Computational Geometry
[csci 3250]

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[csci 3250]

Orthogonal line segment intersection

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Line segment intersection

- The problem (what)
- Applications (why)
- Algorithms (how)
  - A special case: Orthogonal line segments
  - General case and Bentley-Otman line sweep algorithm

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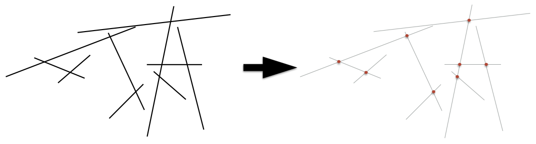
Line segment intersection

Problem: Given a set of line segments in 2D, find all their pairwise intersections.



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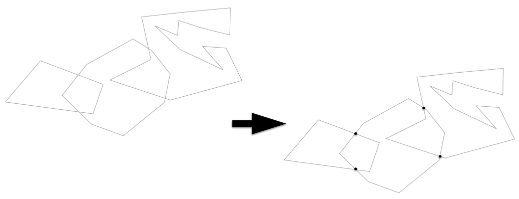

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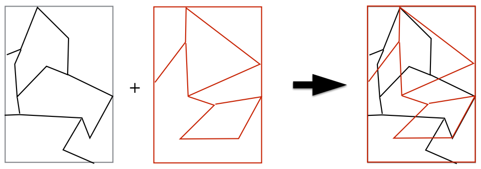

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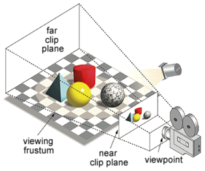
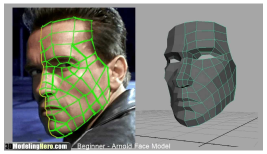
Applications

Graphics: rendering => hidden surfaces ==> intersections





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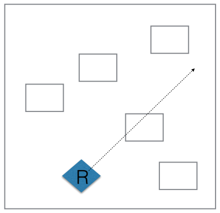
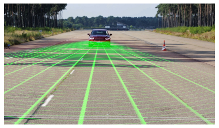
Applications

Motion planning and collision detection in autonomous systems/robotics





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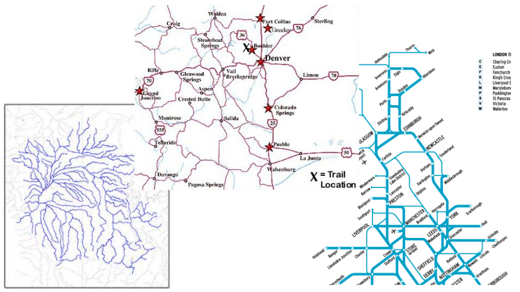
Applications

Geographical data: River networks, road networks, railways, ..



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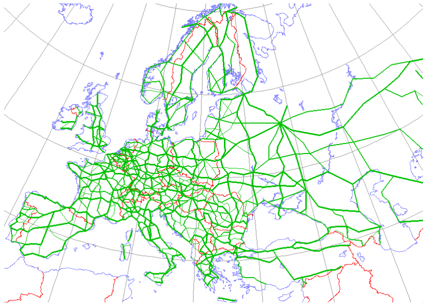

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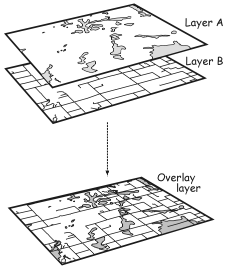
Applications

Map overlay in GIS



from: www.geo.hunter.cuny.edu/aierulli/gis2/lectures/Lecture2/fig9-30_raster_overlay.gif

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Applications

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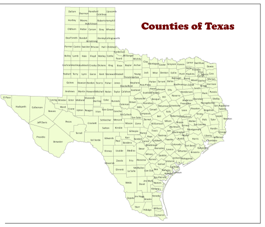

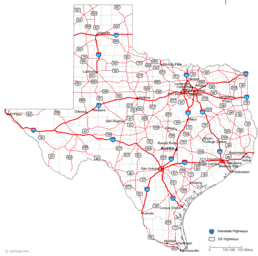

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Algorithms

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Naive

Notation

- n: size of the input (number of segments)
- k: size of output (number of intersections)

Problem: Given a set of n line segments in 2D, find all their pairwise intersections.

Exercises:

- Give upper and lower bounds for k, draw examples that achieve these bounds.
- Give a straightforward algorithm that computes all intersections and analyze its running time. Give scenarios when this algorithm is efficient/inefficient.
- What is your intuition of an upper bound for this problem? (how fast would you hope to be able to solve it?)

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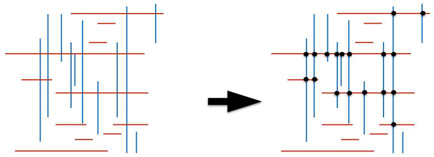
A special case: Orthogonal line segment intersection



Exercises

- Come up with a straightforward algorithm and analyze its time
- Improved algorithm?

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Balanced Binary Search Trees
- review -

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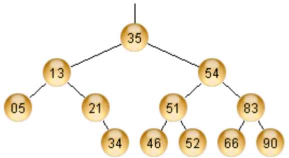
Binary Search Trees (BST)

- Operations
  - insert
  - delete
  - search
  - successor, predecessor
  - traversals (in order, ..)
  - min, max



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Balanced Binary Search Trees (BBST)

- Binary search trees + invariants that constrain the tree to be balanced (and thus have logarithmic height)
- These invariants have to be maintained when inserting and deleting (so we can think of the tree as self-balancing)

- BBST variants
  - red-black trees
  - AVL trees
  - B-trees
  - (a,b) trees
  - …

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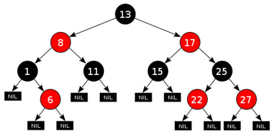
Example: Red-Black trees

- Binary search tree, and
  - Each node is Red or Black
  - The children of a Red node must be Black
  - The number of Black nodes on any path from the root to any node that does not have two children must be the same



Note:
- easier to conceptualize the tree as containing explicit NULL leaves, all Black
- the number of Black nodes on any root-to-leaf path must be the same

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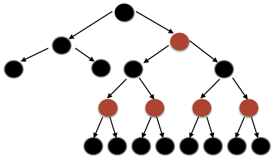
Example: Red-Black trees

- Theorem:
  - A Red-Black tree of n nodes has height Theta(lg n).



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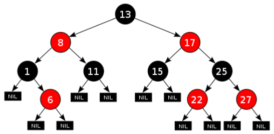
Example: Red-Black trees

- Theorem:
  - After an insertion or a deletion, the RB tree invariants can be maintained in additional O(lg n) time. This is done by performing rotations and recoloring nodes on the path from the inserted/deleted node to the root.



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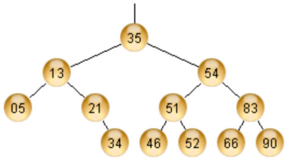
Binary Search Trees

- Operations
  - insert
  - delete
  - search
  - successor, predecessor
  - traversals (in order, ..)
  - min, max
  - range search (1D)



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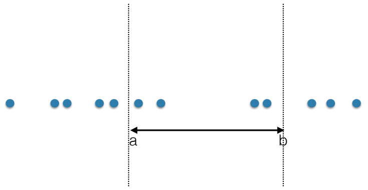
1D Range Searching

- Given a set of values P = {x_1, x_2, x3, …x_n }
- Pre-process it in order to answer

  rangeSearch(a,b): return all elements in P in interval (a,b)



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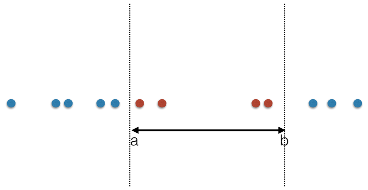

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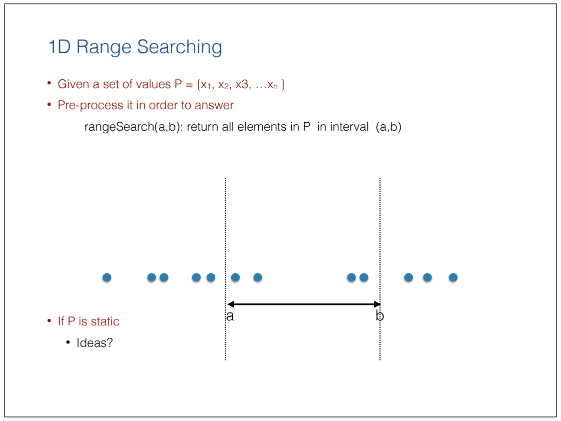

1D Range Searching

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- If P is static
  - Ideas?

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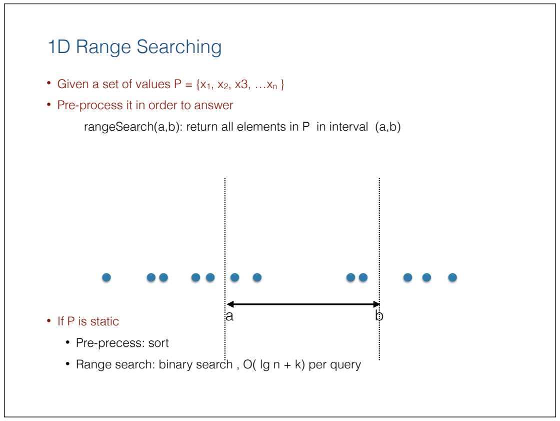

1D Range Searching

- Given a set of values $P = \{x_1, x_2, x3, \ldots x_n\}$
- Pre-process it in order to answer

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- If P is static
  - Pre-precess: sort
  - Range search: binary search , O(lg n + k) per query

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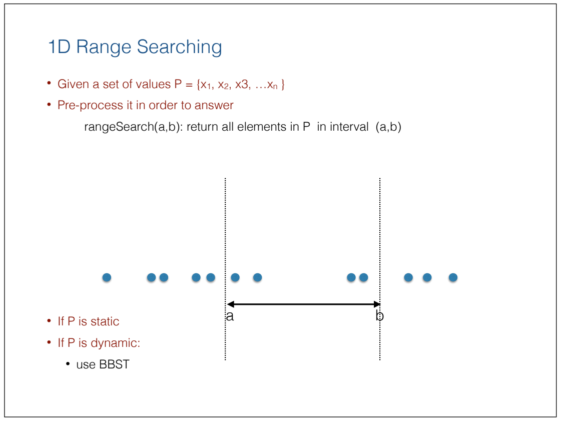

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- Pre-process it in order to answer

  rangeSearch(a,b): return all elements in P in interval (a,b)

- If P is static
- If P is dynamic:
  - use BBST

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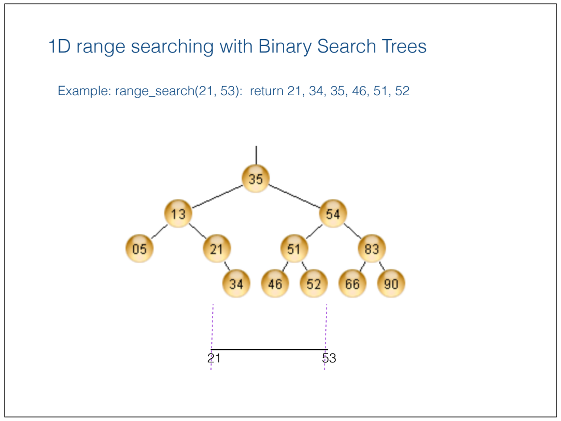

1D range searching with Binary Search Trees

Example: range_search(21, 53): return 21, 34, 35, 46, 51, 52

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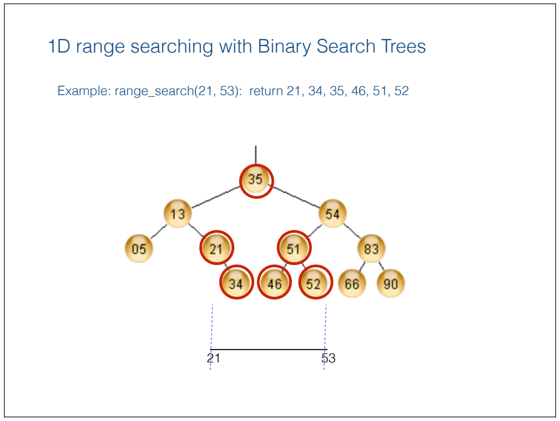

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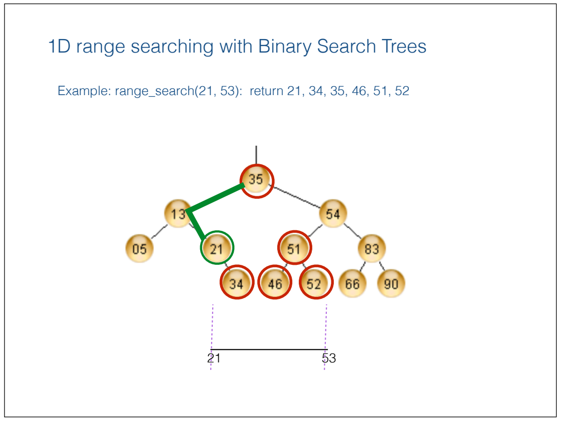

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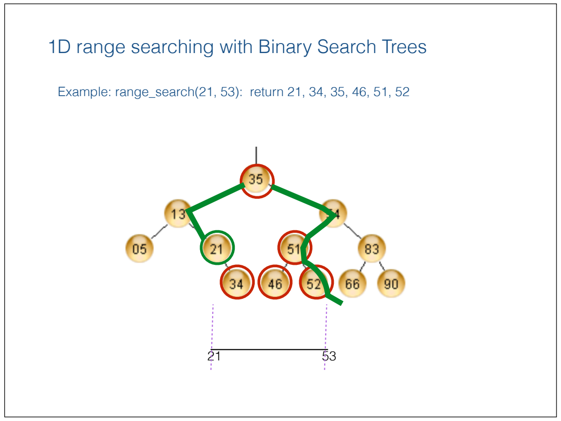

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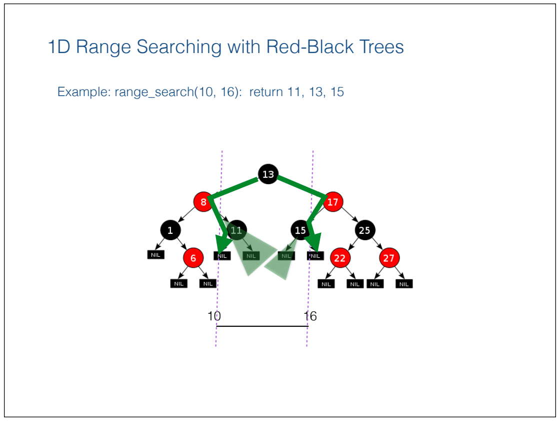

1D Range Searching with Red-Black Trees

Example: range_search(10, 16): return 11, 13, 15

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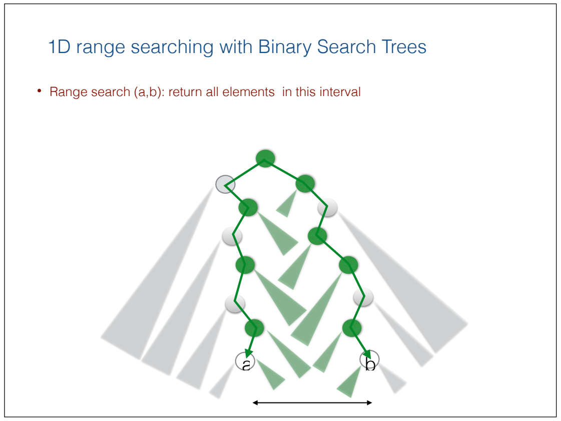

1D range searching with Binary Search Trees

- Range search (a,b): return all elements in this interval

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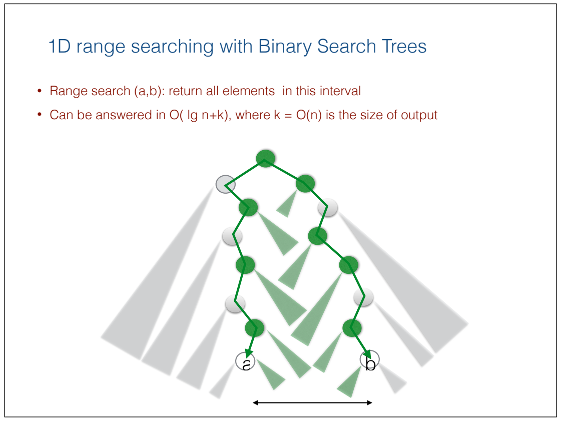

1D range searching with Binary Search Trees

- Range search (a,b): return all elements in this interval
- Can be answered in O(lg n+k), where k = O(n) is the size of output

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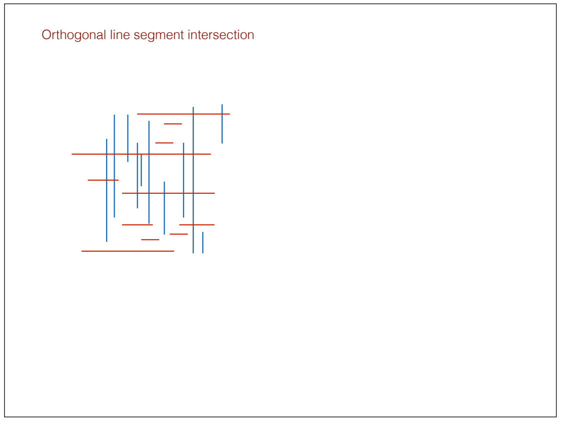

Orthogonal line segment intersection

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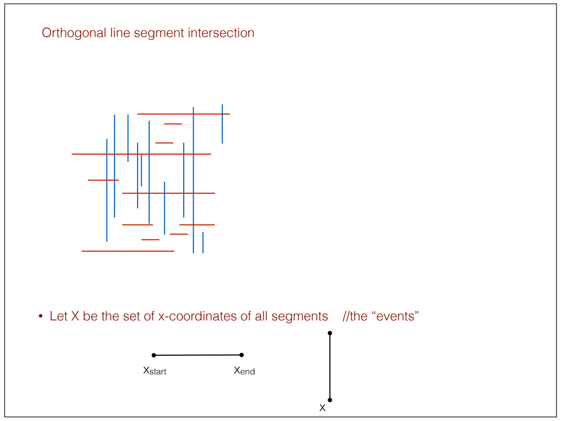

Orthogonal line segment intersection

- Let X be the set of x-coordinates of all segments //the “events”

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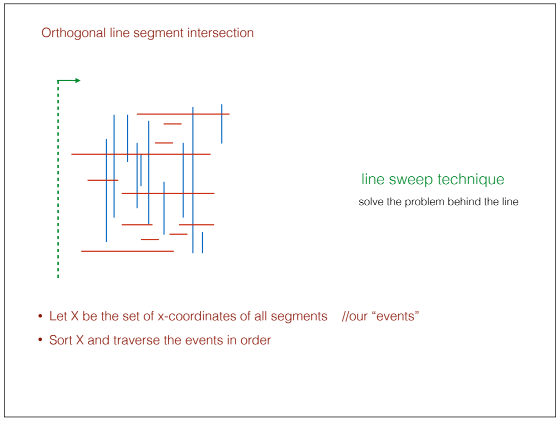

Orthogonal line segment intersection

line sweep technique

solve the problem behind the line

- Let X be the set of x-coordinates of all segments //our “events”
- Sort X and traverse the events in order

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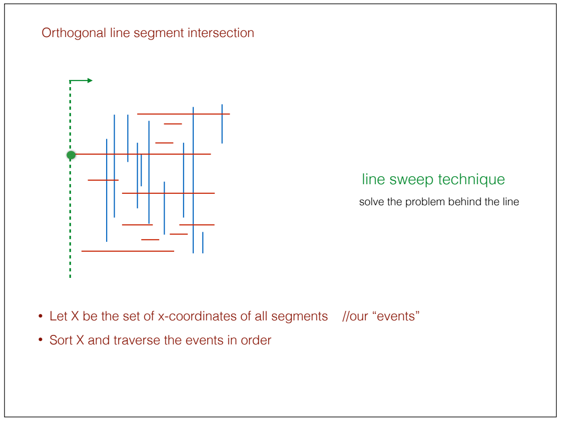

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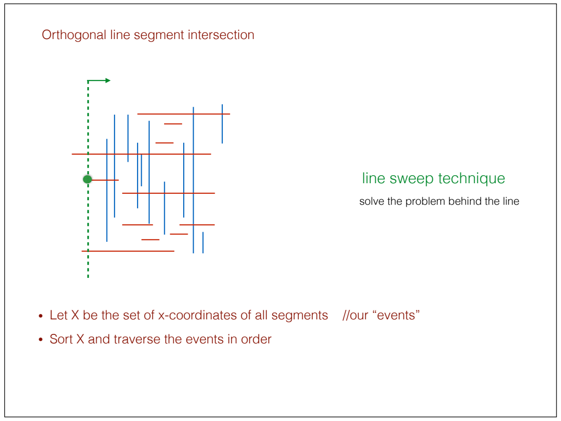

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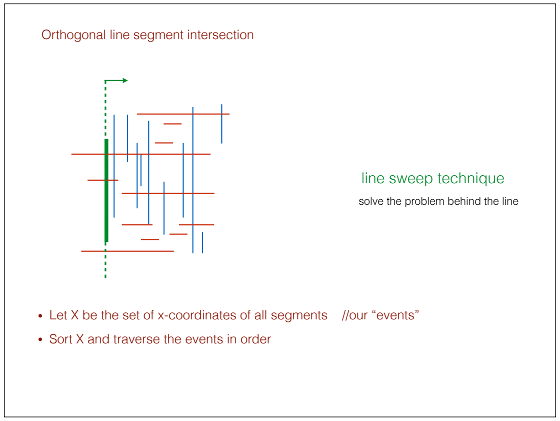

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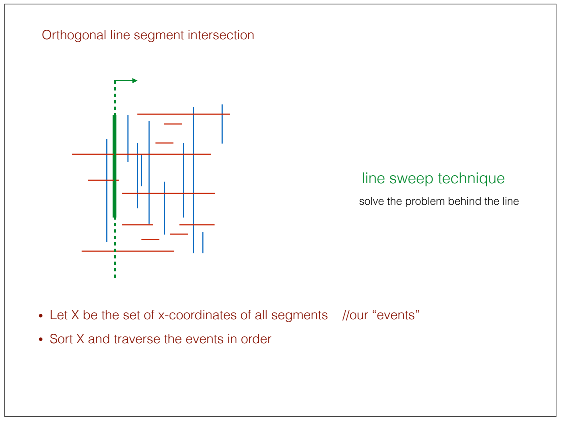

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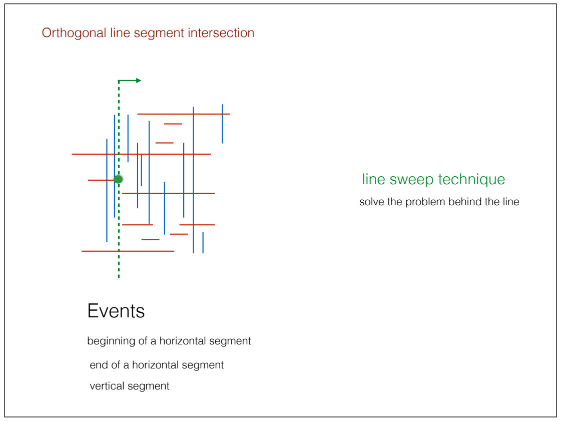

Orthogonal line segment intersection

line sweep technique

solve the problem behind the line

Events

beginning of a horizontal segment

end of a horizontal segment

vertical segment

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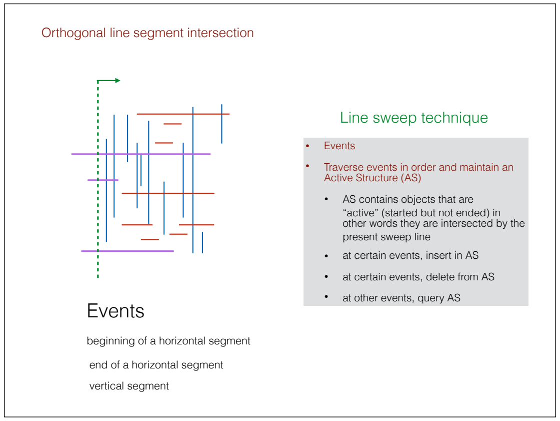

Orthogonal line segment intersection

Line sweep technique

- Events
- Traverse events in order and maintain an Active Structure (AS)
    - AS contains objects that are "active" (started but not ended) in other words they are intersected by the present sweep line
    - at certain events, insert in AS
    - at certain events, delete from AS
    - at other events, query AS

Events

beginning of a horizontal segment

end of a horizontal segment

vertical segment

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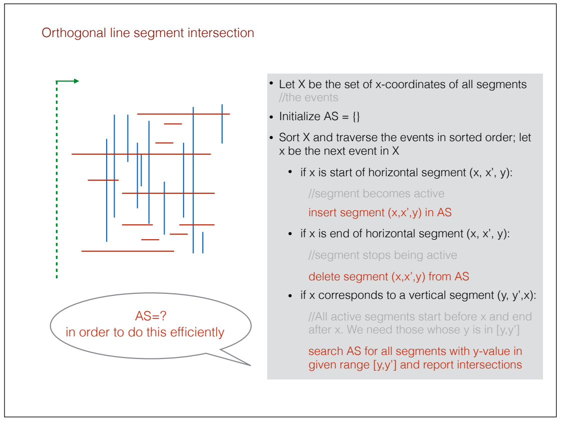

Orthogonal line segment intersection

AS=?
in order to do this efficiently

- Let X be the set of x-coordinates of all segments //the events
- Initialize AS = {}
- Sort X and traverse the events in sorted order; let x be the next event in X
    - if x is start of horizontal segment (x, x', y):

      //segment becomes active

      insert segment (x,x',y) in AS
    - if x is end of horizontal segment (x, x', y):

      //segment stops being active

      delete segment (x,x',y) from AS
    - if x corresponds to a vertical segment (y, y',x):

      //All active segments start before x and end after x. We need those whose y is in [y,y']

      search AS for all segments with y-value in given range [y,y'] and report intersections

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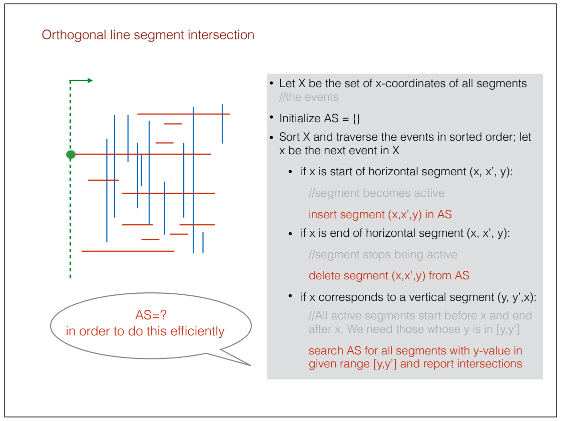

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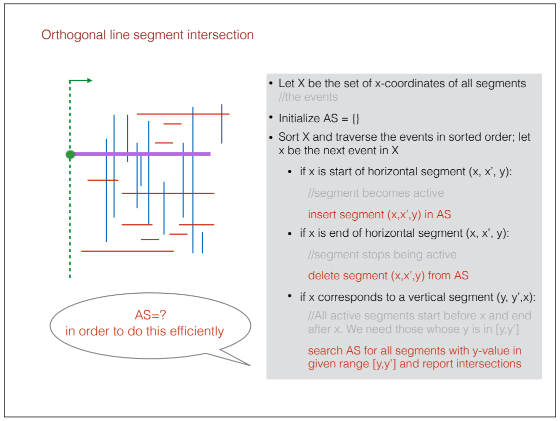

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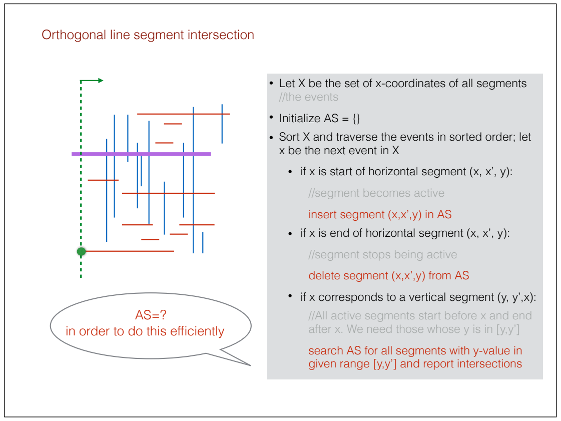

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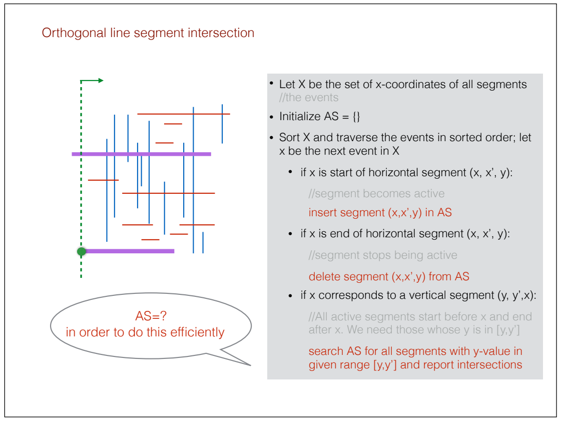

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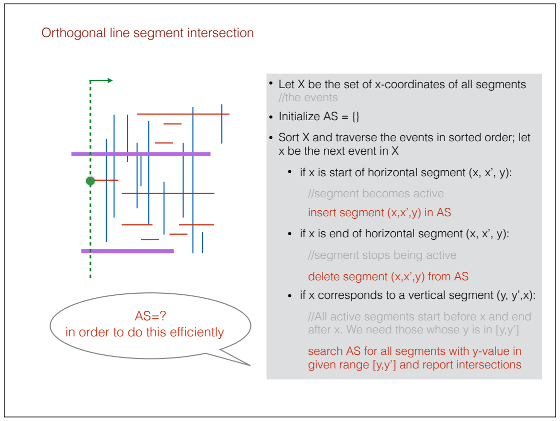

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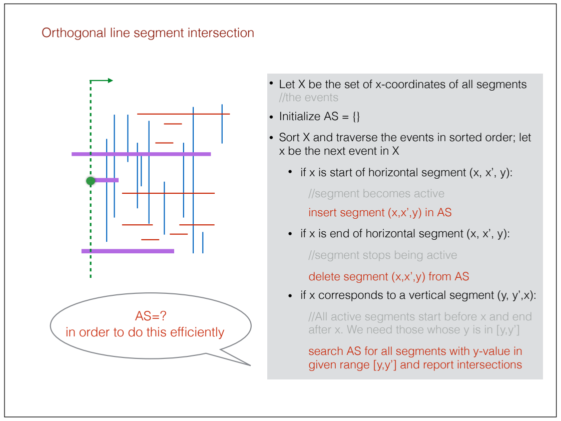
    search AS for all segments with y-value in given range [y,y'] and report intersections

AS=?
in order to do this efficiently

50



Orthogonal line segment intersection

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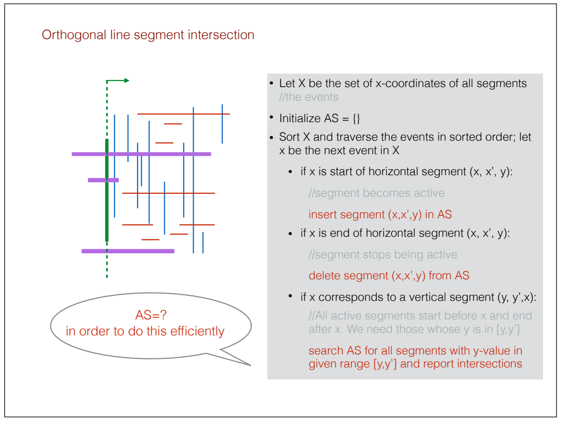

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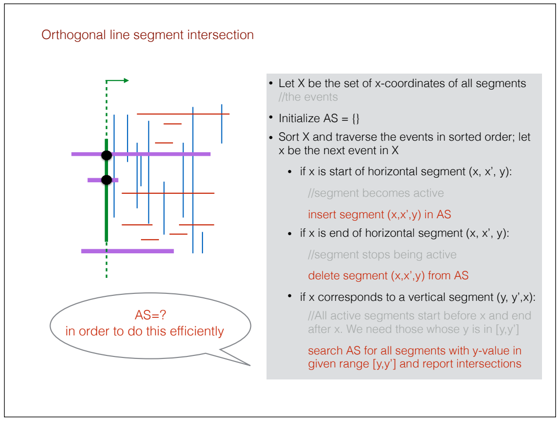

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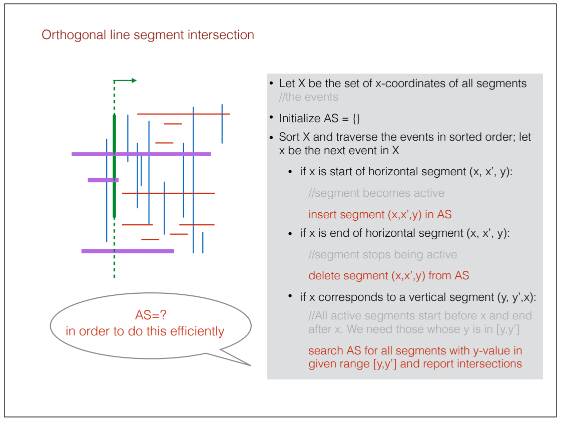

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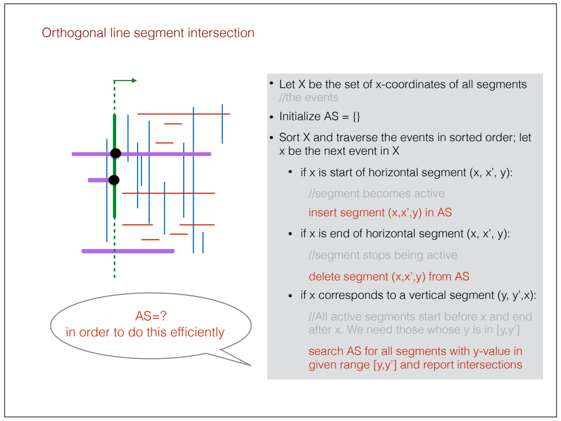

Orthogonal line segment intersection

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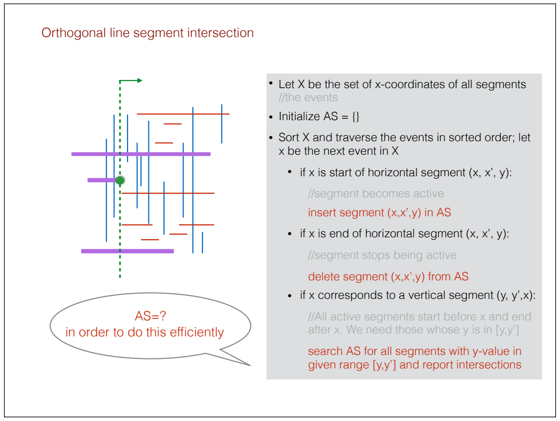

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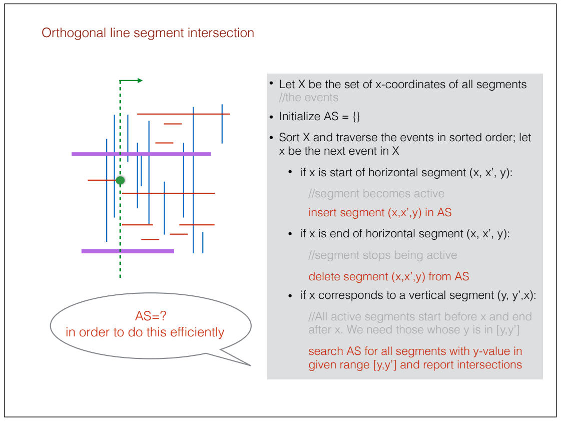

Orthogonal line segment intersection

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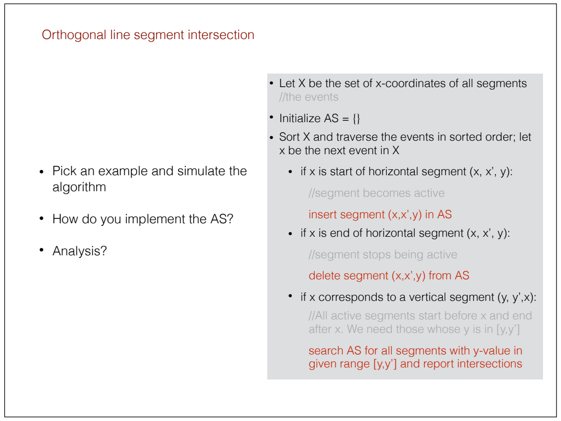

Orthogonal line segment intersection

- Pick an example and simulate the algorithm
- How do you implement the AS?
- Analysis?

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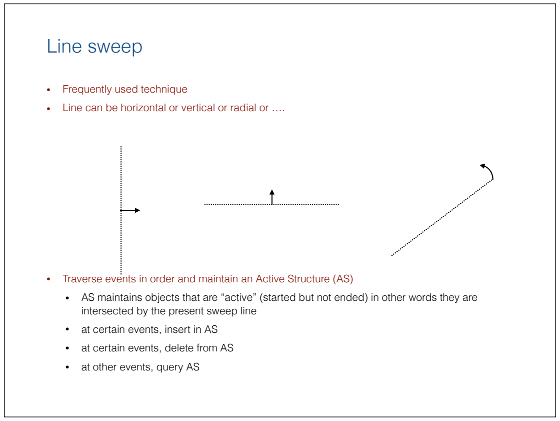

Line sweep

- Frequently used technique
- Line can be horizontal or vertical or radial or ….
- Traverse events in order and maintain an Active Structure (AS)
  - AS maintains objects that are "active" (started but not ended) in other words they are intersected by the present sweep line
  - at certain events, insert in AS
  - at certain events, delete from AS
  - at other events, query AS