Delaunay Triangulations

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Let $P = \{p_1, p_2, ..., p_n\}$ be a set of points in the plane, in general position (i.e. no four are collinear).

Triangulations

- A *triangulation* or P is a planar subdivision whose faces are triangles and whose verties are the points of P.
- P has many triangulations and they all have the same number of triangles and edges. More precisely, assuming that the number of points on the convex hull of P is k, then any triangulation of P has $2n - k - 2$ triangles and $3n - k - 3$ edges.
- Question: Come up with an algorithm to build an arbitrary triangulation. How long does it take?
- In practice triangulations with fat triangles are better. Skinny triangles and small angles give numerical issues with geometric predicates like leftof(), etc. Also interpolation is better on fat meshes.
- We'll see that the Delaunay triangulation is the one that avoids small angles, to the extent possible.

The Delanay triangulation

- The DT of P is defined as the straight-line dual of the Voronoi diagram of P. Specifically, for every pair of Voronoi regions cell(u) and cell(v), we draw the line segment between u and v.
- Theorem [Delaunay, 1934]: The straight-line dual of Vor(P) is planar, and is a triangulation.

Properties

- **The empty circle property of DT:** Since every Voronoi vertex is the center of a circle that has 3 sites on its boundary and no other sites inside $\rightarrow$ this implies that for every triangle in the DT, its circumcircle does not contain any other points of P inside.
- Delaunay showed that it's true the other way around as well: If every triangle in a triangulation of P has the empty circle property, then the triangulation is the Delaunay triangulation (Delaunay lemma).

- Note that a local property on each triangle implies a global condition. This is nice because it will lead to simple algorithms.

DT via edge flipping

- The empty-circle property of triangles can be stated in terms of edges. Suppose that an edge ab is shared by the triangles abc and abd.
- If abc has the empty-circle property then d must be outside the circle through abc. Turns out the condition is symmetric: If d is outside the circle through abc then c is outside the circle that goes through abd, and the other way around. In this case edge ab is called *legal*.
- An edge ab is called *illegal* if it's not legal. In other words, if d is inside the circle through abc. Again, it turns out the condition is symmetric: If d is inside the circle through abc then c is inside the circle that goes through abd, and the other way around.
- **Edge flip:** As above, consider an edge ab shared by the triangles abc and abd. Flipping edge ab to edge cd will create triangles cda and cdb.
- Claim: If edge ab is illegal then edge cd is legal, and the other way around.
- We can start with an arbitrary triangulation of P and convert it to the Delaunay triangulation by flipping all illegal edge. A triangulation where all edges are legal must be the Delaunay triangulation, by the theorem.

```
Algorithm EdgeFlipDelaunay(P)
    construct an arbitrary triangulation T
    push all edges in T onto a stack and mark them
    while stack is not empty do
        pop edge ab from stack and unmark it
        if ab is not legal
            flip edge and update T
            for each new edge ac, ad, bc, bd: if not marked, push onto
            stack and mark it
```

- It's not at all obvious why does the algorithm terminates.
- **Angle vector:** Let T be a triangulation of P. The angle vector $A(T)$ of T is the vector of all angles of the triangles of T, sorted in increasing order. Recall that all triangulations on P have the same number of triangles, therefore they will all have an angle vector of the same size.
- We compare angle vectors using lexicographic order.
- Claim: When we flip an illegal edge to a legal edge, the angle vector increases.

- Claim: The angle vector of the Delaunay triangulation is the maximum angle vector among all triangulations of P.
- In other words, DT is the triangulation that has the largest minimum angle. If there is more than one triangulation with the same minimum angle, DT is that triangulation with the largest second-smallest angle. etc.
- When P is so that no four points are co-circular, there exists a unique legal triangulation which is the DT. The DT maximizes the minimum angle.
- **Termination and run time:** Each time we flip an edge, the angle vector increases. There are $O(n^2)$ edges, and each edge can be flipped at most once (once an edge is legal it stays legal). So the algorithm runs in $O(n^2)$.

Randomized Incremental construction (RIC)

- Idea: Add one point at a time, in random order.
- When adding a point p, locate the triangle abc that contains it, split it into three new triangles abp, bcp and cap, and flip all edges that are illegal, recursively. Any illegal edge that's flipped becomes incident to p. Therefore the cost of edge flipping is proportional to the degree of p.
- Since a triangulation has $3n - k - 3$ edges, the average degree of a point is 3, and therefore only a constant number of flips at each step [....] the overall cost of the algorithm is $O(n \lg n)$ expected if points are inserted in random order.

Connection DT (2D) and Convex hull (3D)

- Consider the paraboloid $z = x^2 + y^2$. Lift every point p from P onto this paraboloid to obtain $p' = (p_x, p_y, p_x^2 + p_y^2$. Mapping the points from 2D to this paraboloid is called the *lifting map*.
- Checking if d is inCircle (a, b, c) is the same as checking if d' is below the plane a', b', c'.
- This means that abc has the empty circle property if and only if all sites are above the plane $a'b'c'$. This means $a'b'c'$ is a face of the lower 3D CH.
- Therefore DT(P) and 3D CH are equivalent

Constructing DT: summary

- Dual of Voronoi: Construct the Voronoi diagram with Fortune's plane sweep in $O(n \lg n)$, then build the dual in $O(n)$ time
- Via edge flipping in $O(n^2)$ time
- RIC in $O(n \lg n)$ expected
- Dual of 3D convex hull: Lift points on the paraboloid, construct lower 3D hull, and project back onto xy-plane.

Applications/More properties

- DT used in high-quality meshing because it's angle optimal
- Finding all-nearest-neighbors on P: Theorem: The nearest neighbor graph of P is a subset of DT(P).
- Euclidian minimum spanning tree: Theorem: The Euclidian MST of P is a subset of DT(P).
- Euclidian traveling salesman problem: A 2-approximation can be obtained via the Euclidian MST (which can be obtained from DT(P).