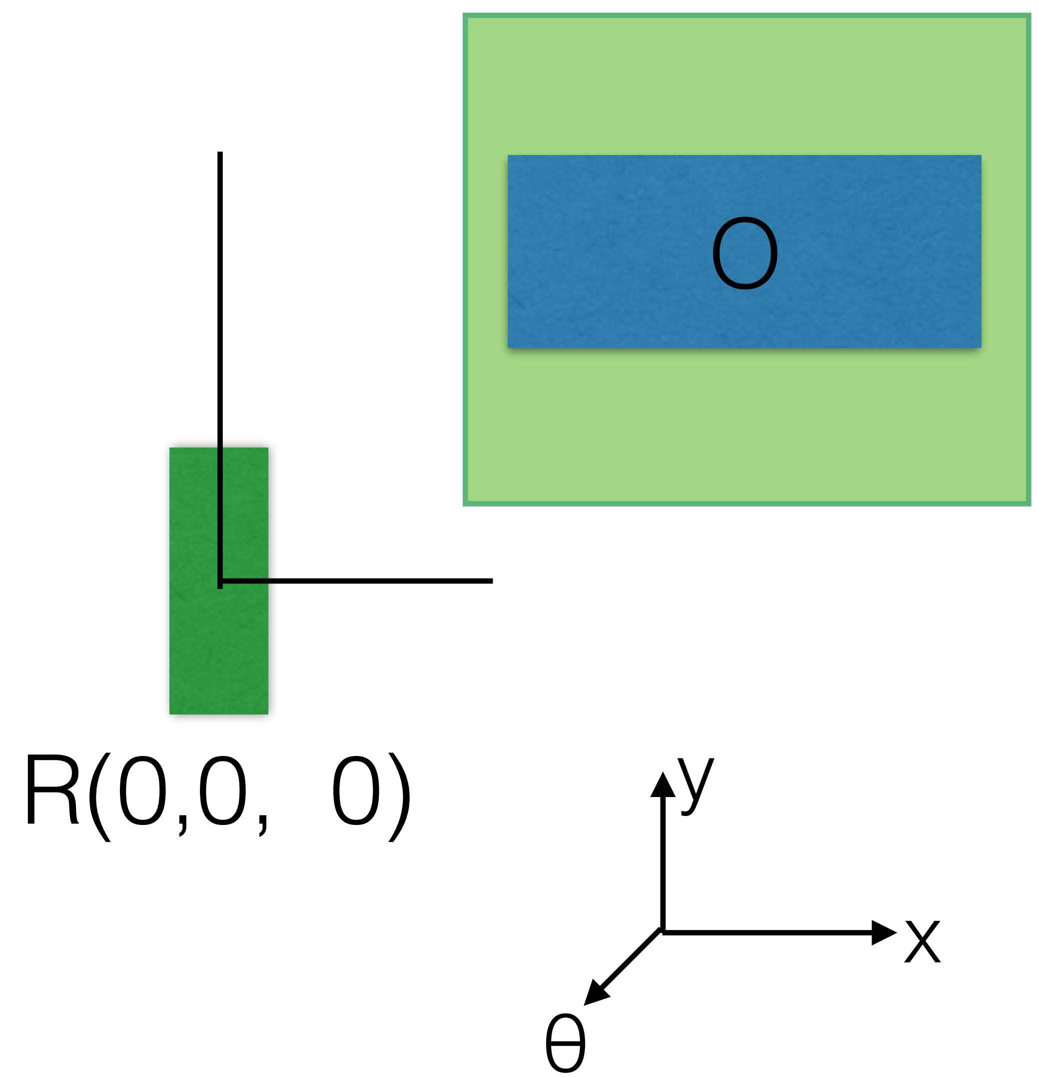
Polygonal robot in 2D with rotations

- What does a C-obstacle look like when rotations are allowed?



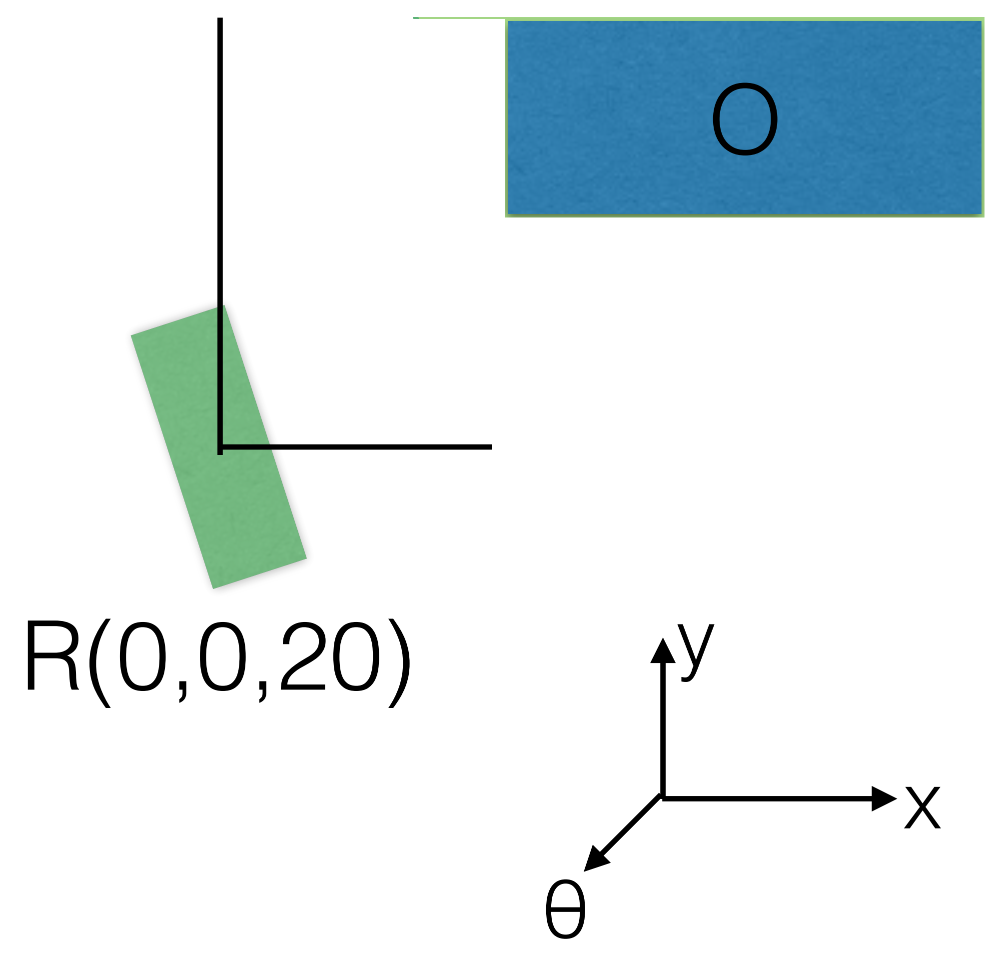
Polygonal robot in 2D with rotations

- What does a C-obstacle look like when rotations are allowed?



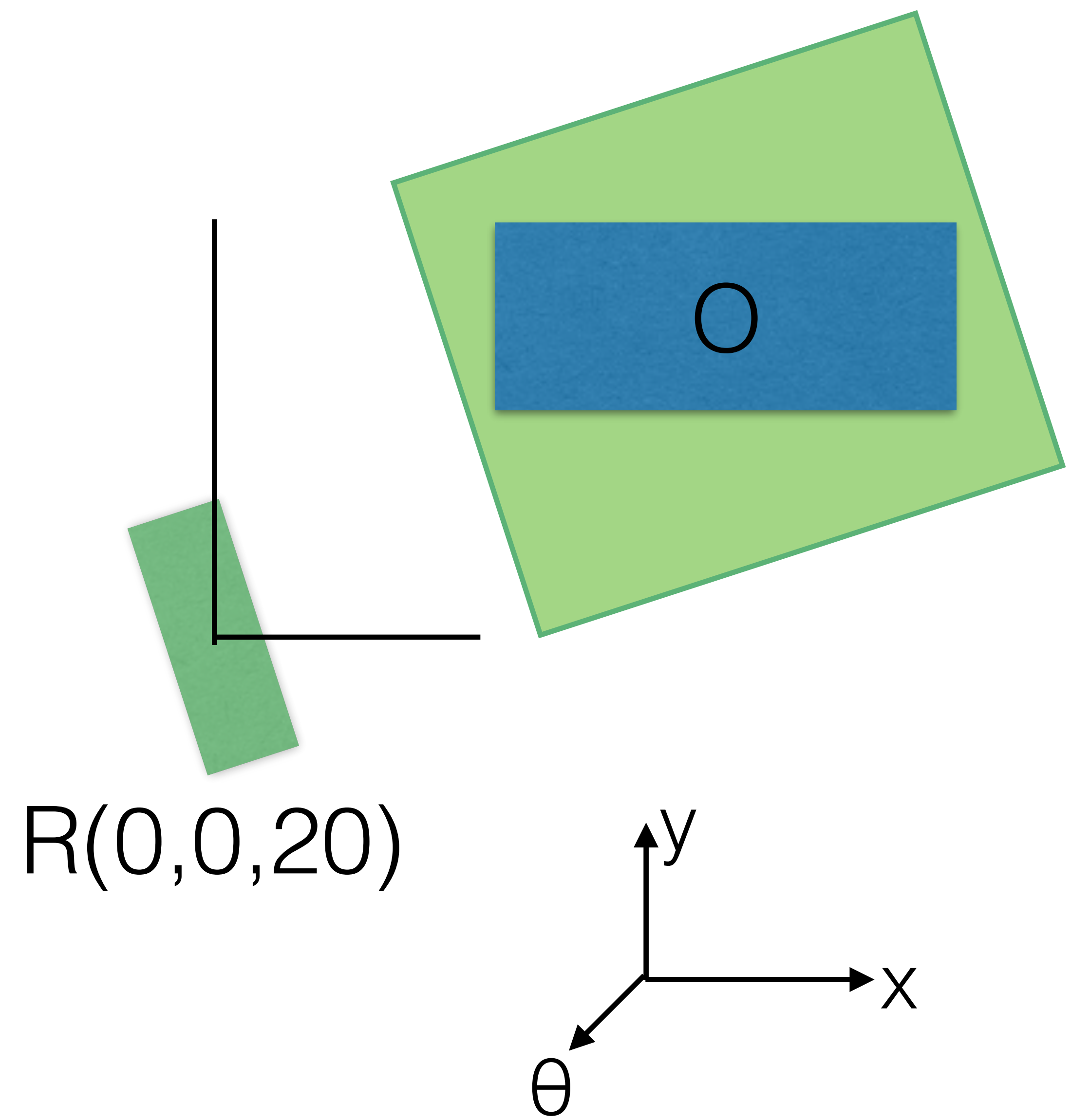
Polygonal robot in 2D with rotations

- What does a C-obstacle look like when rotations are allowed?



Polygonal robot in 2D with rotations

- What does a C-obstacle look like when rotations are allowed?



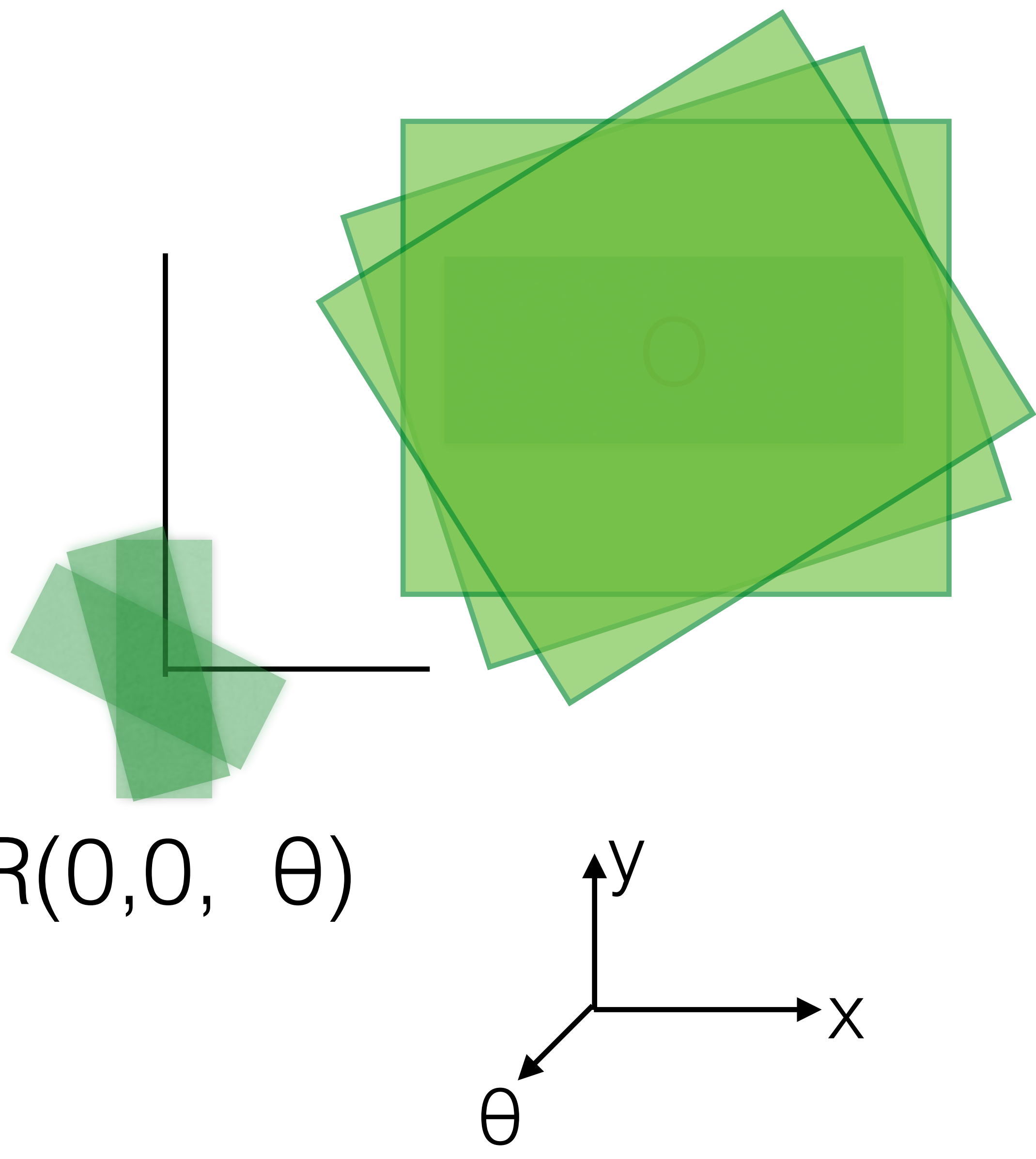
Polygonal robot in 2D with rotations

A C-obstacle is a 3D shape.

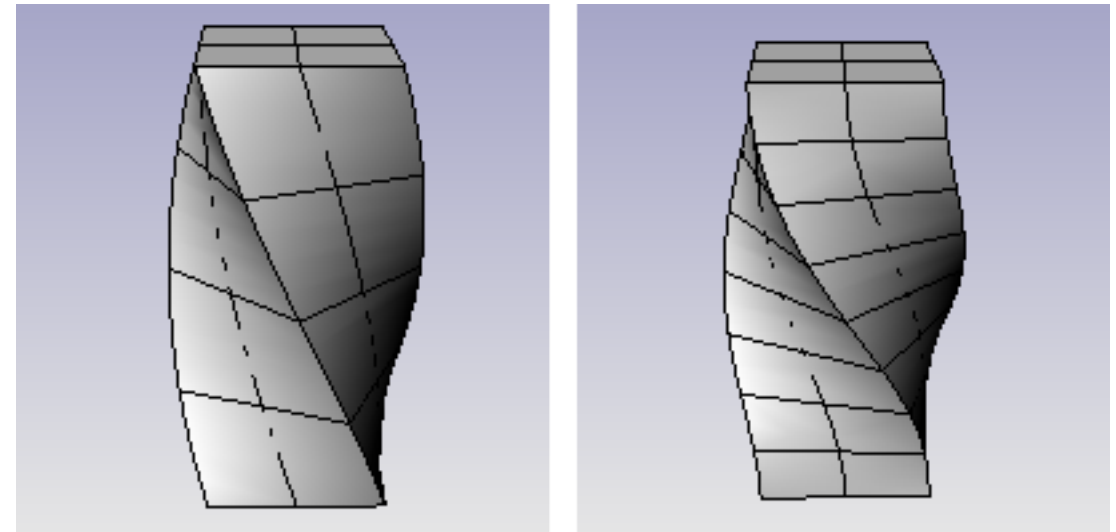
Imagine moving a horizontal plane vertically through C-space.

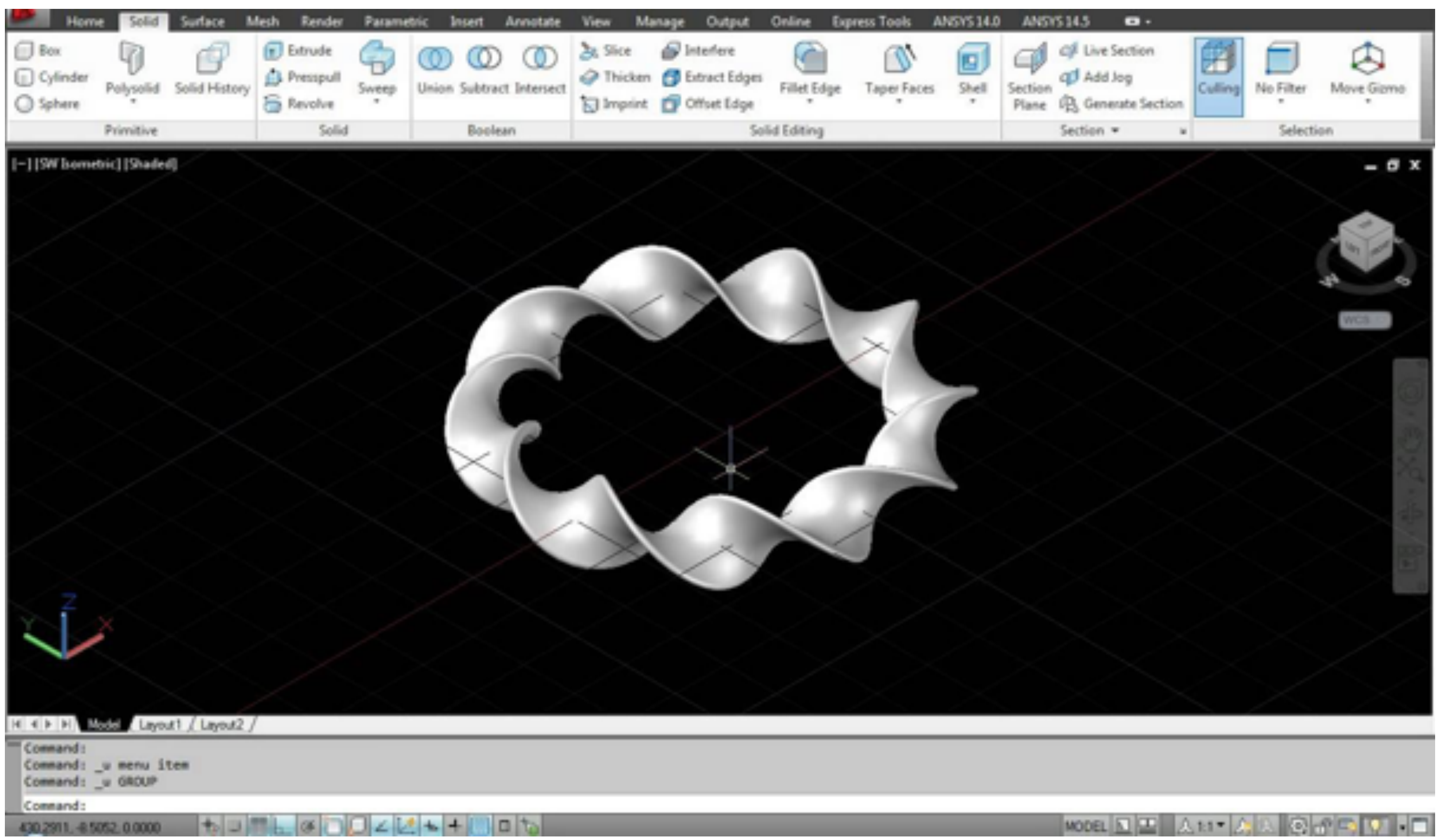
Each cross-section of the C-obstacle is a Minkowski sum O ⊕ -R (0,0,θ)

=> twisted pillar



the closest i could find ..





Polygonal robot in 2D with rotations

What's known:

- C-space is 3D
- Boundary of free space is curved, not polygonal.
- Combinatorial complexity of free space is $O(n^2)$ for convex, $O(n^3)$ for non-convex robot

Polygonal robot in 2D with rotations

What's known:

- C-space is 3D
- Boundary of free space is curved, not polygonal.
- Combinatorial complexity of free space is $O(n^2)$ for convex, $O(n^3)$ for non-convex robot

- Extend same approach:
    1. Compute C-obstacles and C-free
    2. Compute a decomposition of free space into simple cells
    3. Construct a roadmap
    4. BFS on roadmap

space is 3D

Difficult to construct a good cell decomposition for curved 3D space

Polygonal robot in 2D with rotations

- Difficult to construct a good cell decomposition for curved 3D space
- A possible approach:
  - If angle is fixed: you got translational motion planning
  - Discretize rotation angle and compute a finite number of slices, one for each angle
  - Construct a trapezoidal decomposition for each slice and its roadmap
  - Link them into a 3D roadmap
    - Add “vertical” edges between slices to allow robot to move up/down between slices; these correspond to rotational moves.

- Example: Consider two angles a and b. If placement (x,y) is in free space in slice a, and (x,y) is in free space in slice b, then the 3D roadmap should contain a vertical edge between slice a and b at that position

Is this complete ?

Combinatorial path planning: Summary

- **Idea: Compute free C-space combinatorially (= exact)**
- **Approach**
  - Reduce (robot, obstacles) => (point robot, C-obstacles)
  - Compute roadmap of free C-space
    - any path: trapezoidal decomposition or triangulation
    - shortest path: visibility graph

- **Comments**
  - Complete
  - Works beautifully in 2D and for some cases in 3D
    - Worst-case bound for combinatorial complexity of C-objects in 3D is high
  - Unfeasible/intractable for high #DOF
    - A complete planner in 3D runs in $O(2^{n\wedge\#DOF})$