Computational Geometry
[csci 3250]

Line segment intersection

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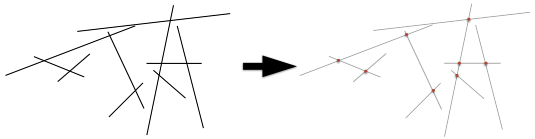
Bowdoin College

1

Line segment intersection

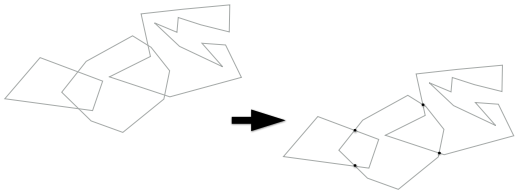
Problem: Given a set of line segments in 2D, find all their pairwise intersections.



2

Line segment intersection

Problem: Given a set of line segments in 2D, find all their pairwise intersections.



3

Line segment intersection

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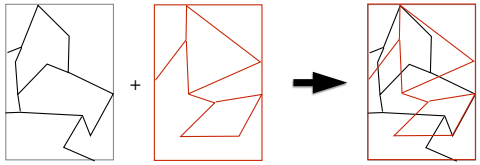
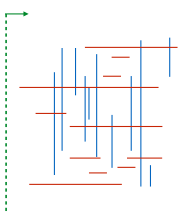


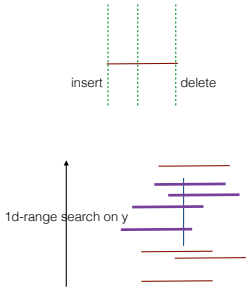
4

Orthogonal line segment intersection



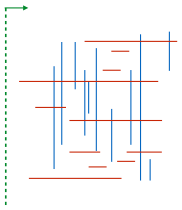
line sweep technique

solve the problem behind the line



5

Orthogonal line segment intersection



line sweep technique

solve the problem behind the line

- n: size of the input (number of segments)
- k: size of output (number of intersections)

Result: The intersections of a set of n orthogonal segments in the plane can be found in O(n lg n + k) time.

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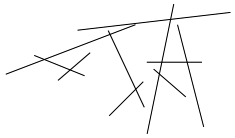
General line segment intersection

- n: size of the input (number of segments)
- k: size of output (number of intersections)



- Extend sweep line idea
- We'll get an overall bound of O(n lg n + k lg n) which improves on the naive $O(n^2)$ when k is small
- The algorithm was developed by Jon Bentley and Thomas Ottmann in 1979
- Simple (once you see it) and practical

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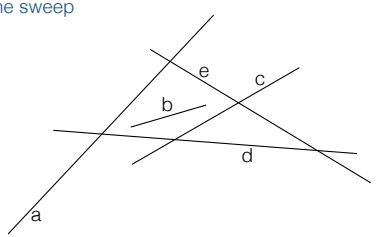
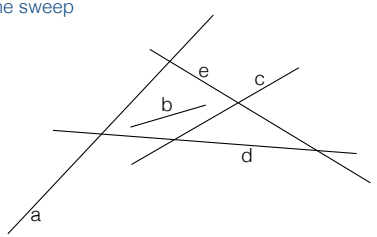
The sweep



8

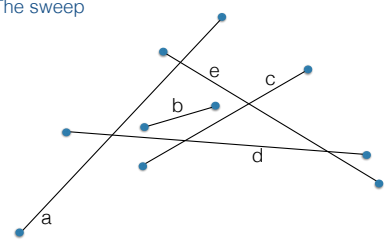
The sweep



- Let X be the set of all x-coords of segments

9

The sweep



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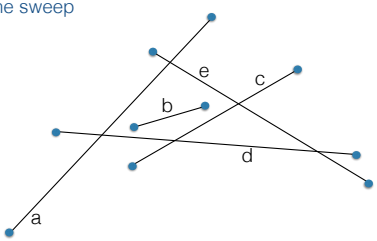
10

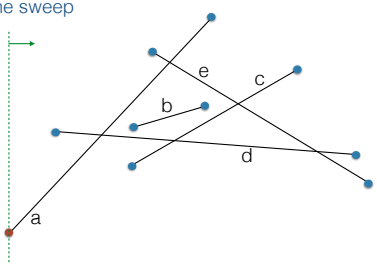
The sweep



- Let X be the set of all x-coords of segments
- Traverse the events in X in order

11

The sweep



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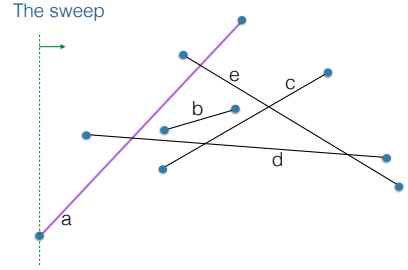
12

The sweep



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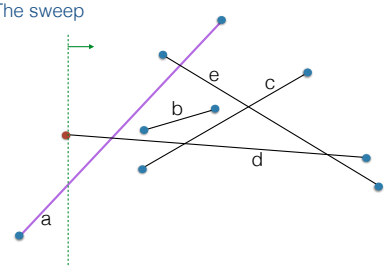
13

The sweep



- Let X be the set of all x-coords of segments
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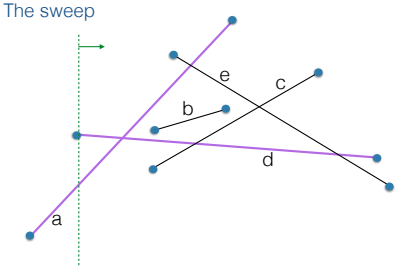
14

The sweep



- Let X be the set of all x-coords of segments
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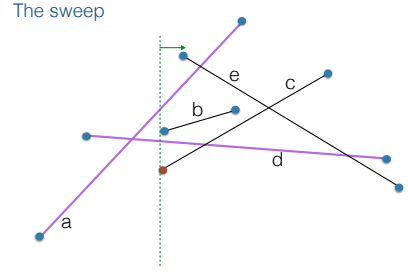
15

The sweep



- Let X be the set of all x-coords of segments
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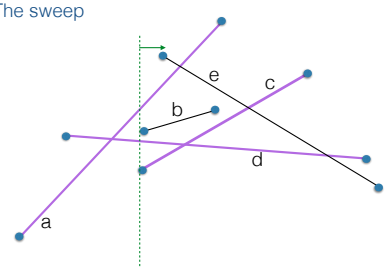
16

The sweep



- Let X be the set of all x-coords of segments
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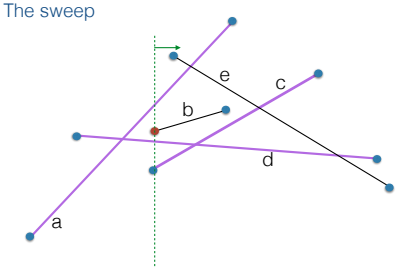
17

The sweep



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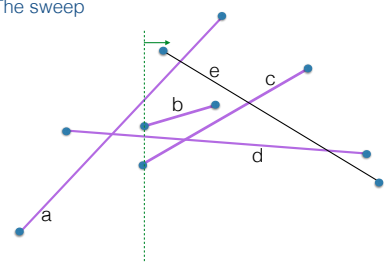
18

The sweep



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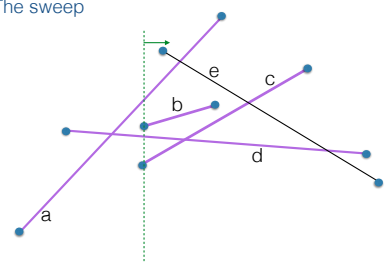
19

The sweep



- 4 segments are active
- How do we order these segments?
- How do we detect intersections?

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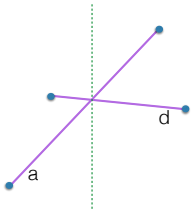
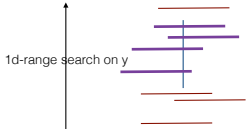
How do we order these segments?



a below d

a above d

general segments



orthogonal segments

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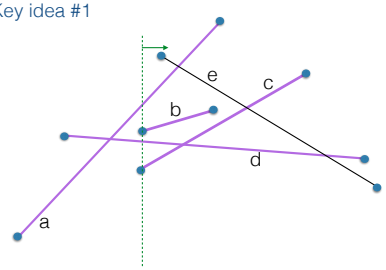
Key idea #1



- How do we order the active segments?
  - Use above-below order
  - Order will flip at intersection point

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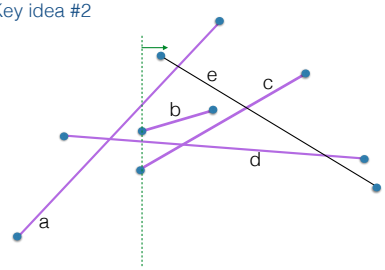
Key idea #2



- How do we detect intersections?
  - Segments that intersect are consecutive in above-below order just before they intersect

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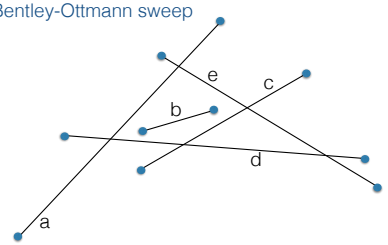
Bentley-Ottmann sweep



- Let X be the set of all x-coords of segments
- Initialize AS = {}
- Traverse events in order

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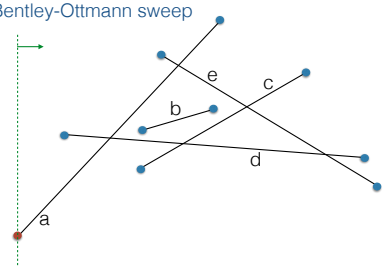
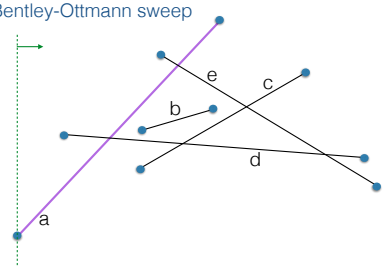
Bentley-Ottmann sweep



25

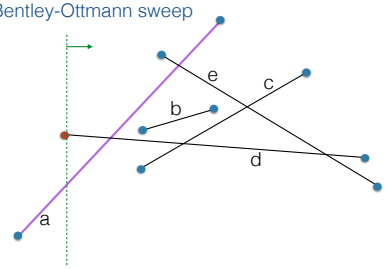
Bentley-Ottmann sweep



- this event is start of segment a:
  - insert a in AS: a

26

Bentley-Ottmann sweep



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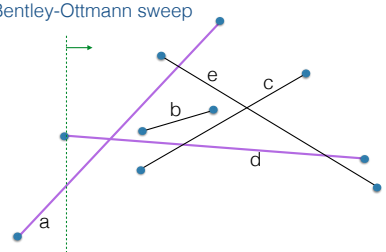
Bentley-Ottmann sweep



- this event is start of segment d
  - insert d in AS: a < d
  - check if (d,a) intersect to the right of the line; they do; report point and insert it in the list of future events

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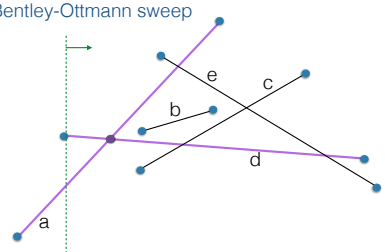
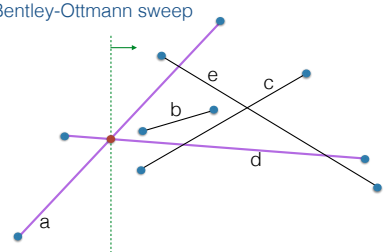
Bentley-Ottmann sweep



- this event is start of segment d
  - insert d in AS, d is above a (a < d)
  - check if (d,a) intersect to the right of the line; they do; report point and insert it in the list of future events

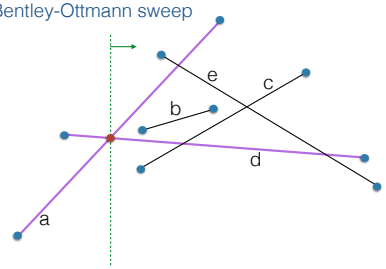
29

Bentley-Ottmann sweep



30

Bentley-Ottmann sweep



- this event is an intersection point of (a,d):
  - flip a and d is AS: a is now above d (d < a)

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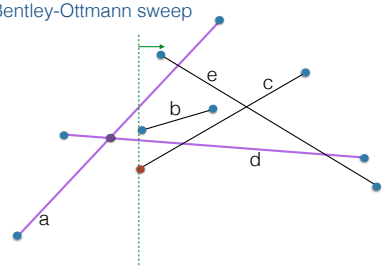
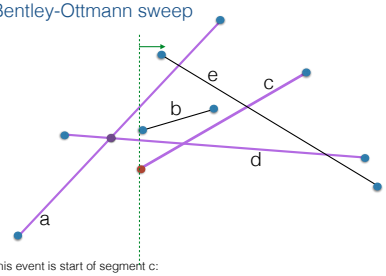
Bentley-Ottmann sweep



32

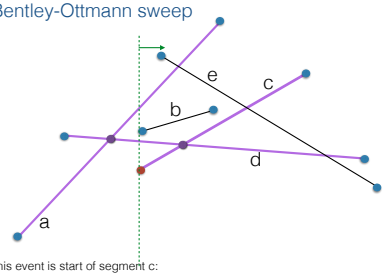
Bentley-Ottmann sweep



- this event is start of segment c:
  - insert c in AS; c is below d (c < d < a)
  - check c with its above and below neighbors for intersection to the right of the sweep line; this detects the intersection point of c and d; report it and insert it as future event

33

Bentley-Ottmann sweep



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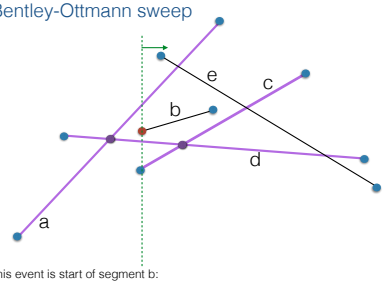
34

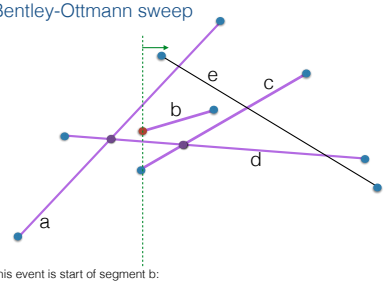
Bentley-Ottmann sweep



- this event is start of segment b:

35

Bentley-Ottmann sweep



- this event is start of segment b:
  - insert b in AS; c < d < b < a
  - check b with its above and below neighbors for intersection to the right of the sweep line; (d,b) don't intersect; (b, a) don't intersect

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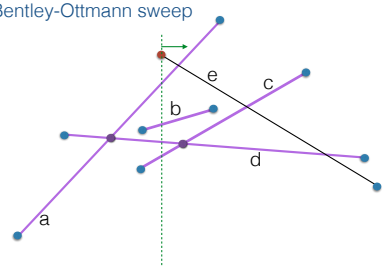
Bentley-Ottmann sweep



- this event is start of segment e:

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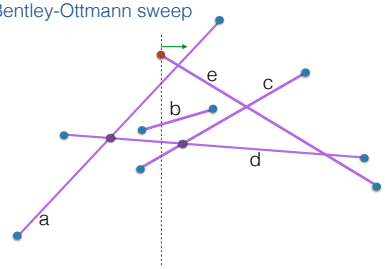
Bentley-Ottmann sweep



- this event is start of segment e:
  - insert e in AS: c < d < b < a < e
  - check e with its above and below neighbors for intersection to the right of the sweep line; this detects intersection point of (a,e); report it and insert it as future event

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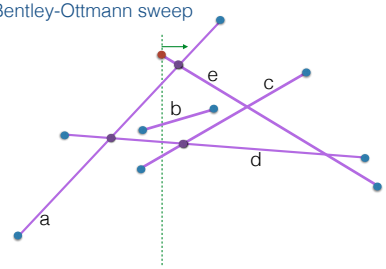
Bentley-Ottmann sweep



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  - insert e in AS: c < d < b < a < e
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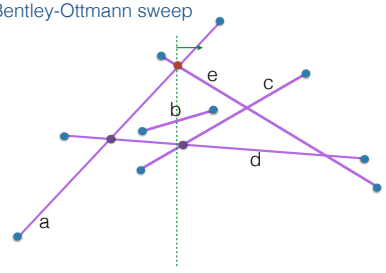
39

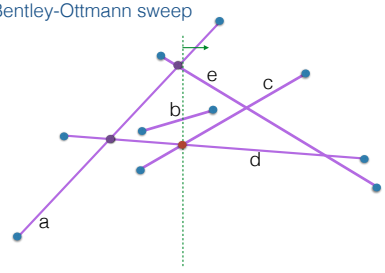
Bentley-Ottmann sweep



- this event is intersection of (a,e):
  - flip a and e: c < d < b < e < a
  - check new neighbors (e,b) for intersection to the right of the sweep line; (e,b) don't intersect

40

Bentley-Ottmann sweep



- this event is intersection of (c,d):
  - flip c and d: d < c < b < e < a
  - check new neighbors (c,b) for intersection to the right of the sweep line; (c,b) don't intersect

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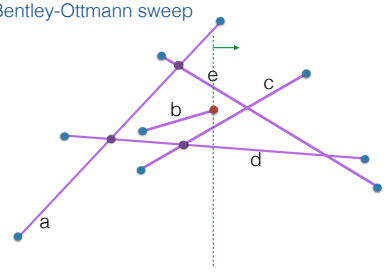
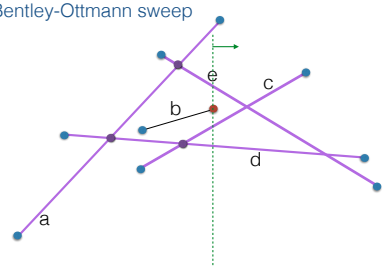
Bentley-Ottmann sweep



- this event is end of segment b:
  - delete b from AS: d < c < e < a
  - check new neighbors (c,e) for intersection to the right of the sweep line; this detects the intersection point of (c,e); report it and insert it as future event

42

Bentley-Ottmann sweep



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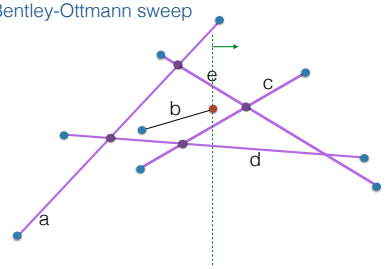
43

Bentley-Ottmann sweep



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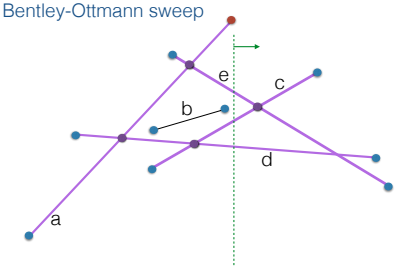
  - delete b from AS: d < c < e < a
  - check new neighbors (c,e) for intersection to the right of the sweep line; this detects the intersection point of (c,e); report it and insert it as future event

44

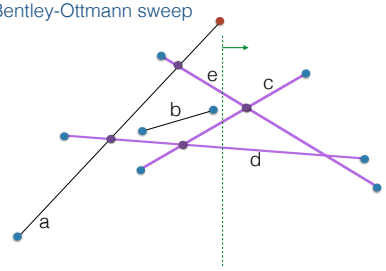
Bentley-Ottmann sweep



- this event is end of segment a:
  - delete a from AS: d < c < e
  - no new neighbors

45

Bentley-Ottmann sweep



- this event is end of segment a:
  - delete a from AS: d < c < e
  - no new neighbors

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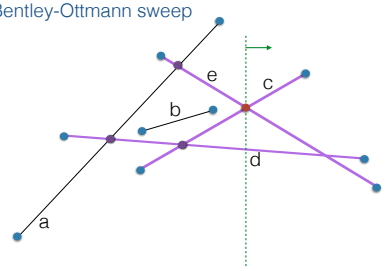
Bentley-Ottmann sweep



- this event is the intersection of (c,e):
  - flip c,e in AS: d < e < c
  - check new neighbors (d,e) for intersection to the right of the sweep line; this detects the intersection of (d,e); report it and insert it as future event

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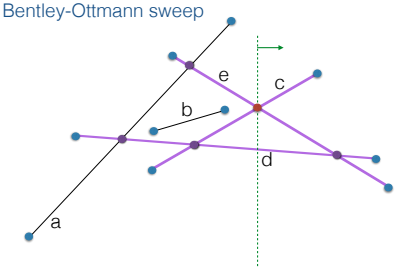
Bentley-Ottmann sweep



- this event is the intersection of (c,e):
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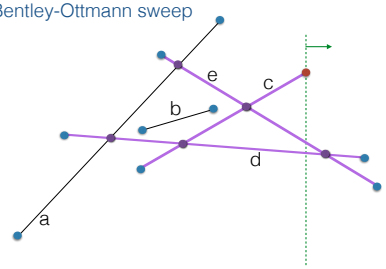
Bentley-Ottmann sweep



- this event is end of segment c:
  - delete c in AS: d < e
  - no new neighbors

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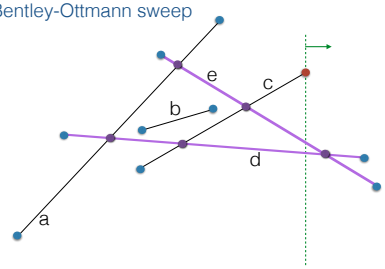
Bentley-Ottmann sweep



- this event is end of segment c:
  - delete c in AS: d < e
  - no new neighbors

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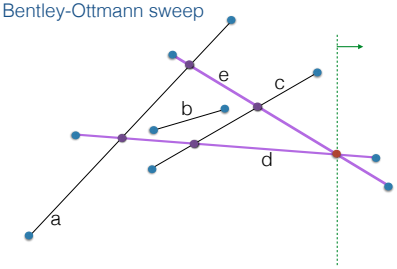
Bentley-Ottmann sweep



- this event is the intersection of (d,e):
  - flip d,e in AS: e < d
  - no new neighbors

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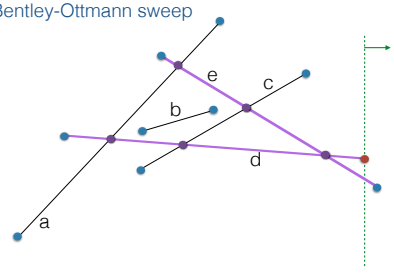
Bentley-Ottmann sweep



- this event is the end of d:
  - delete d in AS: e
  - no new neighbors

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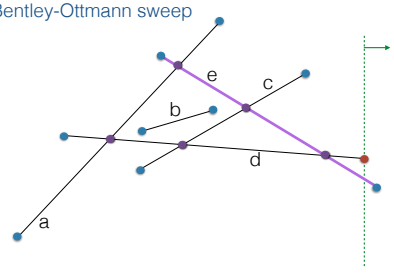
Bentley-Ottmann sweep



- this event is the end of d:
  - delete d in AS: e
  - no new neighbors

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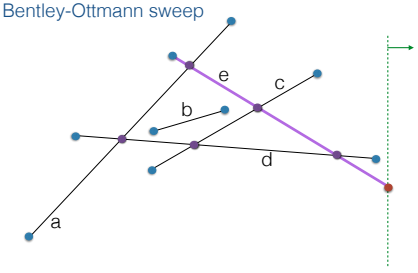
Bentley-Ottmann sweep



- this event is the end of e:
  - delete e in AS:
  - no new neighbors

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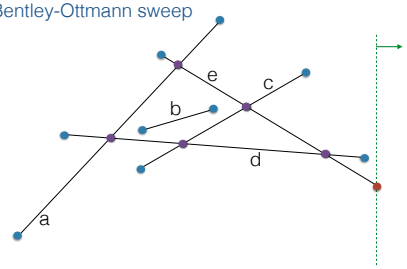
Bentley-Ottmann sweep



- this event is the end of e:
  - delete e in AS:
  - no new neighbors

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Bentley-Ottmann sweep

- Simplifying assumptions
  - no vertical segments
  - no two segments intersect at their endpoints
  - no three (or more) segments have a common intersection
  - all endpoints (of segments) and all intersection points have different x-coordinates
  - no segments overlap
- These assumptions are not realistic for real data..
- But, they don't provide insight into the plane sweep technique, so we omit them
- Real data challenges
  - dealing with degenerate cases
  - dealing with finite precision arithmetic and precision problems

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Bentley-Ottmann sweep

We'll maintain the following invariants during the algorithm:

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Bentley-Ottmann sweep

We'll maintain the following invariants during the algorithm:

- Active structure AS:
  - For any position of the sweep line SL, AS contains all active segments (ie segments that start before SL and end after SL)
  - AS is sorted by their y-coordinates of their intersection with SL

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Bentley-Ottmann sweep

We'll maintain the following invariants during the algorithm:

- Active structure AS:
  - For any position of the sweep line SL, AS contains all active segments (ie segments that start before SL and end after SL)
  - AS is sorted by their y-coordinates of their intersection with SL
- Event list EL:
  - For any position of SL, EL contains segment endpoints to the right of SL, and also the intersections to the right of SL of active segments that were/ are neighbors in SL
  - EL is sorted by x-coordinate

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Bentley-Ottmann sweep

We'll maintain the following invariants during the algorithm:

- Active structure AS:
  - For any position of the sweep line SL, AS contains all active segments (ie segments that start before SL and end after SL)
  - AS is sorted by their y-coordinates of their intersection with SL
- Event list EL:
  - For any position of SL, EL contains segment endpoints to the right of SL, and also the intersections to the right of SL of active segments that were/ are neighbors in SL
  - EL is sorted by x-coordinate
- For any position of the sweep line SL, all pairs of intersecting dead segments have been reported.

Algorithm Bentley-Ottmann (S)

//S is a set of n line segments in the plane
- initialize AS= {}
- sort 2n endpoints of all segments in S by x-coord and store them in EventList
- while EventList not empty
  - let e be the next event from EventList; delete it from EL
    //sweep line moves to x=e.x
  - if e is left endpoint of a segment l
    //l becomes active
    - insert l in AS in the right place
    - check if l intersects with l->prev and l->succ in AS to the right of the sweep line; if they do, insert their intersection point in the EventList
      //optional: since l.prev and l.succ are not neighbors anymore, we check if they intersect and if they do, delete that intersection point from the EventList
  - if e is the right endpoint of a segment
    - …
  - if e is the intersection of two segments
    - …
- end.

Bentley-Ottmann sweep

Questions

- AS
  - What operations do we do on AS?
  - What data structure should we use for AS?
- EL
  - Note that we know a priori the 2n events corresponding to start and end-points of segments, but EL is not static; the events corresponding to intersection points are generated on the fly
  - What operations do we do on EL?
  - What data structure should we use for EL?

Bentley-Ottmann sweep

Running time
- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)
- EventList
  - What is the size of EventList?
    - O(n+k)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

Bentley-Ottmann sweep

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Bentley-Ottmann sweep runs in O((n+k) lg n) time.