Finding closest pair

Computational Geometry [csci 3250]
Laura Toma
Bowdoin College

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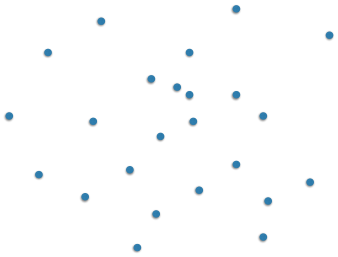

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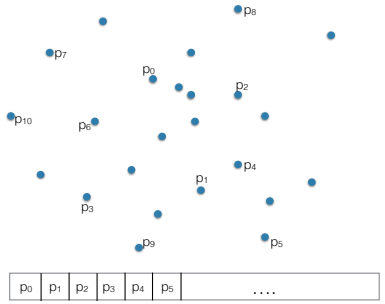
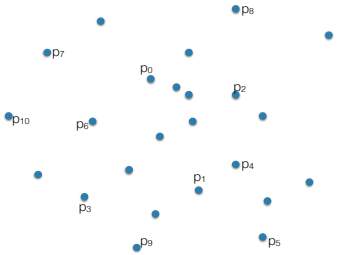


3

Given an array of points in 2D, find the closest pair.



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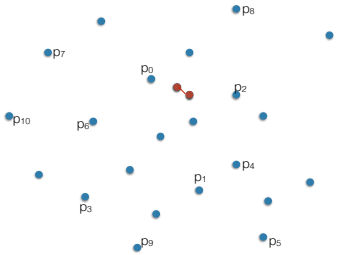

5

Given an array of points in 2D, find the closest pair.

Brute force:

- mindist = VERY_LARGE_VALUE
- for all distinct pairs of points p_i, p_j
  - d = distance (p_i, p_j)
  - if (d< mindist): mindist=d

6

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  - $O(n^2)$ pairs ==> $O(n^2)$ time

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Can we do better than $O(n^2)$?

8

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Can we do better than $O(n^2)$?

Hint: divide-and-conquer

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Divide-and-conquer

mergesort(array A)

- if A has 1 element, there's nothing to sort, so just return it
- else

  //divide input A into two halves, A1 and A2
  - A1 = first half of A
  - A2 = second half of A

  //sort recursively each half
  - sorted_first_half = **mergesort**(array A1)
  - sorted_second_half = **mergesort**(array A2)

  //merge
  - result = merge_sorted_arrays(sorted_first_half, sorted_second_half)
  - return result

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Analysis: T(n) = 2T(n/2) + O(n) => O(n lg n)

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In general

```
DC(input P)
  if P is small, solve and return
  else
    //divide
    divide input P into two halves, P1 and P2
    //recurse
    result1 = DC(P1)
    result2 = DC(P2)
    //merge
    do_something_to_figure_out_result_for_P

    return result
```

Analysis: T(n) = 2T(n/2) + O(merge phase)

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In general

```
DC(input P)
  if P is small, solve and return
  else
    //divide
    divide input P into two halves, P1 and P2
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    result1 = DC(P1)
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    //merge
    do_something_to_figure_out_result_for_P
    return result
```

Analysis: T(n) = 2T(n/2) + O(merge phase)

- if merge phase is **O(n)**: T(n) = 2T(n/2) + **O(n)** => O(n lg n)
- if merge phase is **O(n lg n)**: T(n) = 2T(n/2) + **O(n lg n)** => O(n $\lg^2$n)

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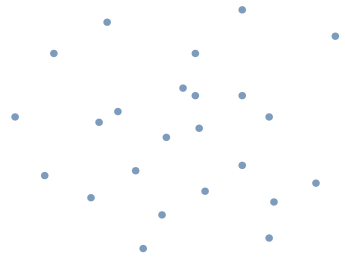
Divide-and-conquer for closest pair



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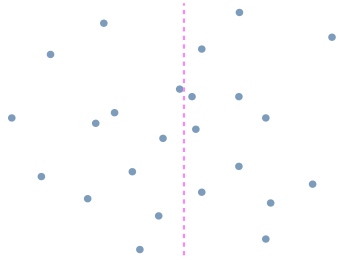
Divide-and-conquer for closest pair

- find vertical line that splits P in half



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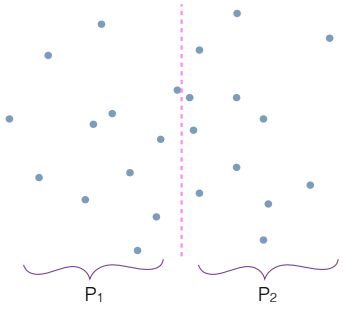
Divide-and-conquer for closest pair

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line



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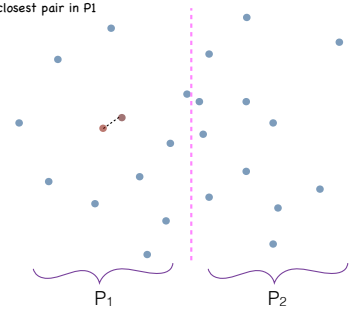
Divide-and-conquer for closest pair

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1



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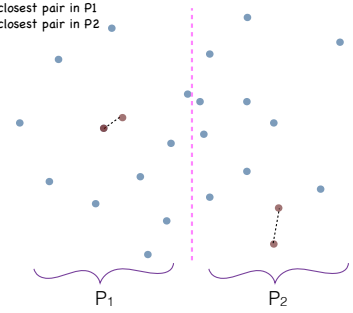
Divide-and-conquer for closest pair

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1
- recursively find closest pair in P2



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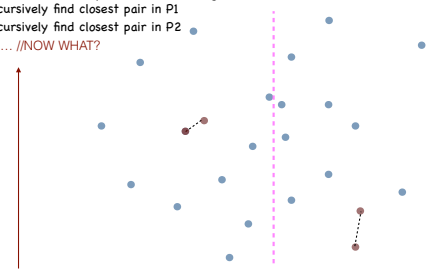
Divide-and-conquer for closest pair

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1
- recursively find closest pair in P2
- ..... //NOW WHAT?



How can you find closest pair in P?

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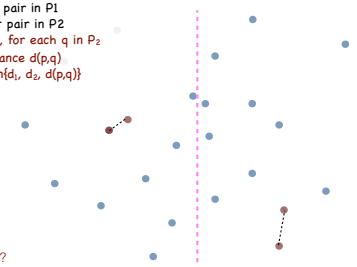
Divide-and-conquer for closest pair

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- d_1 = find closest pair in P1
- d_2 = find closest pair in P2
- for each p in P_1, for each q in P_2
  - compute distance d(p,q)
  - mindist = min{d_1, d_2, d(p,q)}



1. Is this correct?
2. Running time?

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Divide-and-conquer for closest pair

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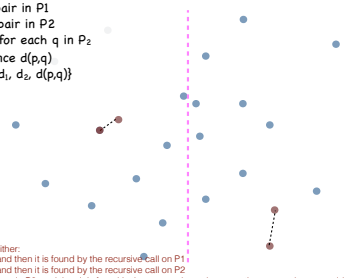

Is this correct?

YES. The closest pair is either:
- both points are in P1, and then it is found by the recursive call on P1
- both points are in P2, and then it is found by the recursive call on P2
- one point is in P1 and one in P2, and then it is found in the merge phase, because the merge phase consider all such pairs

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Divide-and-conquer for closest pair

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- for each p in P_1, for each q in P_2
  - compute distance d(p,q)
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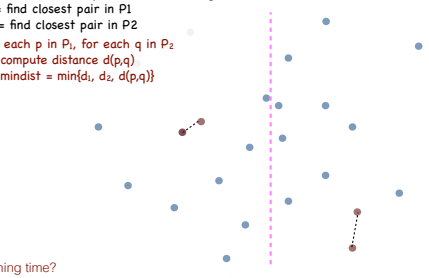

Running time?
- $T(n) = 2T(n/2) + O(n^2)$ => solves to $O(n^2)$

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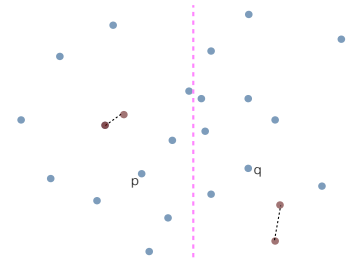
Refining the merge

Do we need to examine all pairs (p,q), with p in P_1, q in P_2?



Can (p,q) be the closest pair?

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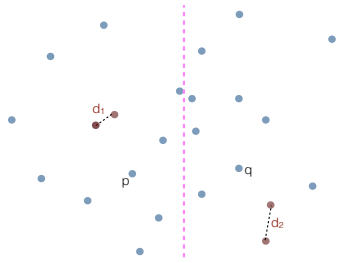
Refining the merge

Do we need to examine all pairs (p,q), with p in P_1, q in P_2?



Can (p,q) be the closest pair?

Why not? Where do p,q need to lie in order to be the closest pair?

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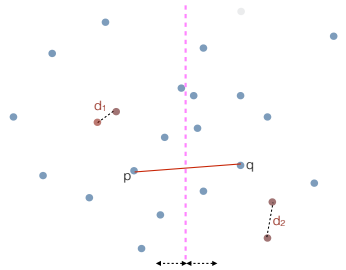
Notation: d = min {d_1, d_2}

In order for dist(p,q) to be smaller than d, it must be that both the horizontal and the vertical distance between p and q must be smaller than d.



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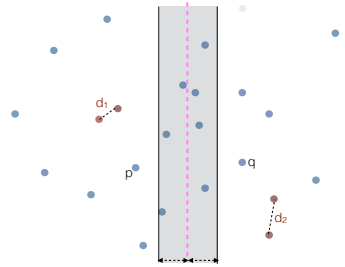
Claim: In order to be candidates for closest pair, points p, q must lie in the d-by-d strip centered at the median.



Proof:

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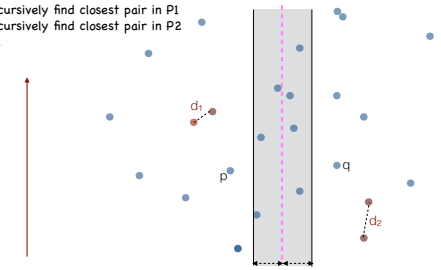
Refining the merge

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1
- recursively find closest pair in P2
- .....



Fill in the details of the new algorithm's merge phase and analyze it.

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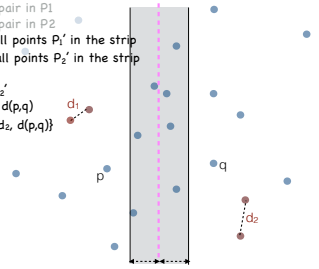
Refining the merge

- find vertical line that splits P in half
- let P1, P2 = set of points to the left/right of line
- recursively find closest pair in P1
- recursively find closest pair in P2
- traverse P_1 and select all points P_1' in the strip
- traverse P_2 and select all points P_2' in the strip
- for each p in P_1'
  - for each point q in P_2'
    - compute distance d(p,q)
    - mindist = min{d_1, d_2, d(p,q)}



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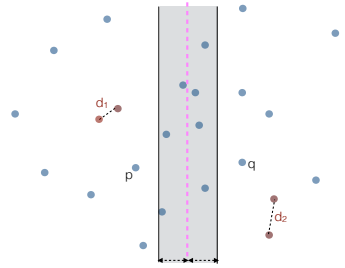
Refining the merge

- Show an example where the strip may contain Omega(n) points.



- What does this imply for the running time?

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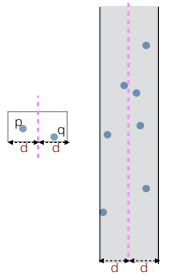
Refining the merge

- Ok, so this is not yet enough
- But … we also know that the vertical distance between p and q cannot be greater than d.



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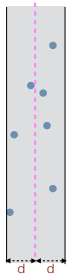
Refining the merge

- Consider a point p in the stripe. How many points below it, at most, could be candidates for the closest pair (p,q)?



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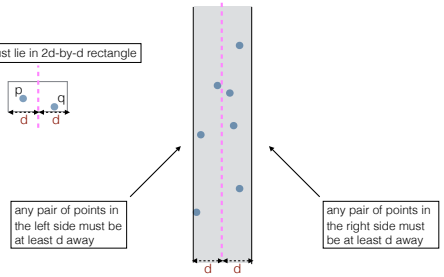
Refining the merge

- Consider a point p in the stripe. How many points below it, at most, could be candidates for the closest pair (p,q)?

p,q must lie in 2d-by-d rectangle



any pair of points in the left side must be at least d away

any pair of points in the right side must be at least d away

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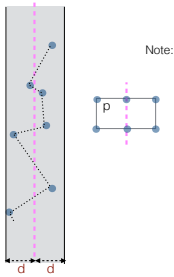
Refining the merge

Claim:
A point p needs to check at most 5 points following p in y-order.



Note: Assume no duplicate points.

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Refining the merge

- Put all these together and write down the algorithm.
- Analyze the running time.

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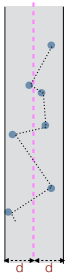
Refining the merge

closestPair(P)

//divide
- find vertical line that splits P in half
- let P_1, P_2 = set of points to the left/right of line
- call closestPair(P_1); let d_1 be the returned closest distance
- call closestPair(P_2); let d_2 be the returned closest distance

//merge
- let d = min{d_1, d_2}
- Strip= empty
- for all p in P_1: if x_p > x_vertical – d: add p to Strip
- for all p in P_2: if x_p < x_vertical + d: add p to Strip
- sort Strip by y-coord
- initialize mindist=d
- for each p in Strip in sorted order
  - compute its distance to the 5 points that come after it in sorted order
  - if any of these is smaller than mindist, update mindist
- return mindist



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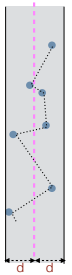

Analysis: ?

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Refining the merge

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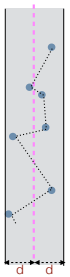

Analysis: T(n) = 2T(n/2) + O(n lg n) => O(n $\lg^2 n$)

Divide-and-conquer for closest pair

- Avoiding the sort

Divide-and-conquer for closest pair

- Describe in full detail how to avoid sorting at every level, and give the detailed pseudocode. Include an explanation for how to find the vertical line that splits P in half.