Voronoi Diagrams

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Let $P = \{p_1, p_2, ..., p_n\}$ be a set of points in the plane.

- The Voronoi cell Vor(p_i) of p_i: the set of all points in the plane that are closer to p_i than to any other site
- The Voronoi diagram Vor(P) of P: the union of all Vor(p_i)

Properties

- Vor(P) defines a partition of the plane (i.e. any point in the plane is in the Voronoi cell of some site)
- Vor(p_i) is convex
- Vor(p_i) as halfplane intersection: Let $H(p_i, p_j)$ be the halfplane determined by the perpendicular bisector of p_i and p_j that contains p_i. Then Vor(p_i) $= \cap_{j,j \neq i} H(p_i, p_j)$
- Voronoi edges: The edges of Vor(P) are segments of perpendicular bisectors. Each Voronoi edge bounds two Voronoi cells, say Vor(p_i) and Vor(p_j) and must lie on the perpendicular bisector of p_i and p_j. Each point on the edge is equidistant to p_i and p_j.
- Voronoi vertex: point where 3 or more Voronoi cells intersect. A Voronoi vertex is equidistant from those sites. This means it's the centre of the circumcircle that goes through those sites. These sites are its nearest neighbors.
- If no 4 points are co-circular, then all Voronoi vertices are the intersection of precisely 3 cells.
- Empty-circle property: For any Voronoi vertex v, the circle $C(v)$ centered at v and going through its 3 sites cannot contain any other sites.

More properties

- The upper bound for the size of a cell in the Vor(P) is $O(n)$
- The total size of Vor(P) is $O(n)$
- A site p_i is on the convex hull of P if and only if its Voronoi cell is unbounded.

Constructing Voronoi diagrams

The standard algorithm is Fortune's plane sweep, which runs in $O(n \lg n)$.

Applications

- Nearest neighbor: Given a point p, find its nearest site. Boils down to finding the cell that contains p. Boils down to computing Vor(P) and putting a point-location structure on top of it.
- Facility location: Where should a new Starbucks be placed? Find the largest empty circle inside the convex hull of P. Claim: the center of the largest empty circle is a Voronoi vertex.
- Many, many others

Some extensions of Voronoi diagrams

- Order-2 Voronoi diagrams
- Farthest point Voronoi diagrams
- Voronoi diagram for a set of segments
- Voronoi diagram for a set of polygons
- Medial axis
- In 3D: Voronoi diagrams have size $O(n^2)$ and can be computed in optimal $O(n^2)$ time. Less useful because of their size.

One final property

Theorem: The straight-line dual of Vor(P) is a triangulation (called the Delaunay triangulation).