

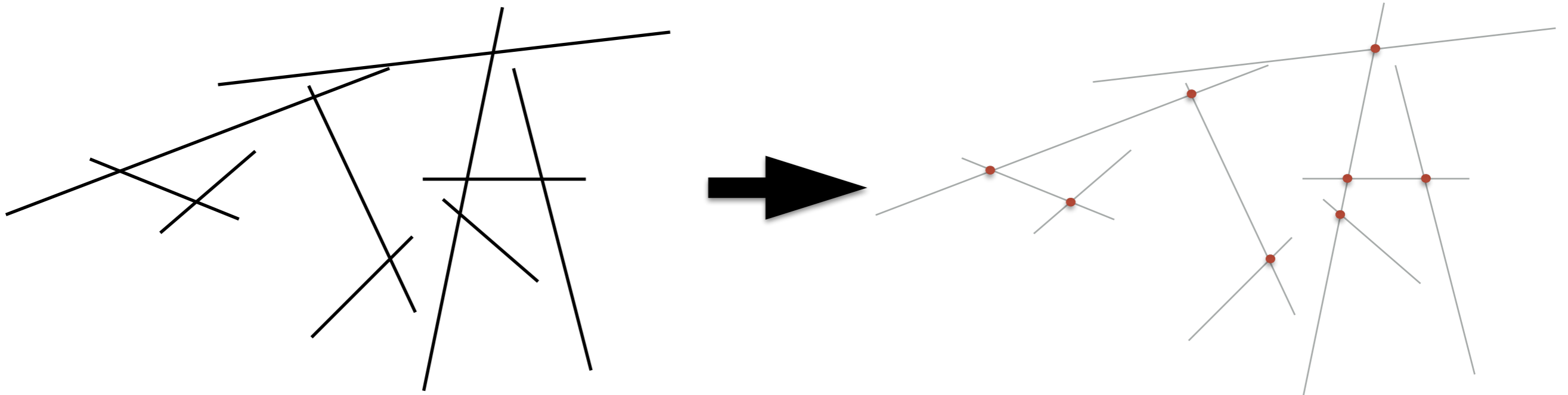
Computational Geometry
[csci 3250]

Line segment intersection

Laura Toma
Bowdoin College

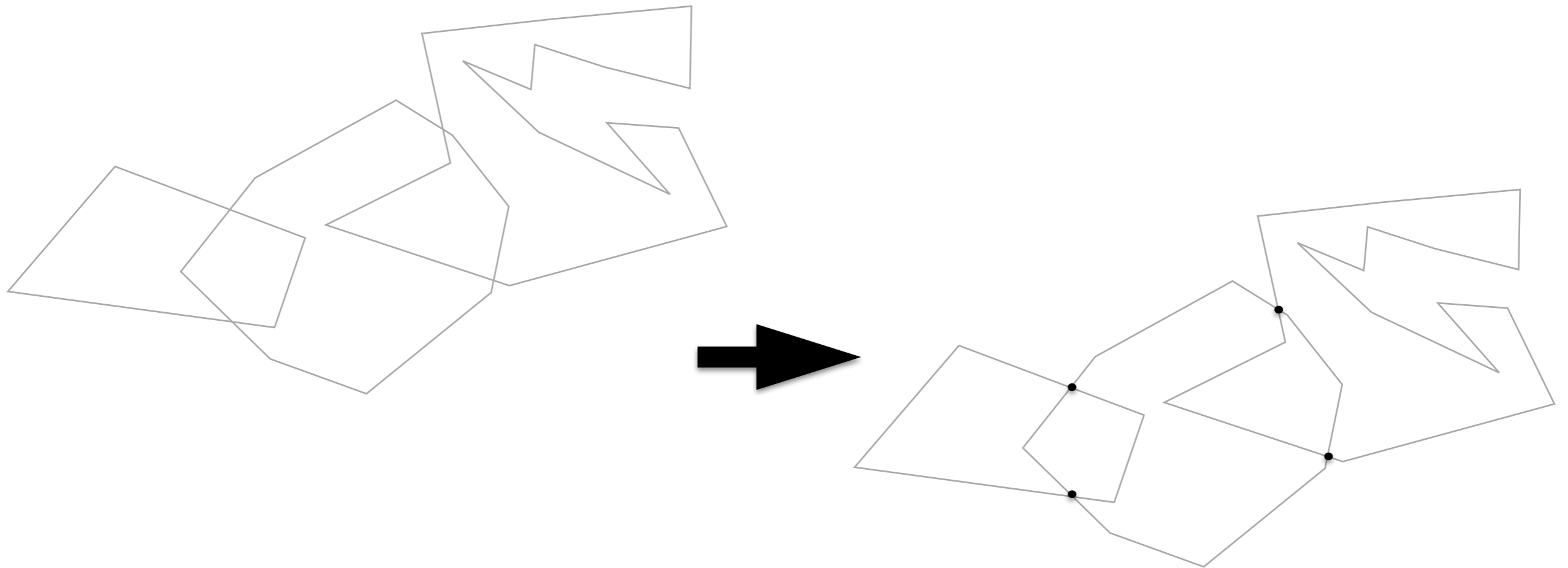
Line segment intersection

Problem: Given a set of line segments in 2D, find all their pairwise intersections.



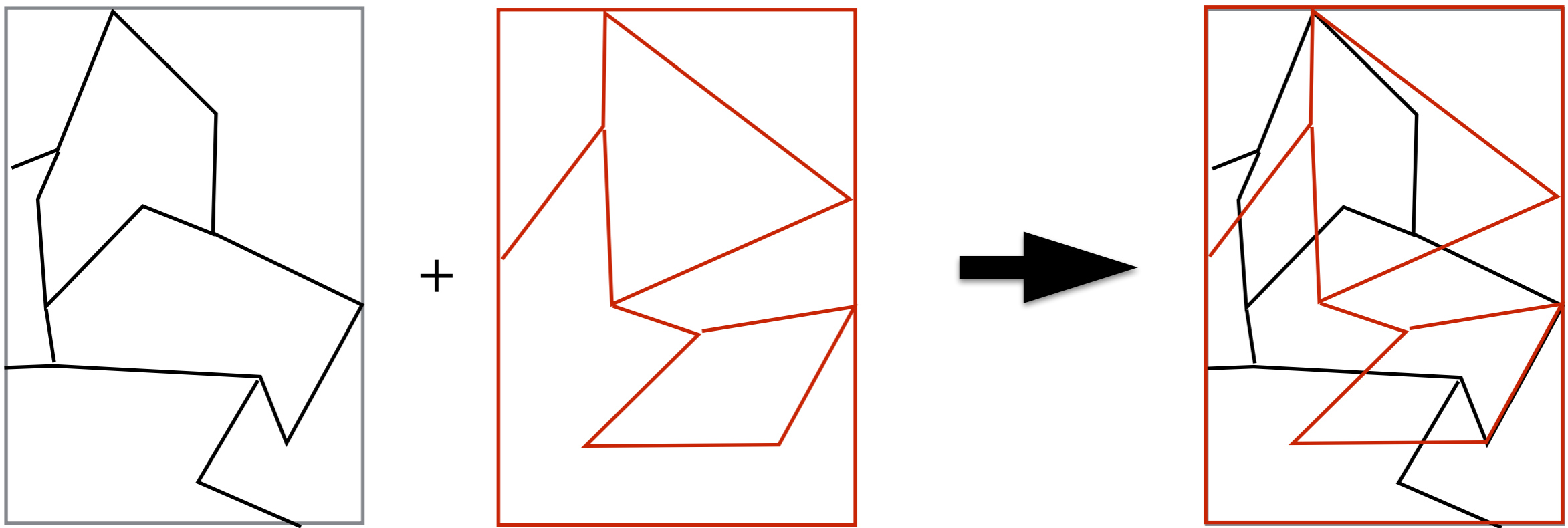
Line segment intersection

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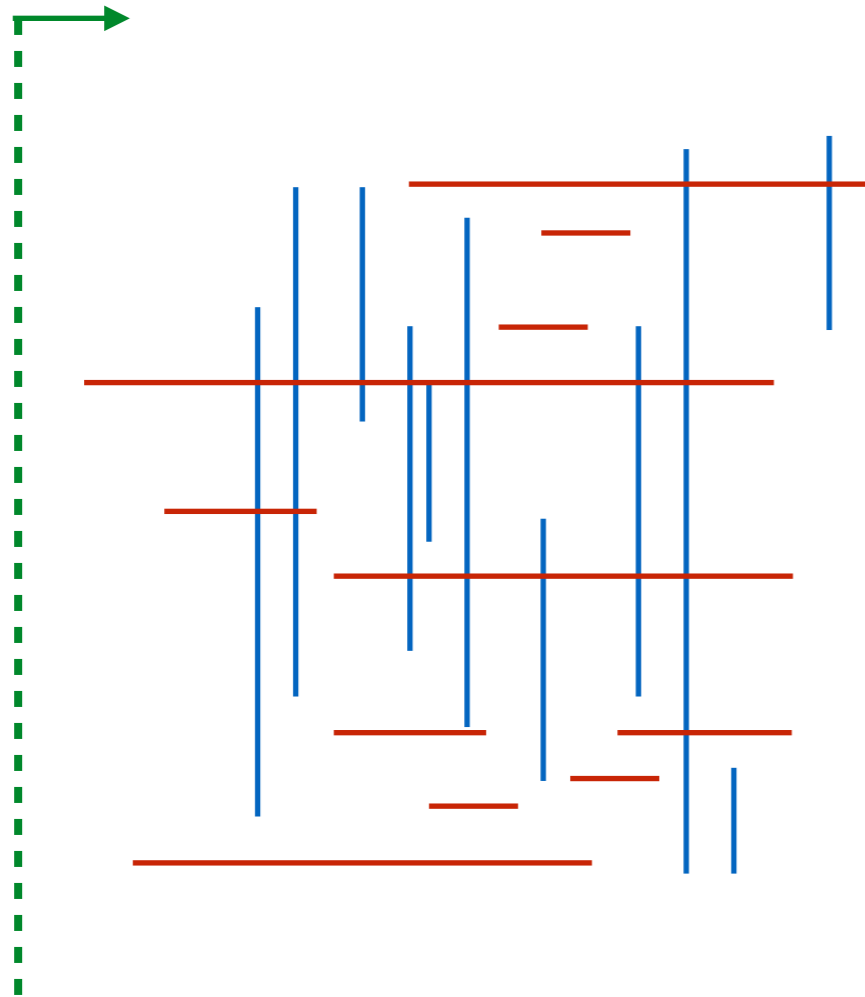


Line segment intersection

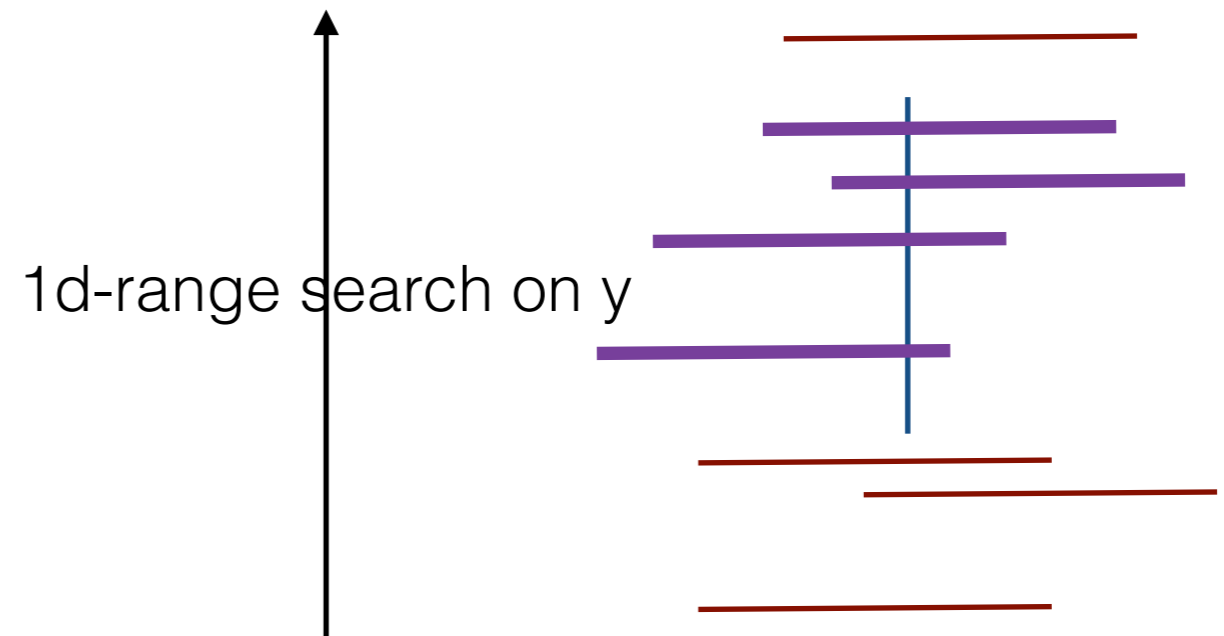
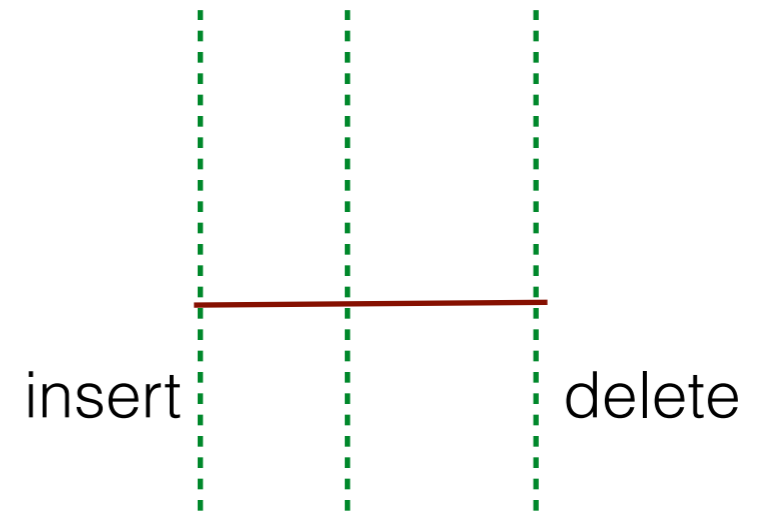
Problem: Given a set of line segments in 2D, find all their pairwise intersections.



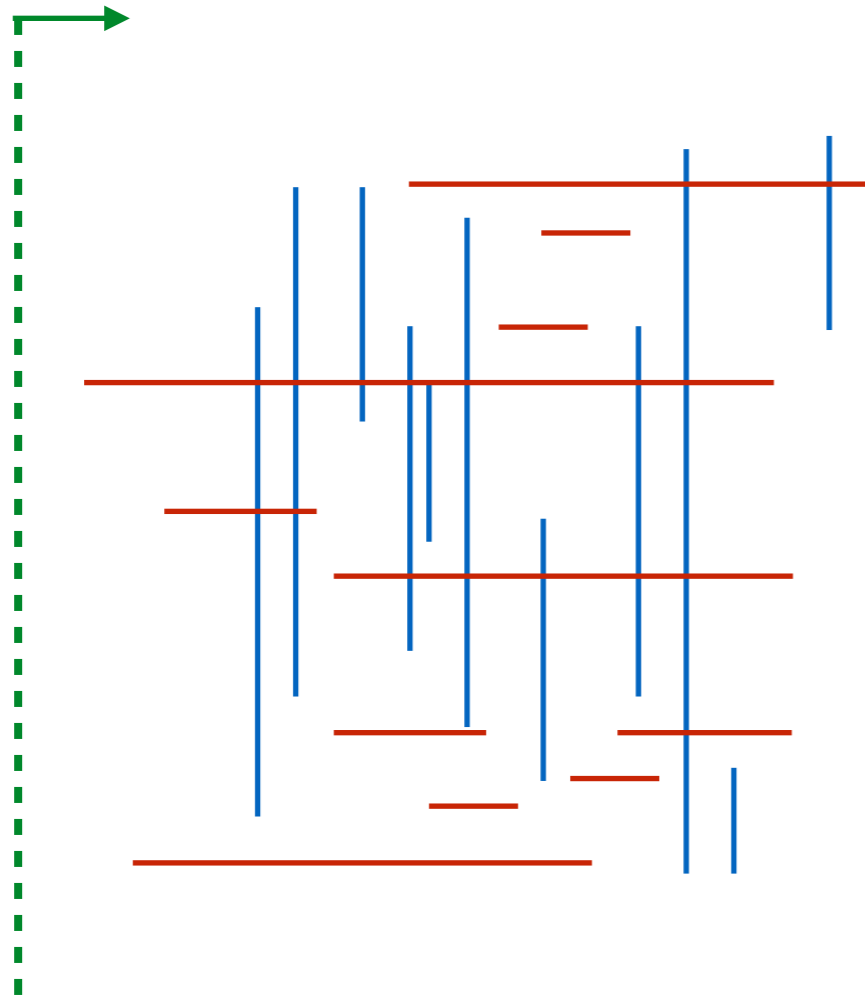
Orthogonal line segment intersection



line sweep technique
solve the problem behind the line



Orthogonal line segment intersection



line sweep technique

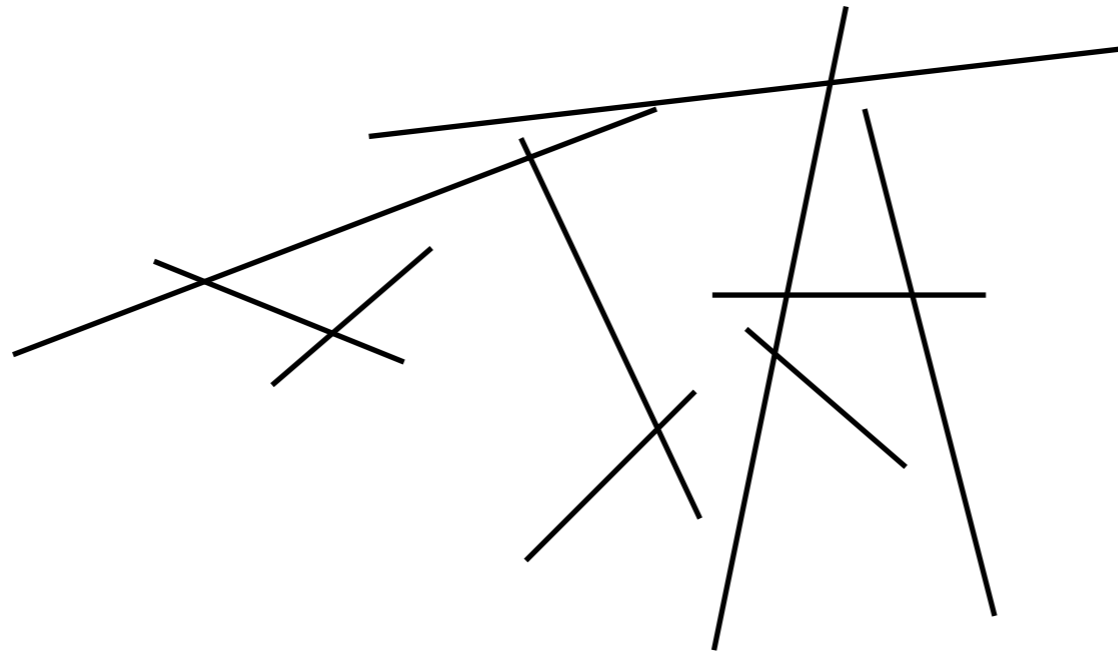
solve the problem behind the line

- n : size of the input (number of segments)
- k : size of output (number of intersections)

Result: The intersections of a set of n orthogonal segments in the plane can be found in $O(n \lg n + k)$ time.

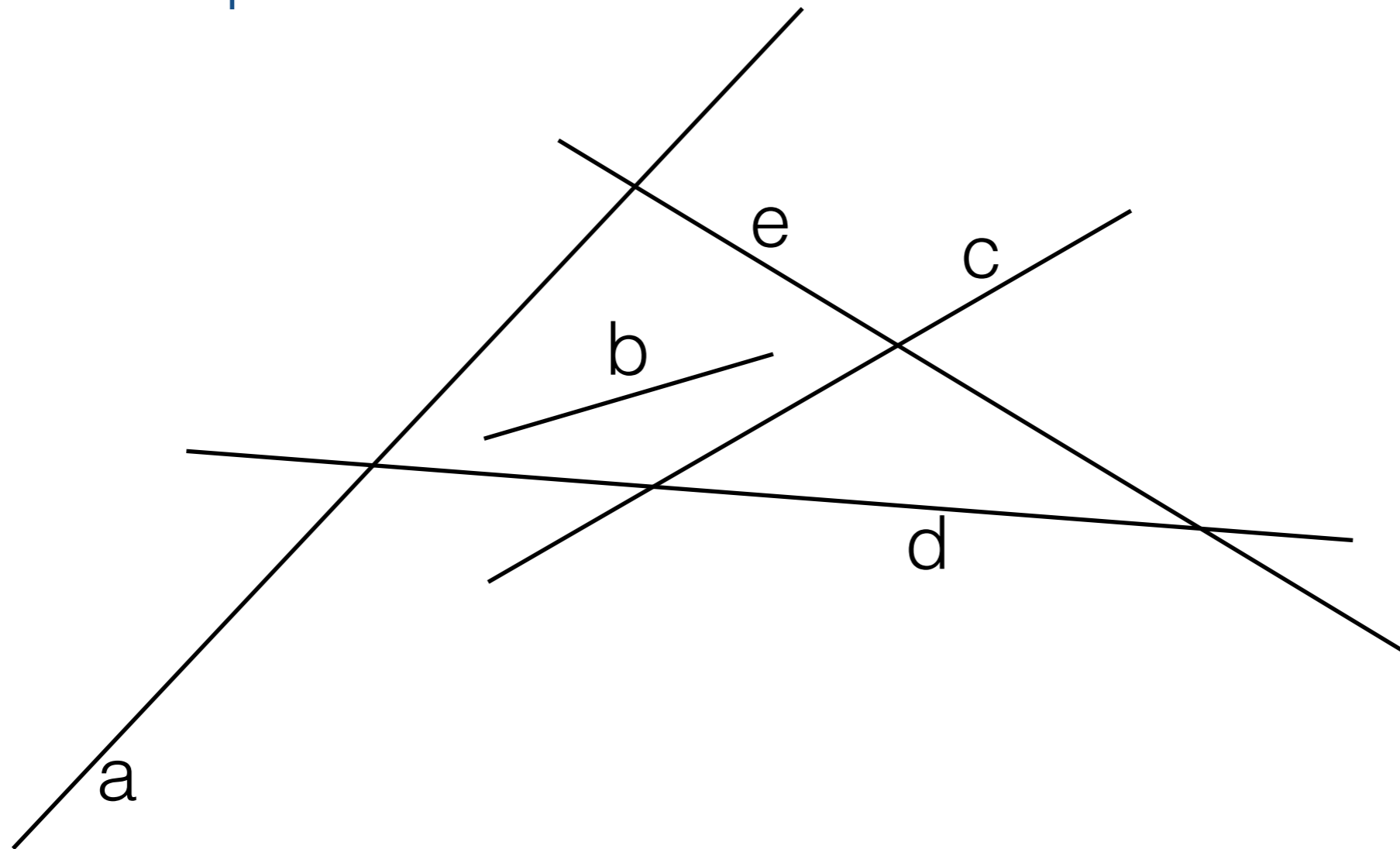
General line segment intersection

- n : size of the input (number of segments)
- k : size of output (number of intersections)

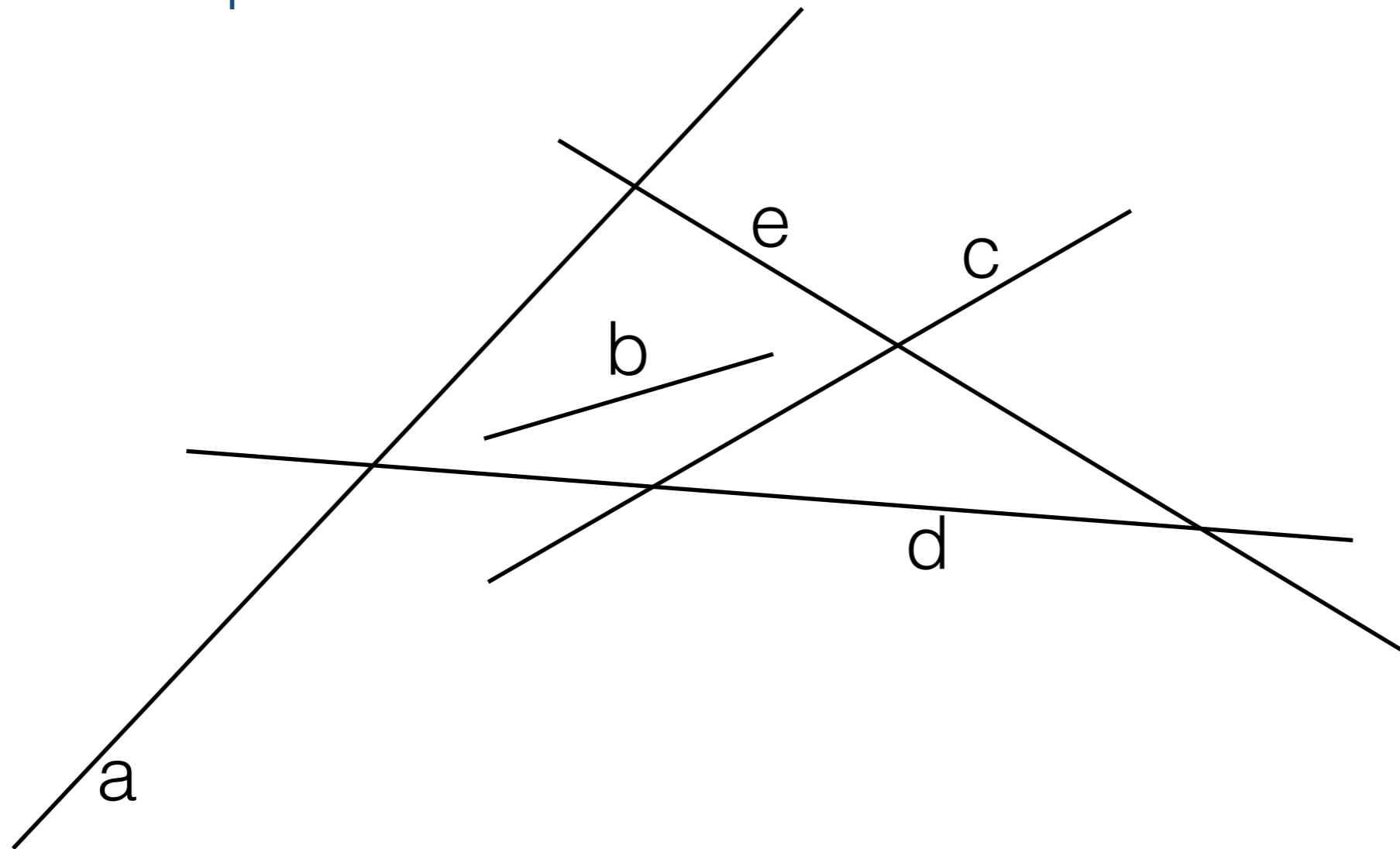


- Extend sweep line idea
- We'll get an overall bound of $O(n \lg n + k \lg n)$ which improves on the naive $O(n^2)$ when k is small
- The algorithm was developed by Jon Bentley and Thomas Ottmann in 1979
- Simple (once you see it) and practical

The sweep

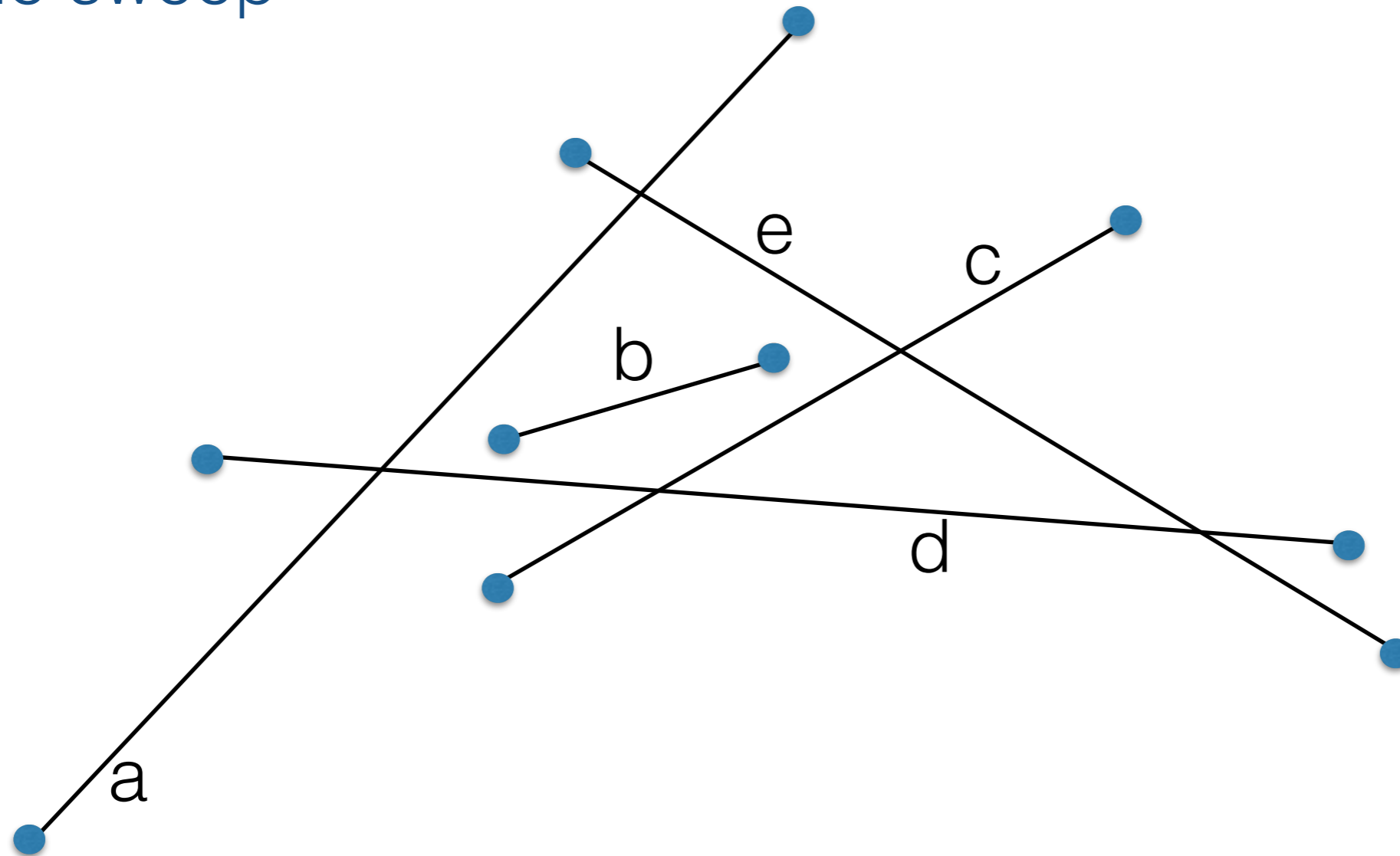


The sweep



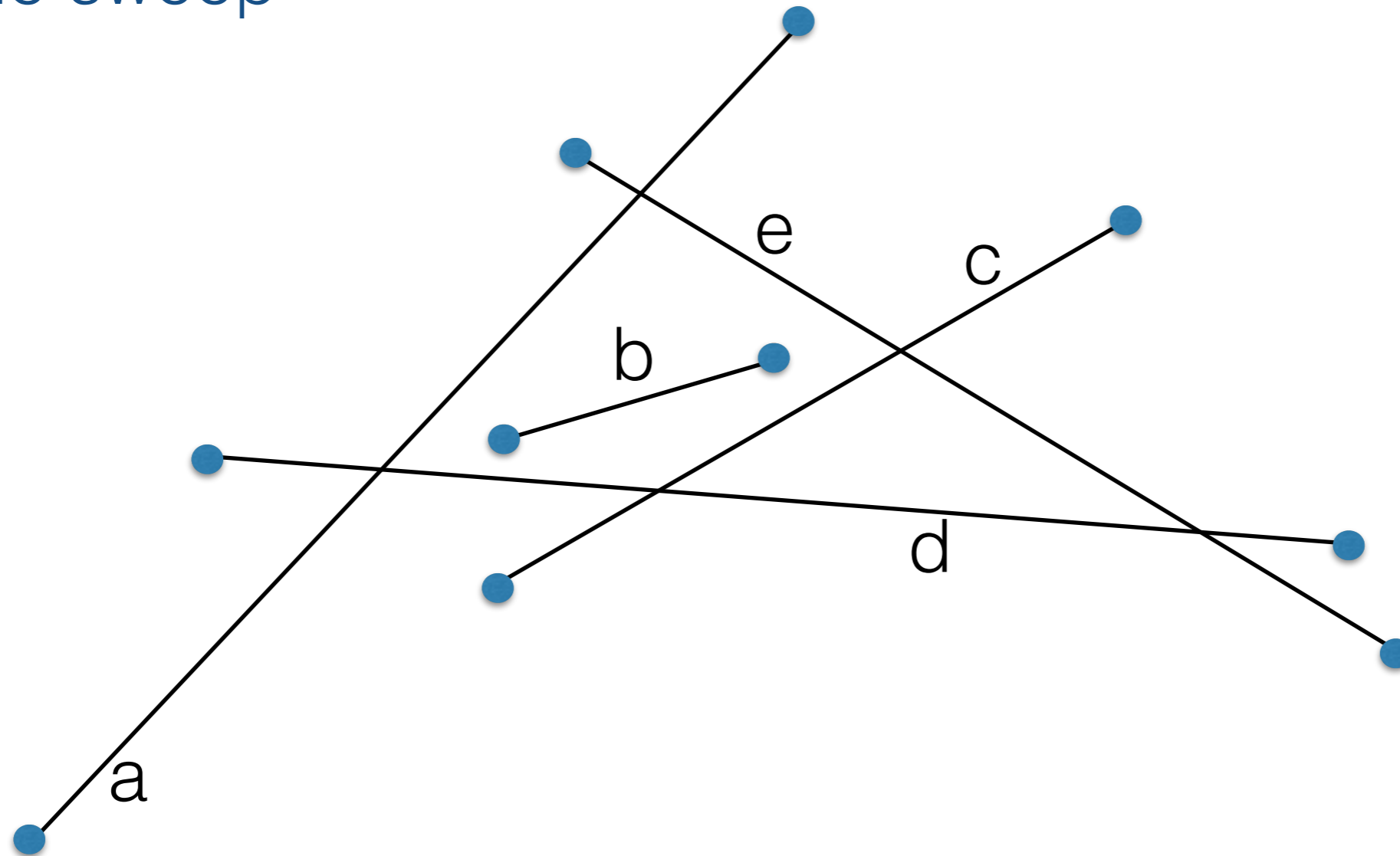
- Let X be the set of all x-coords of segments

The sweep



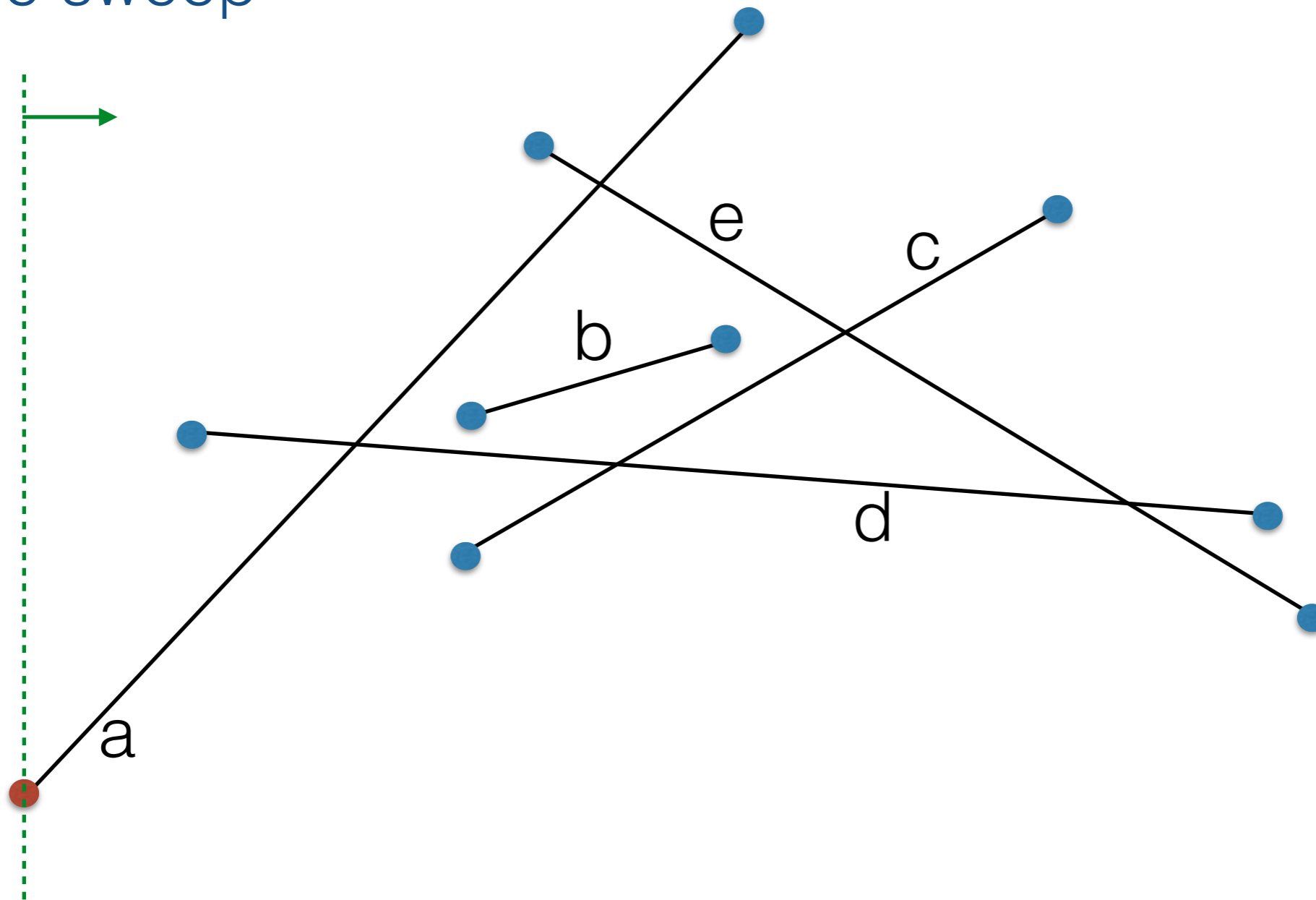
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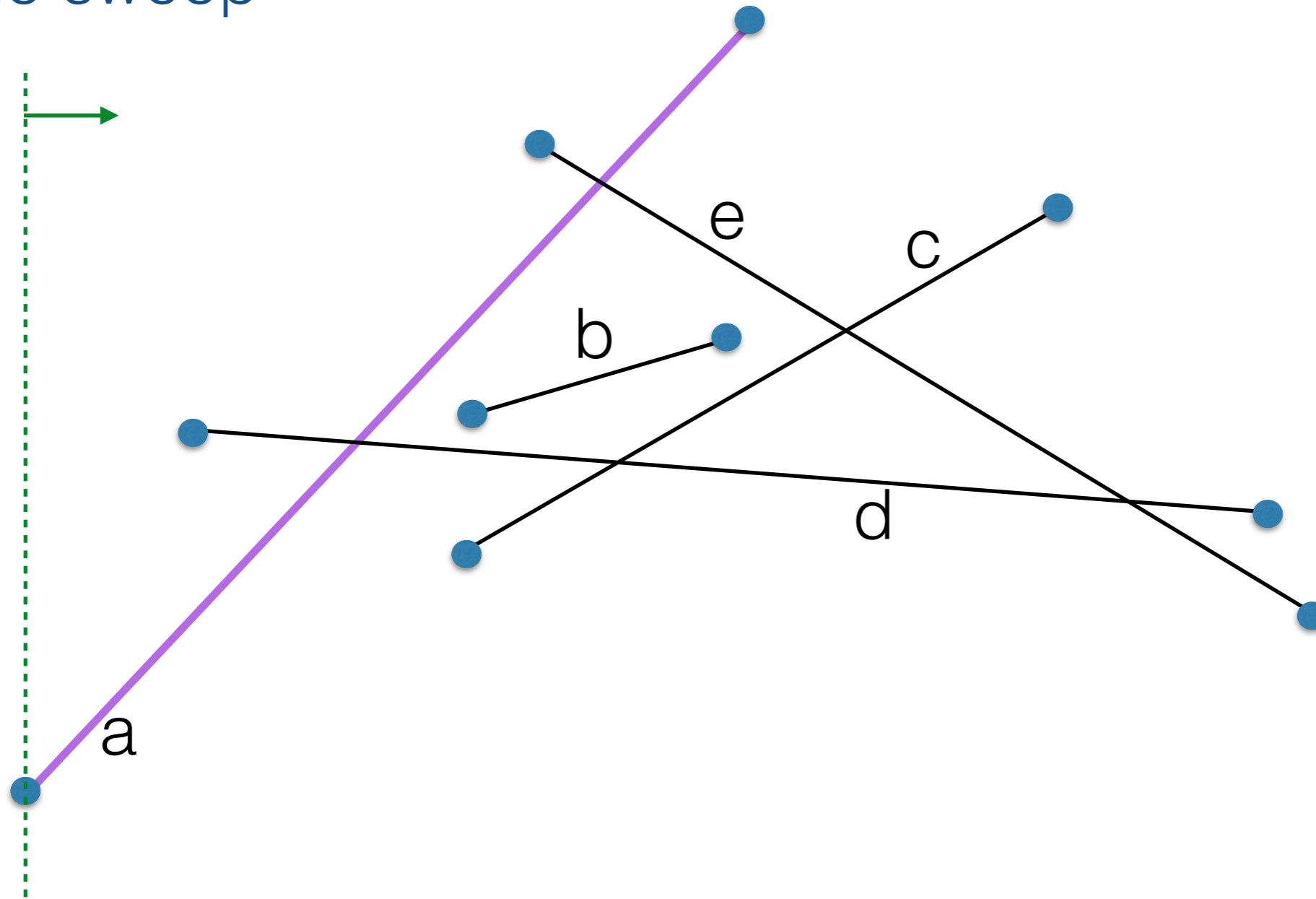
- Let X be the set of all x-coords of segments
- Traverse the events in X in order

The sweep



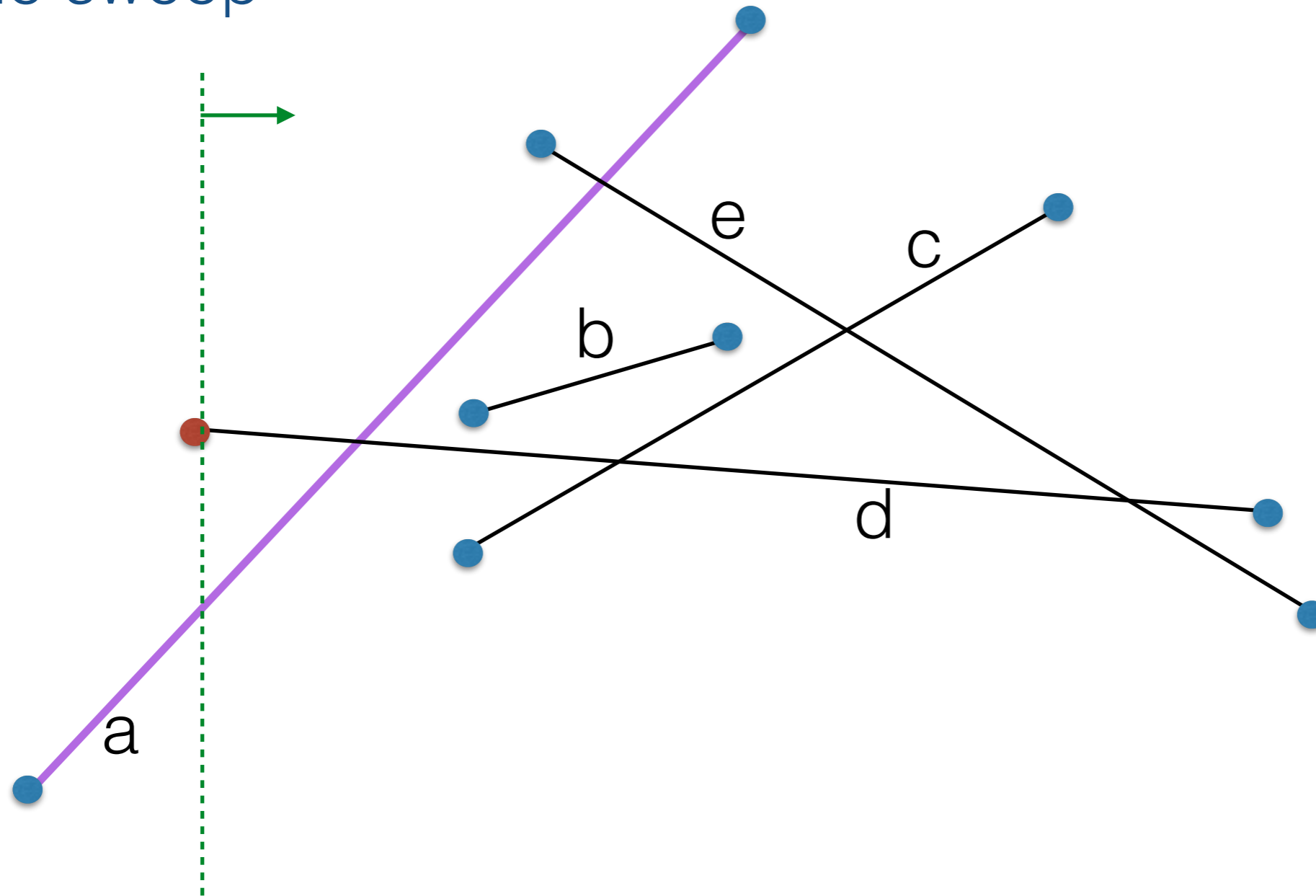
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The sweep



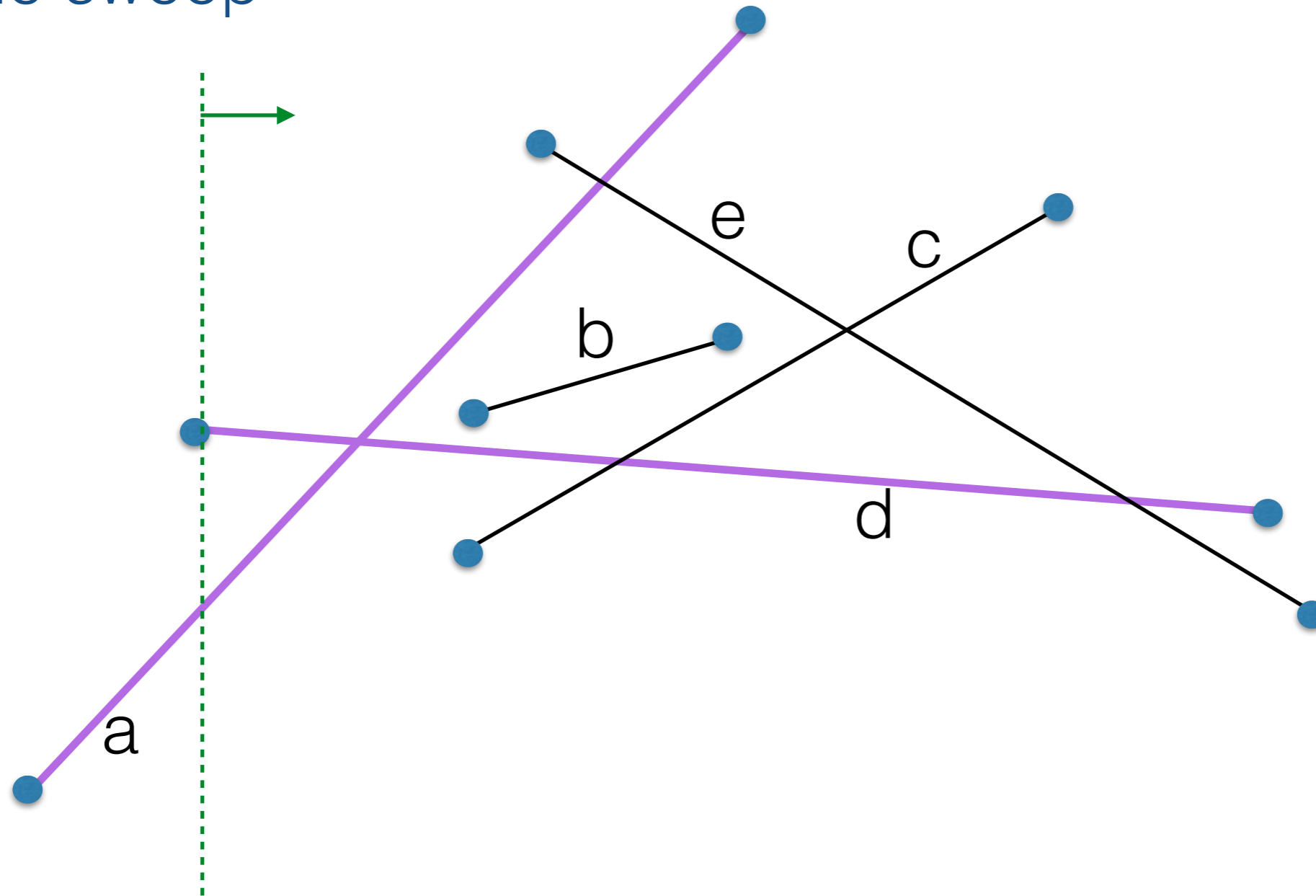
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The sweep



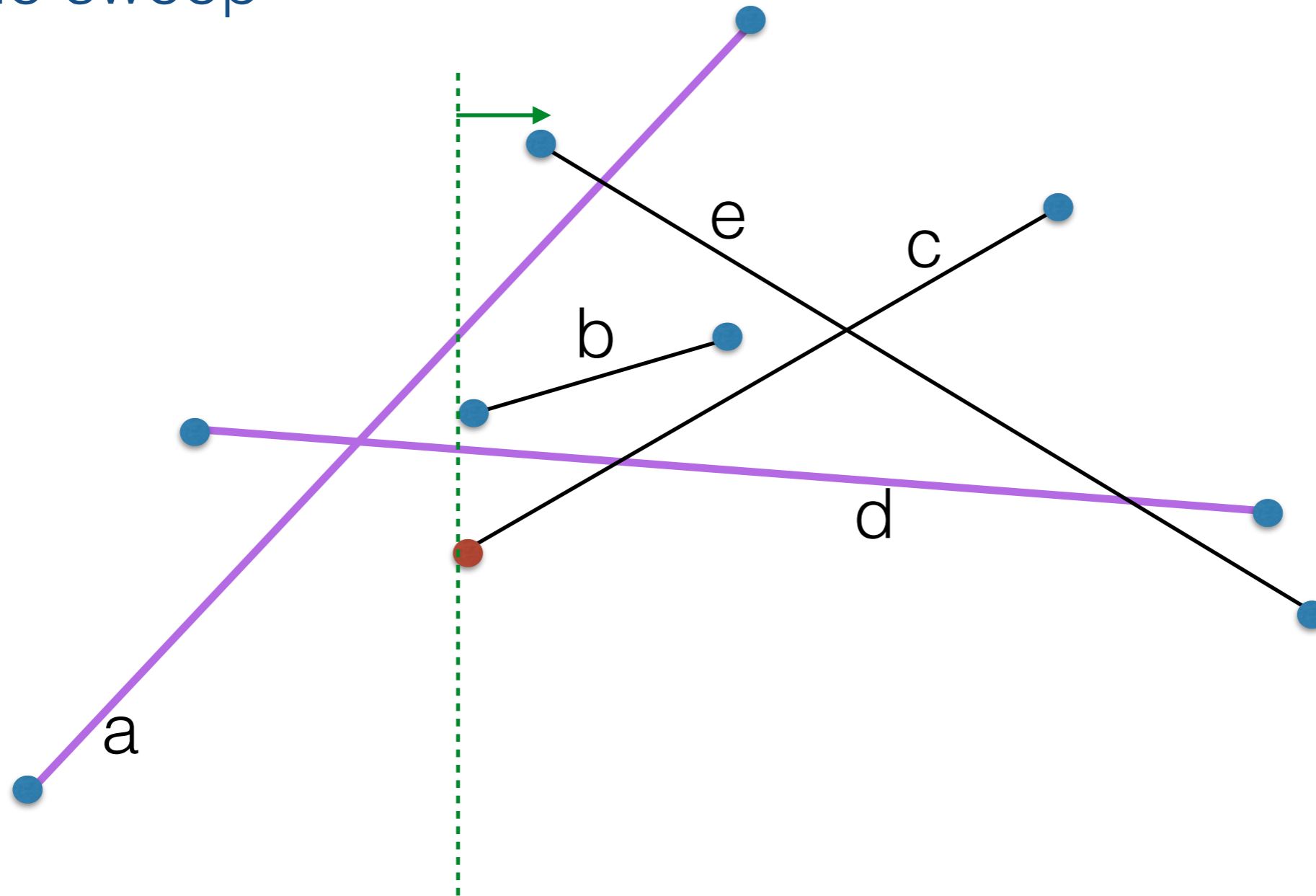
- Let X be the set of all x-coords of segments
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The sweep



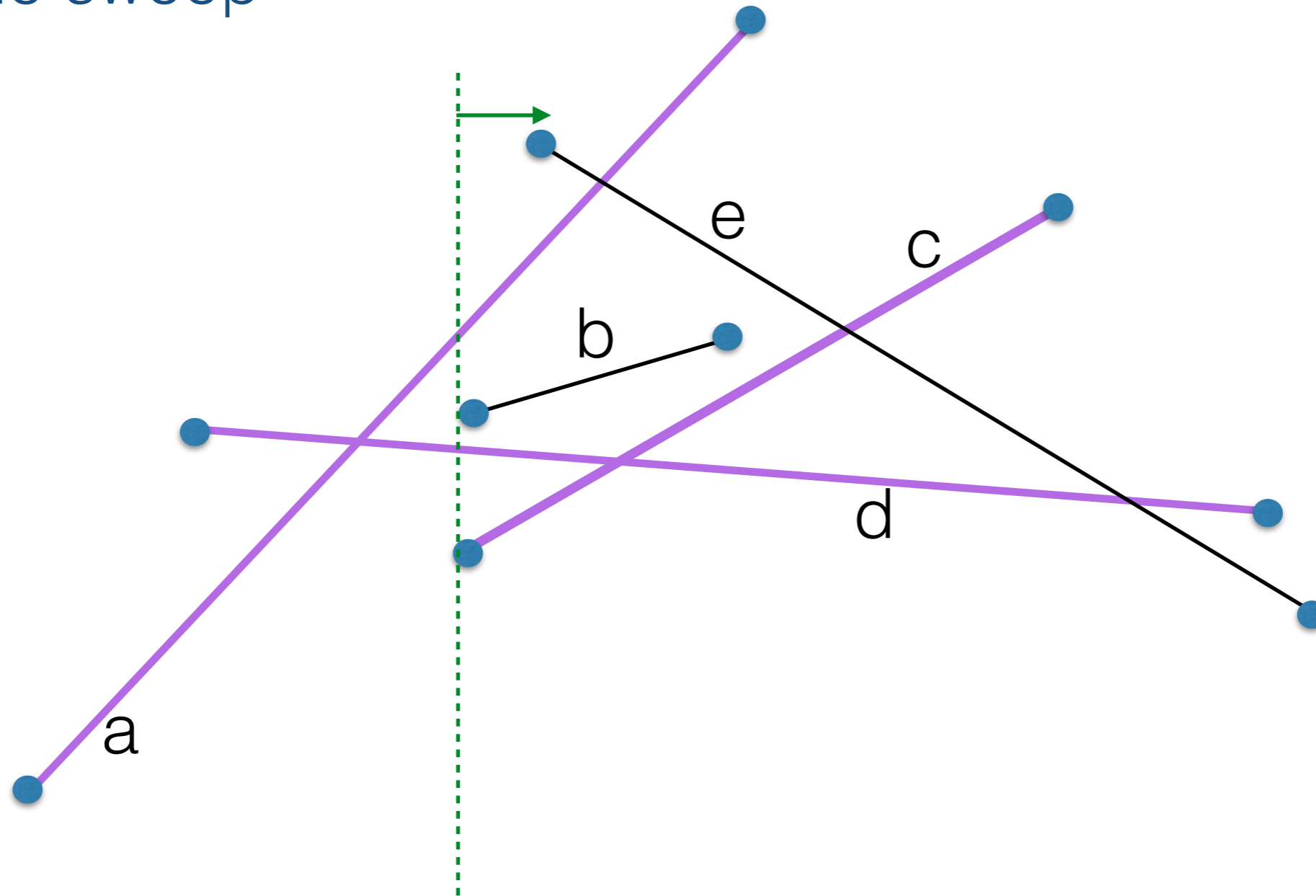
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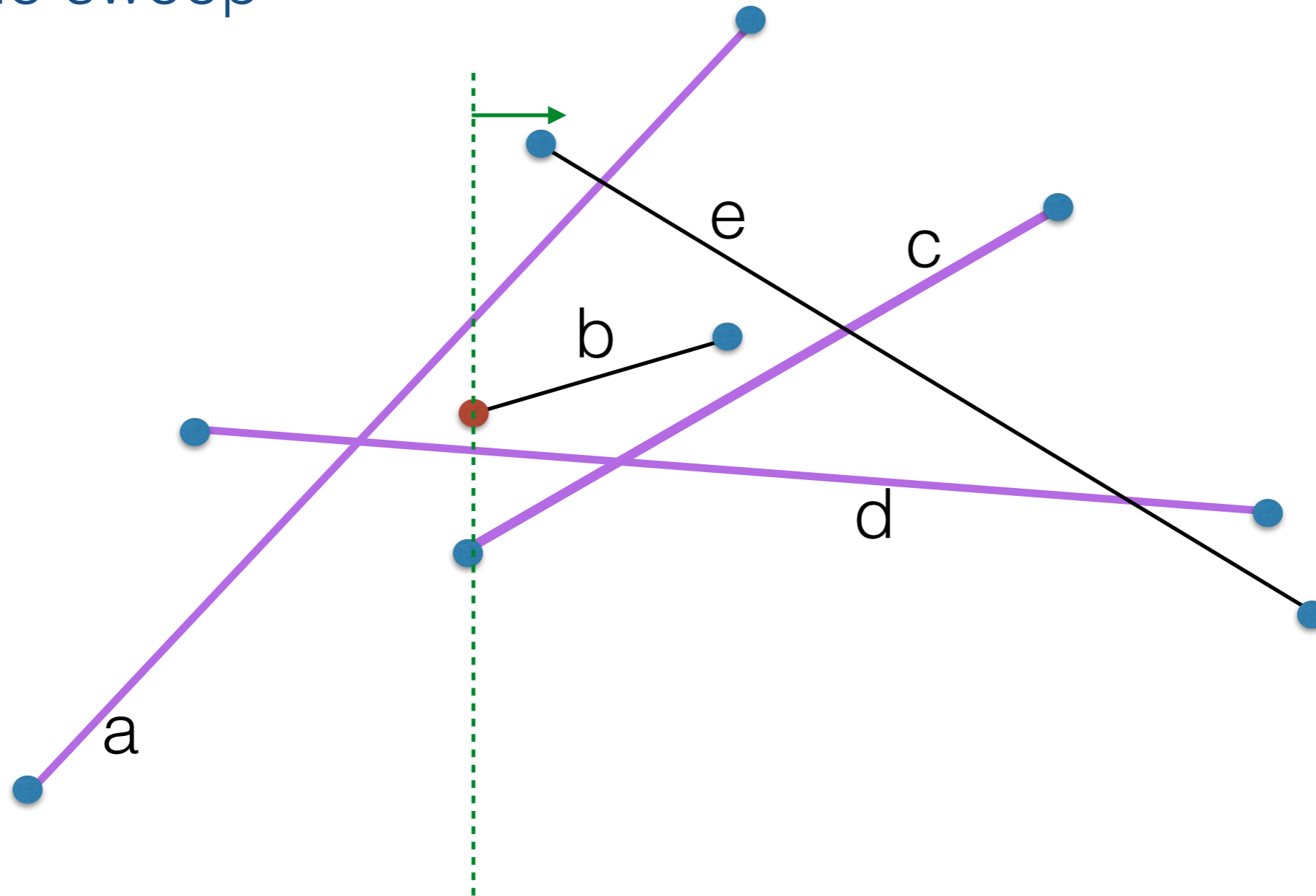
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The sweep



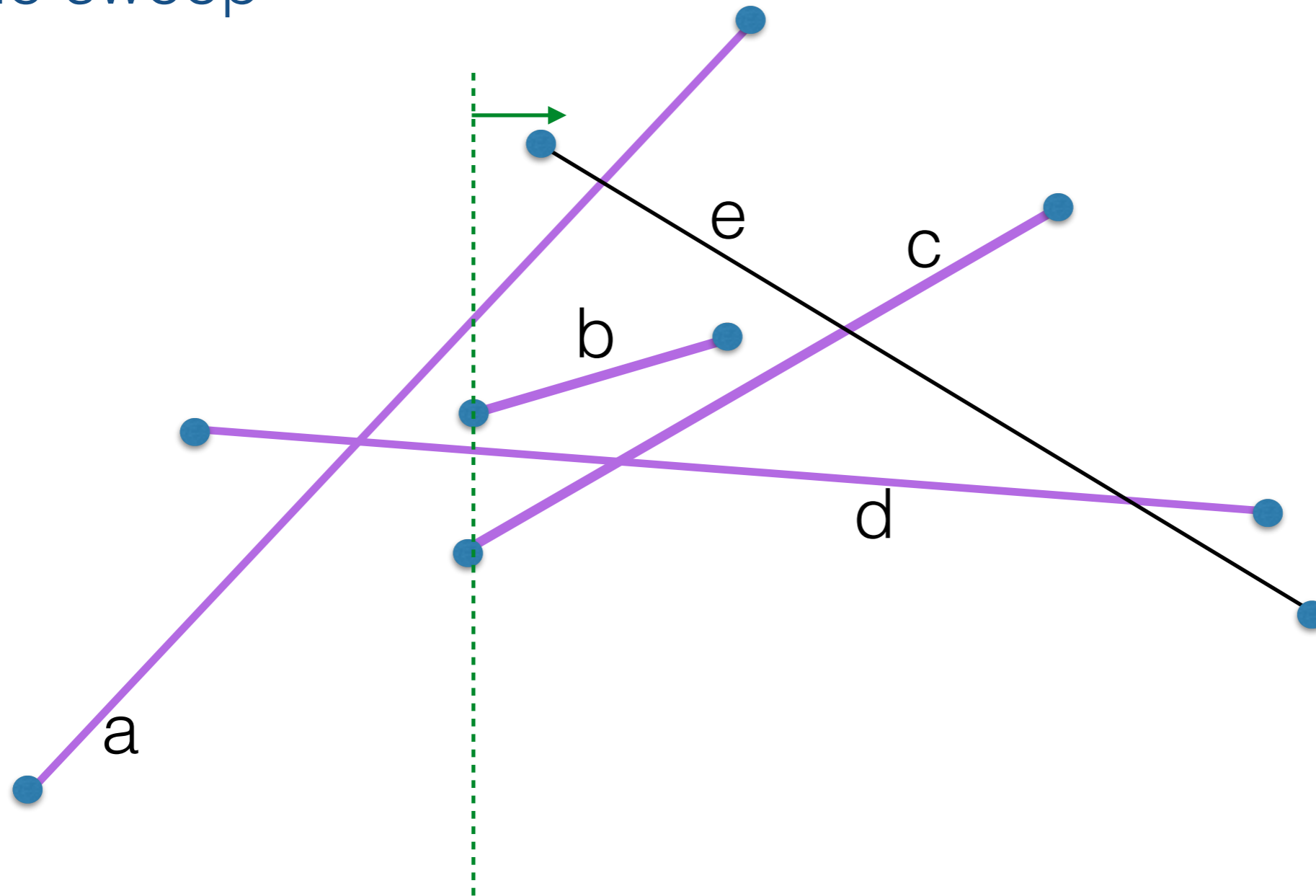
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The sweep



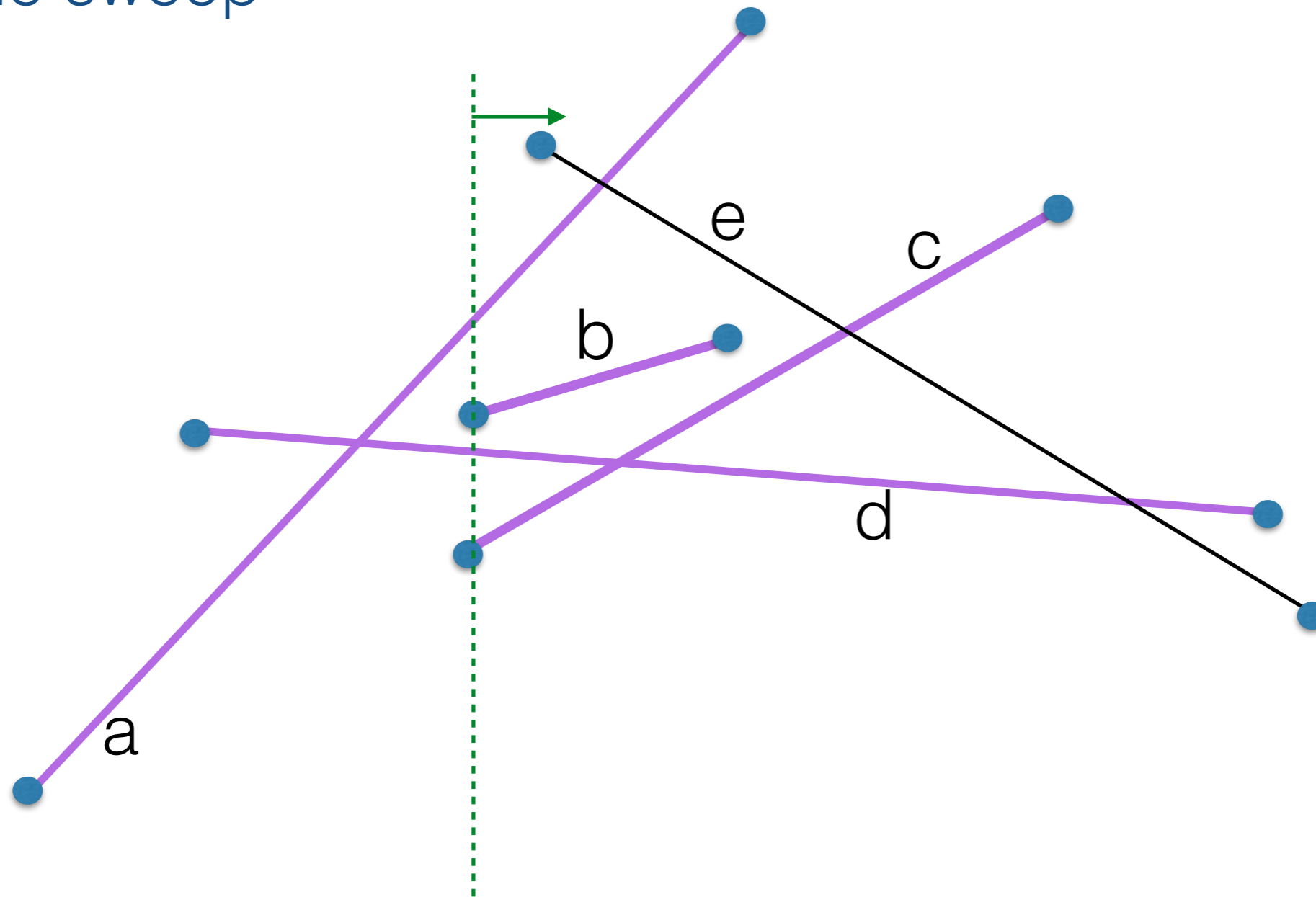
- Let X be the set of all x-coords of segments
- Traverse the events in X in order

The sweep



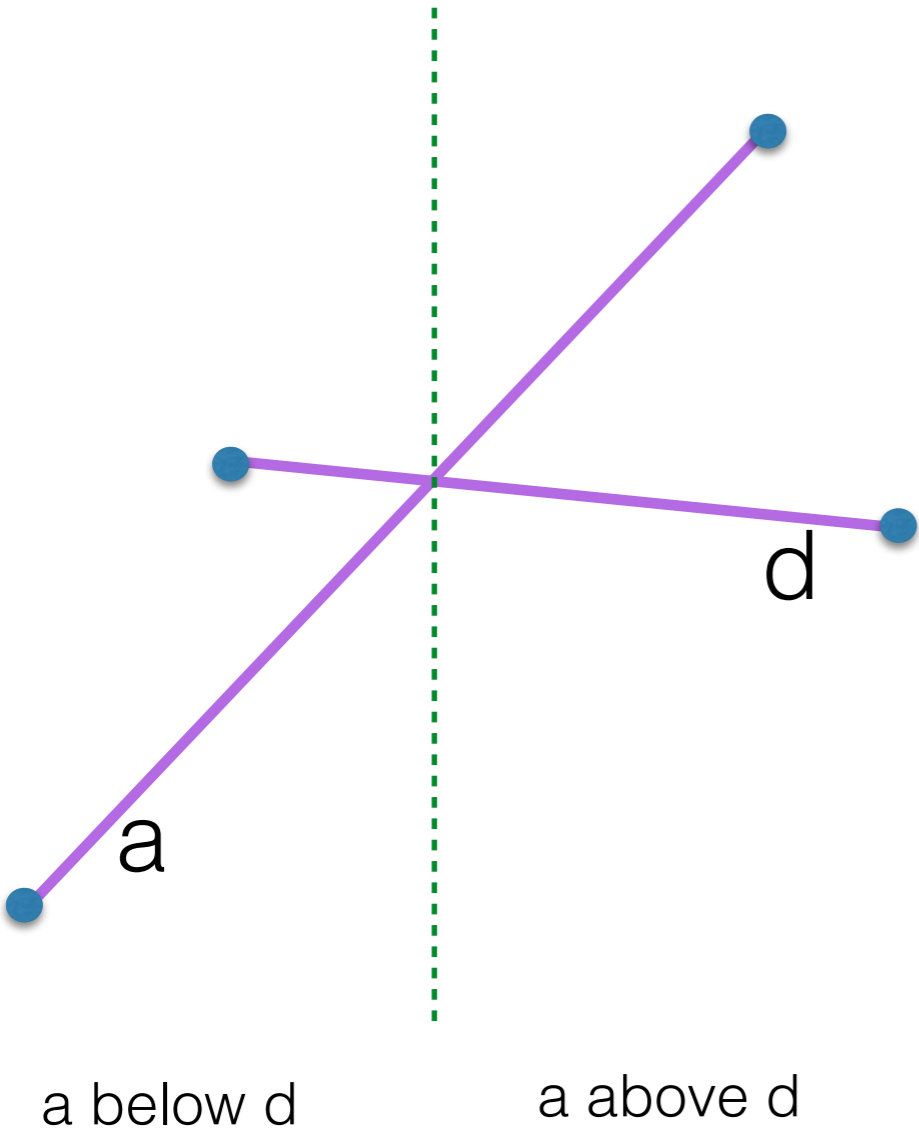
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The sweep

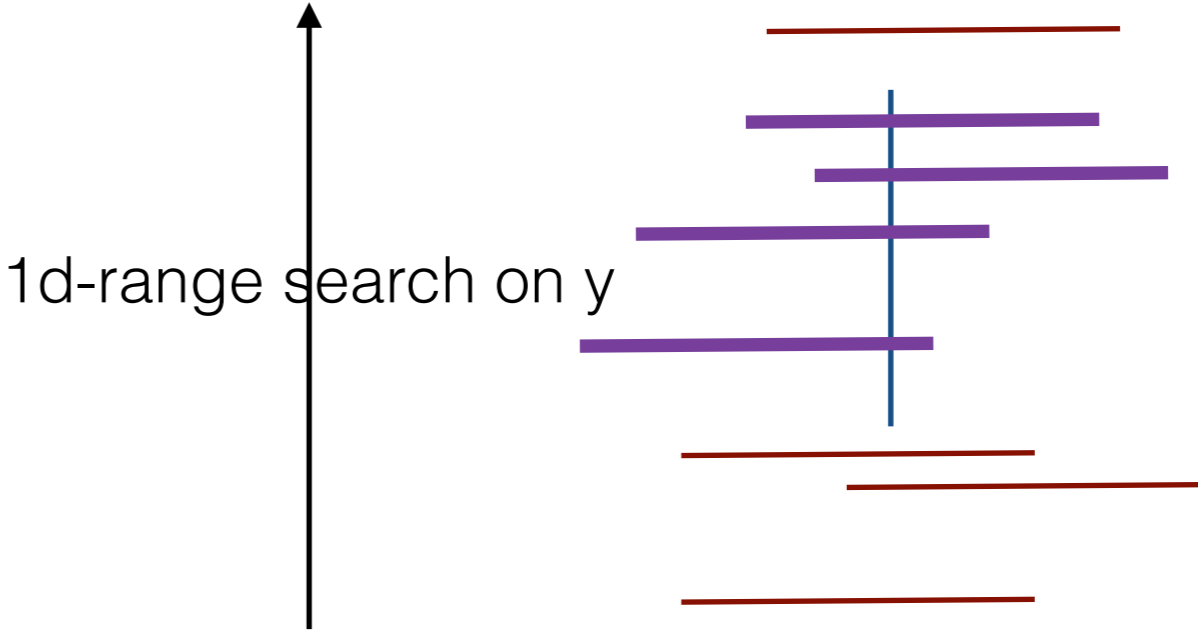


- 4 segments are active
- How do we order these segments?
- How do we detect intersections?

How do we order these segments?

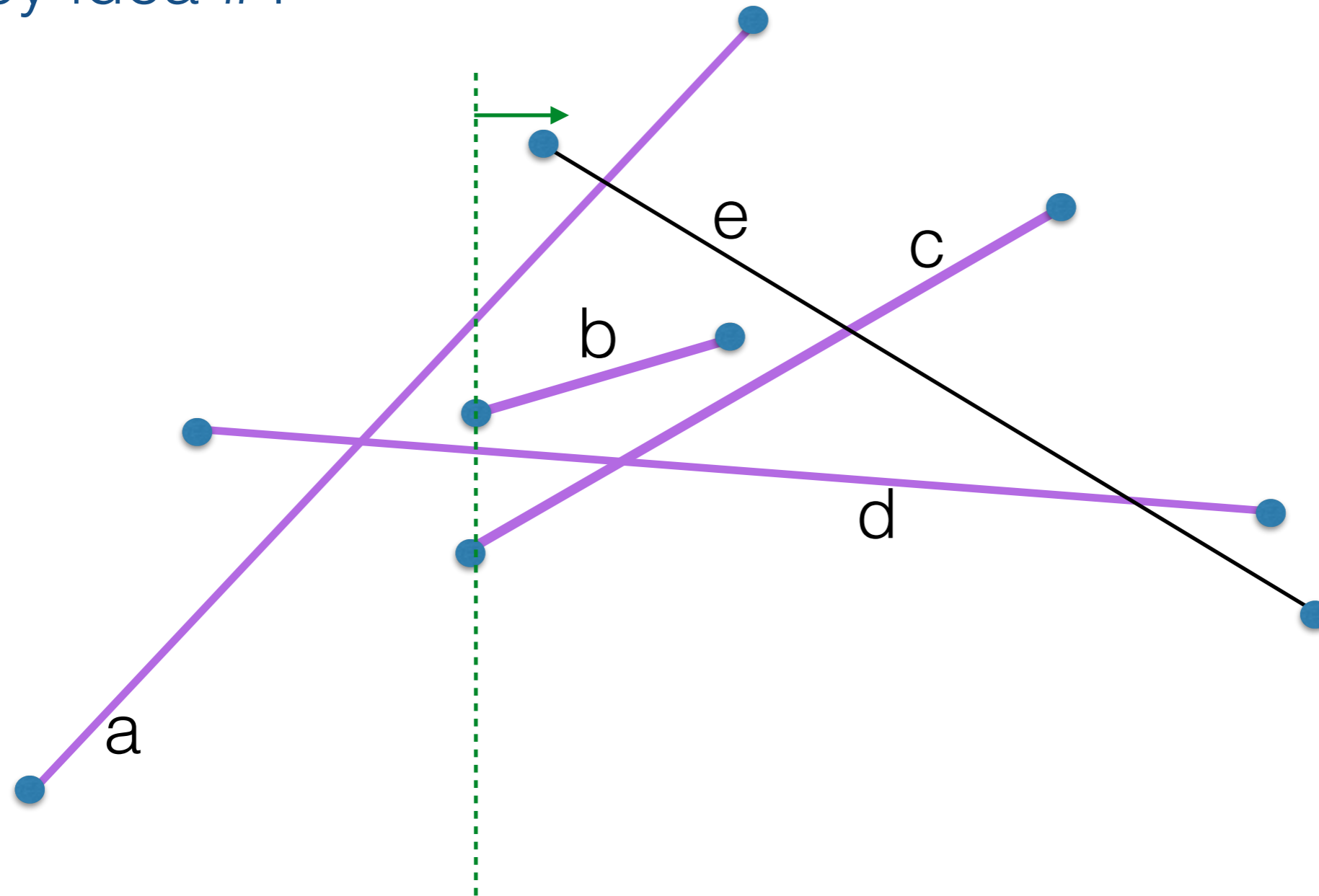


general segments



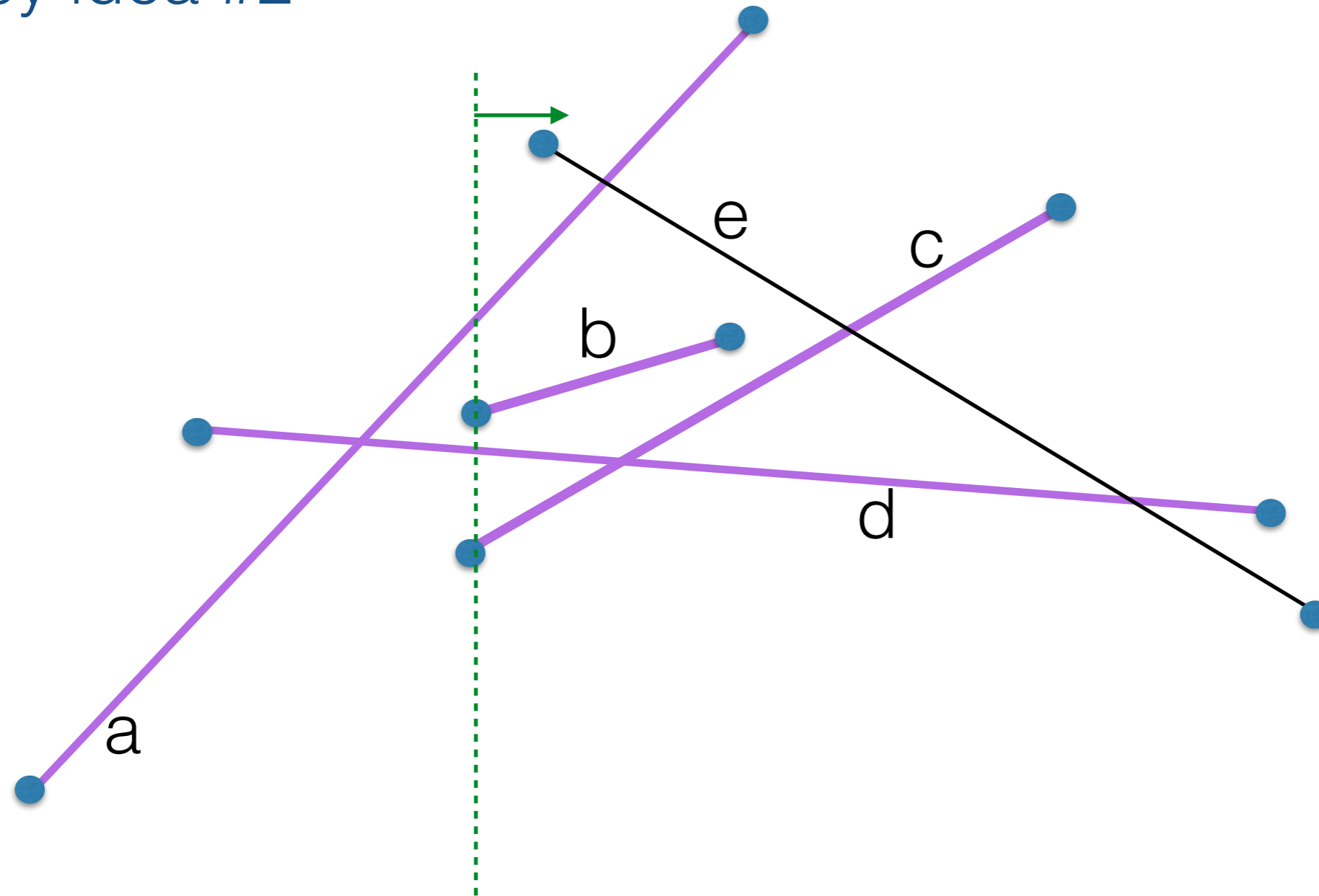
orthogonal segments

Key idea #1



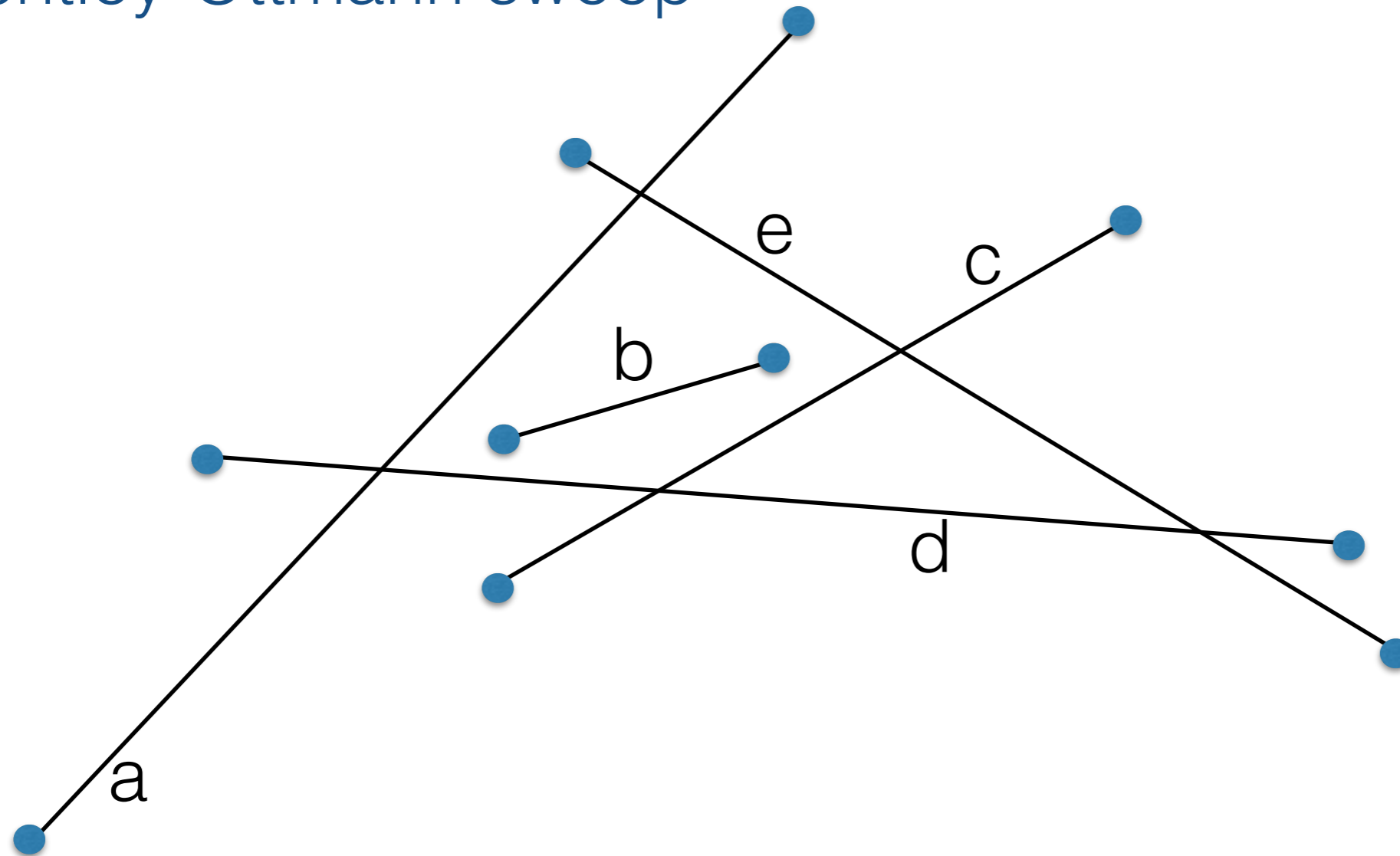
- How do we order the active segments?
 - Use above-below order
 - Order will flip at intersection point

Key idea #2



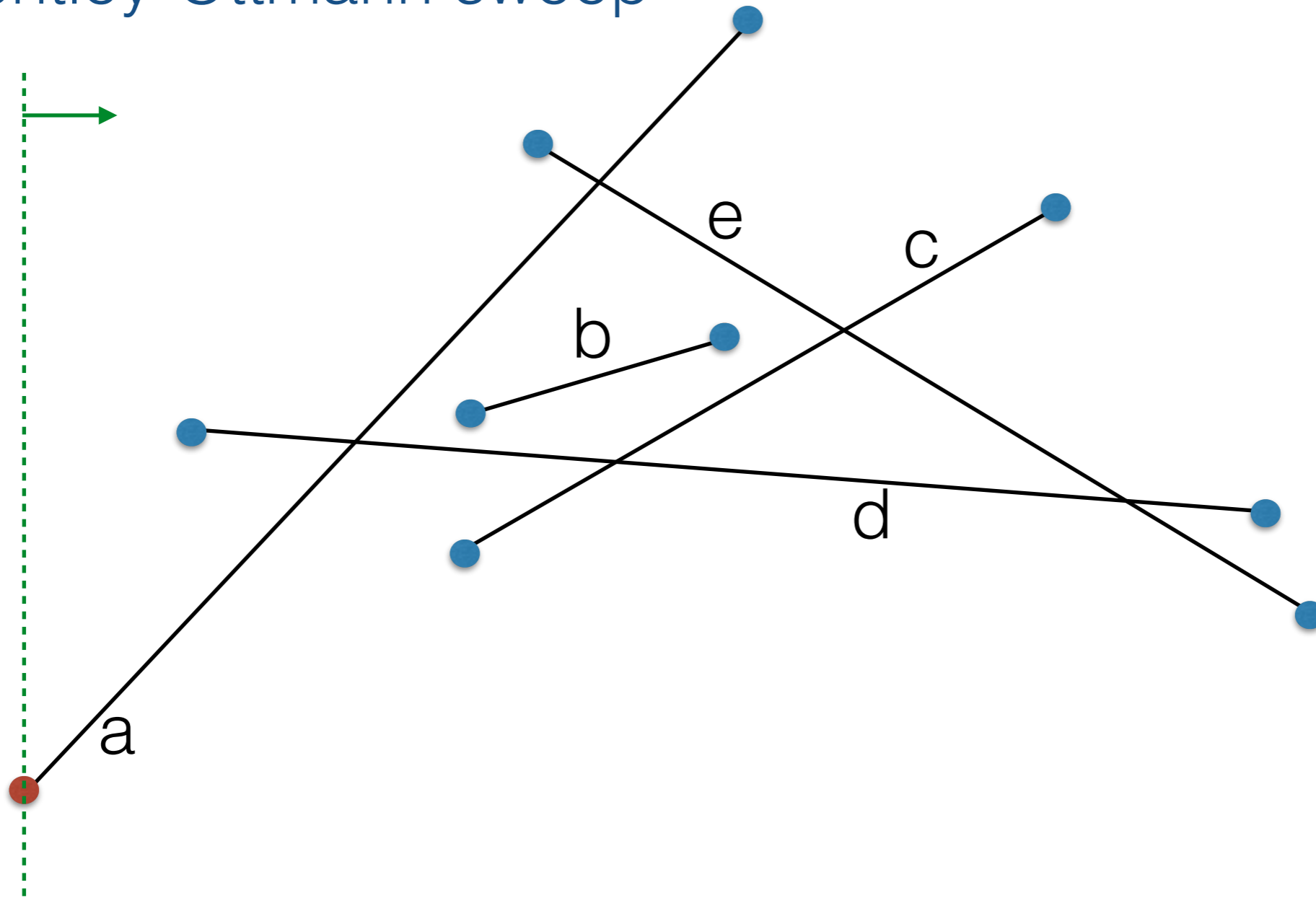
- How do we detect intersections?
 - Segments that intersect are consecutive in above-below order just before they intersect

Bentley-Ottmann sweep

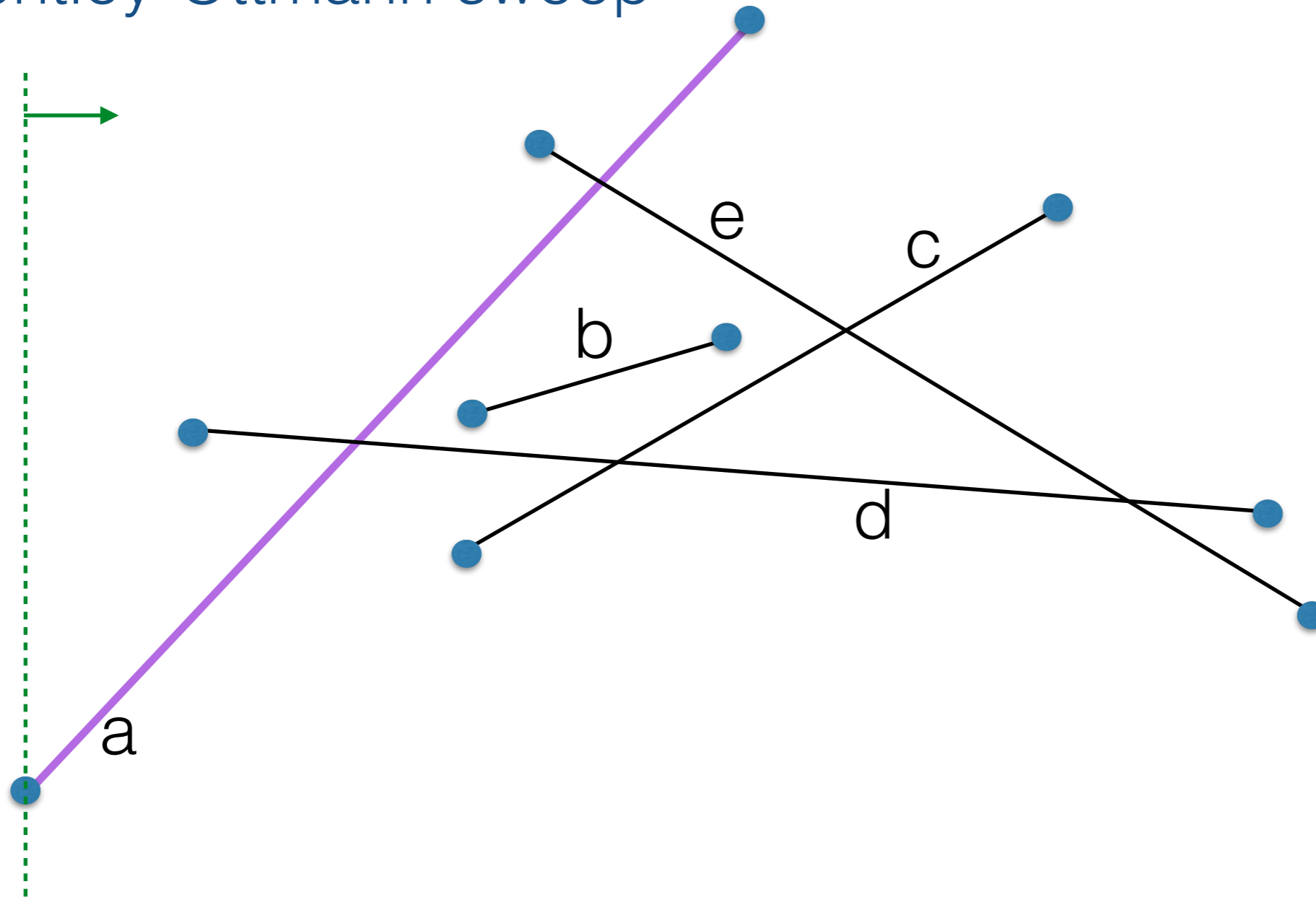


- Let X be the set of all x-coords of segments
- Initialize $AS = \{\}$
- Traverse events in order

Bentley-Ottmann sweep

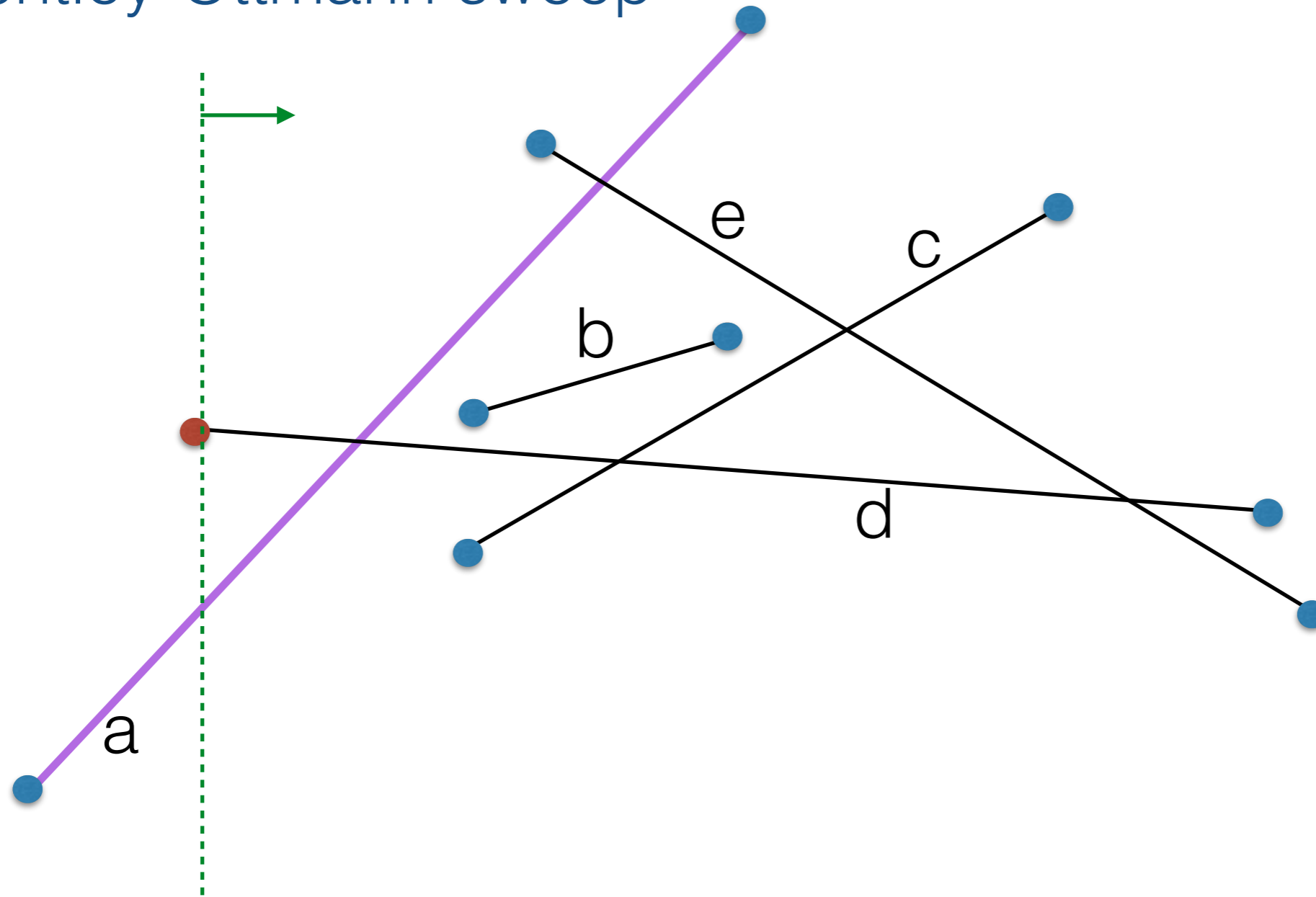


Bentley-Ottmann sweep

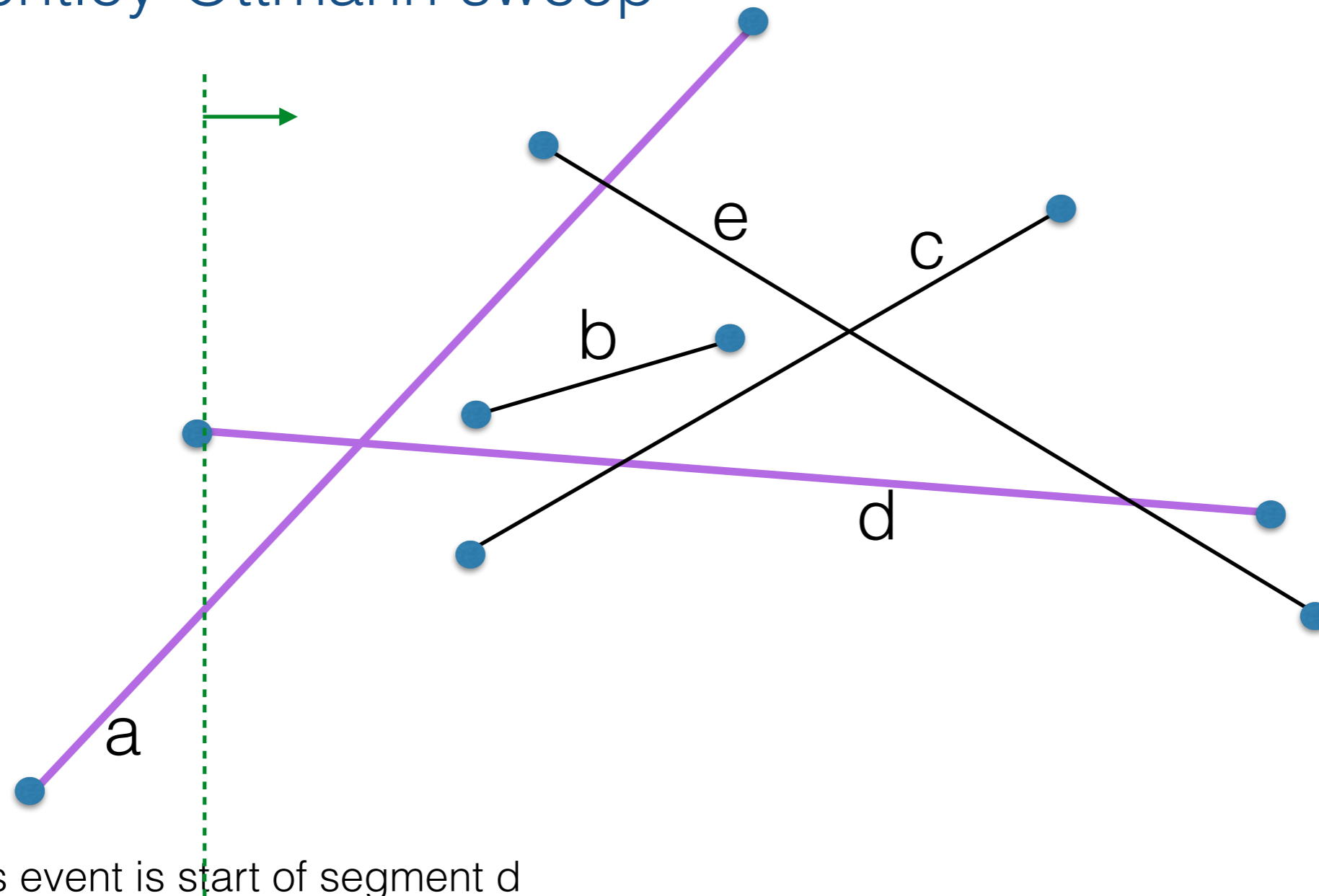


- this event is start of segment a:
 - insert a in AS: a

Bentley-Ottmann sweep

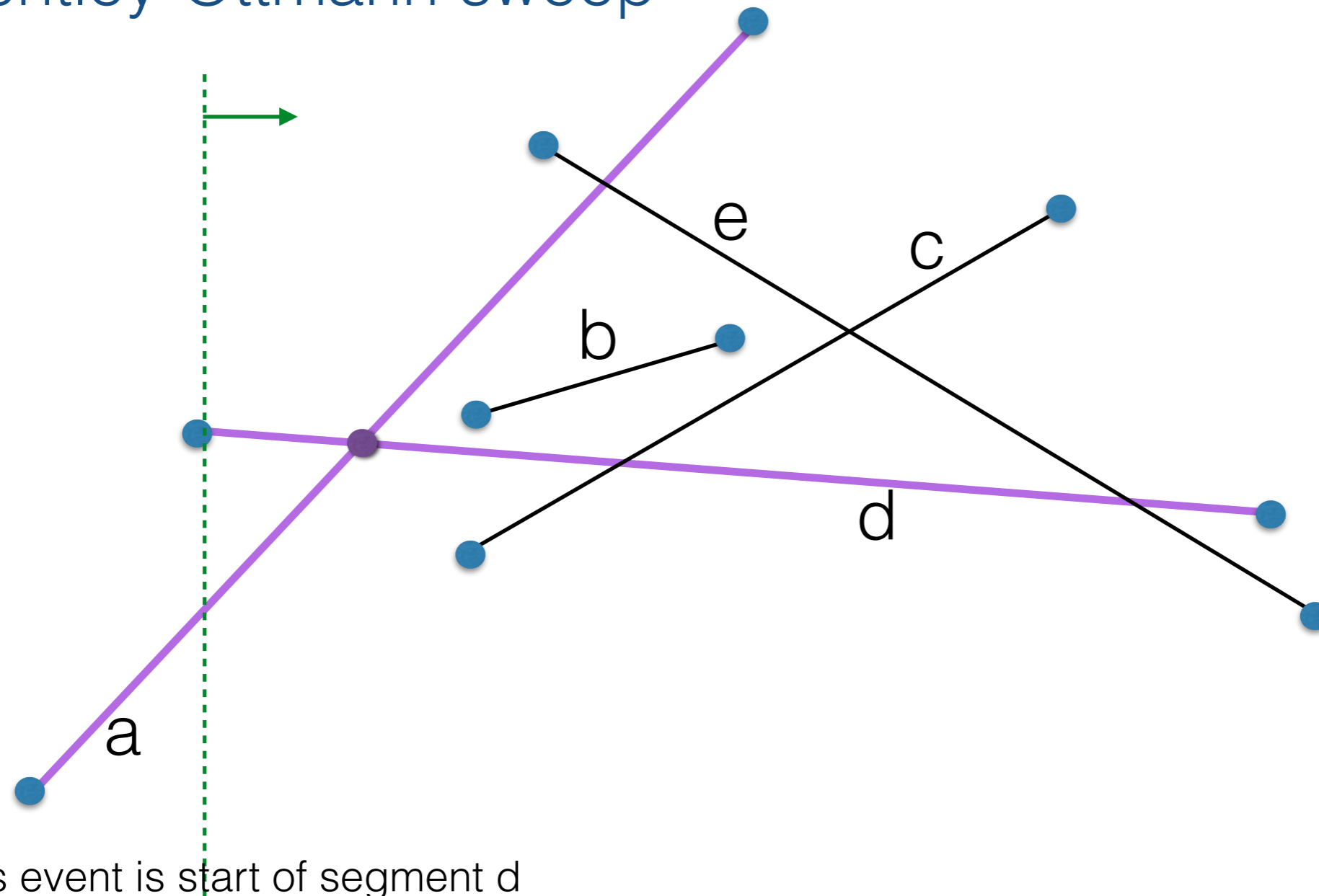


Bentley-Ottmann sweep



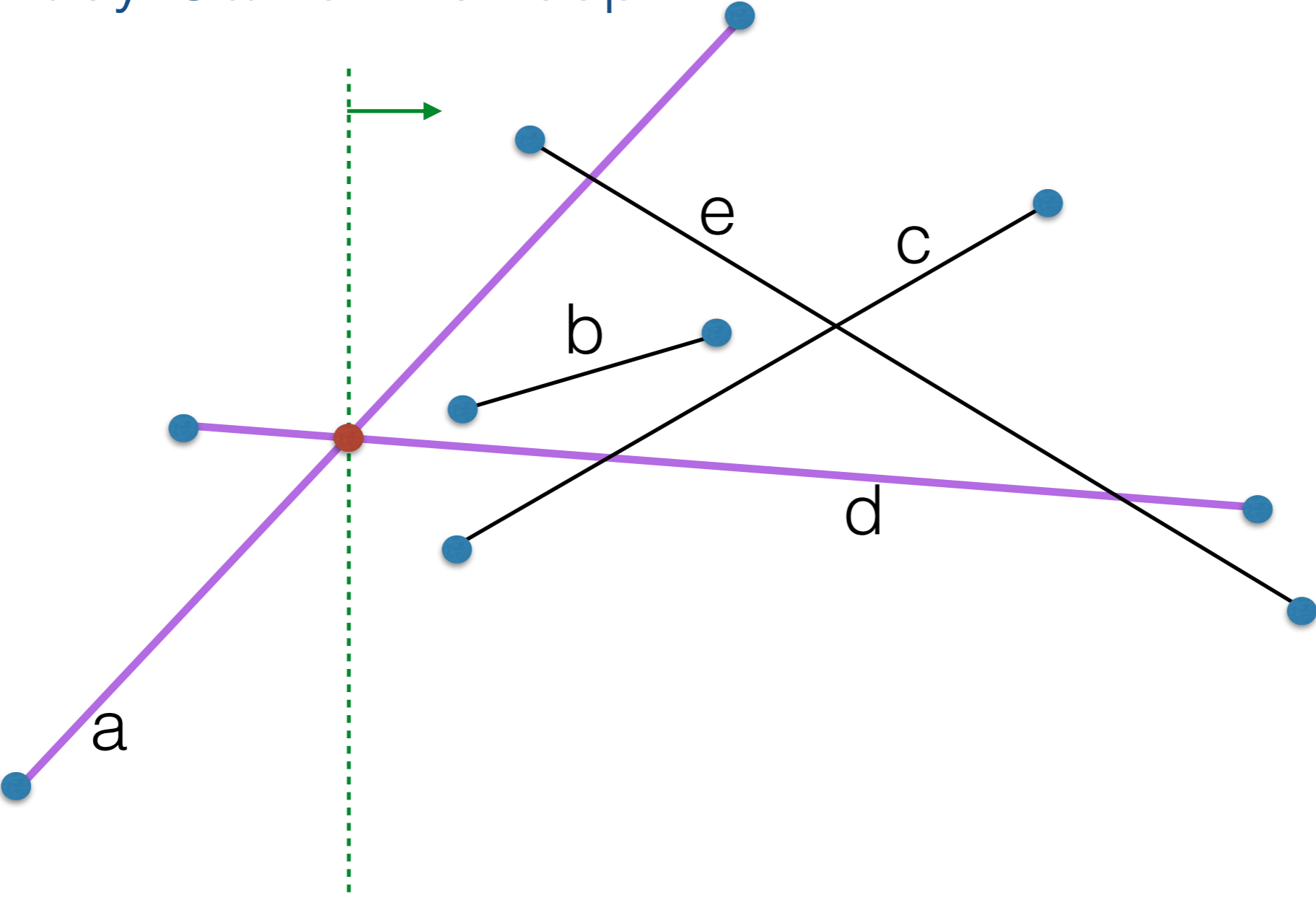
- this event is start of segment d
 - insert d in AS: $a < d$
 - check if (d,a) intersect to the right of the line; they do; report point and insert it in the list of future events

Bentley-Ottmann sweep

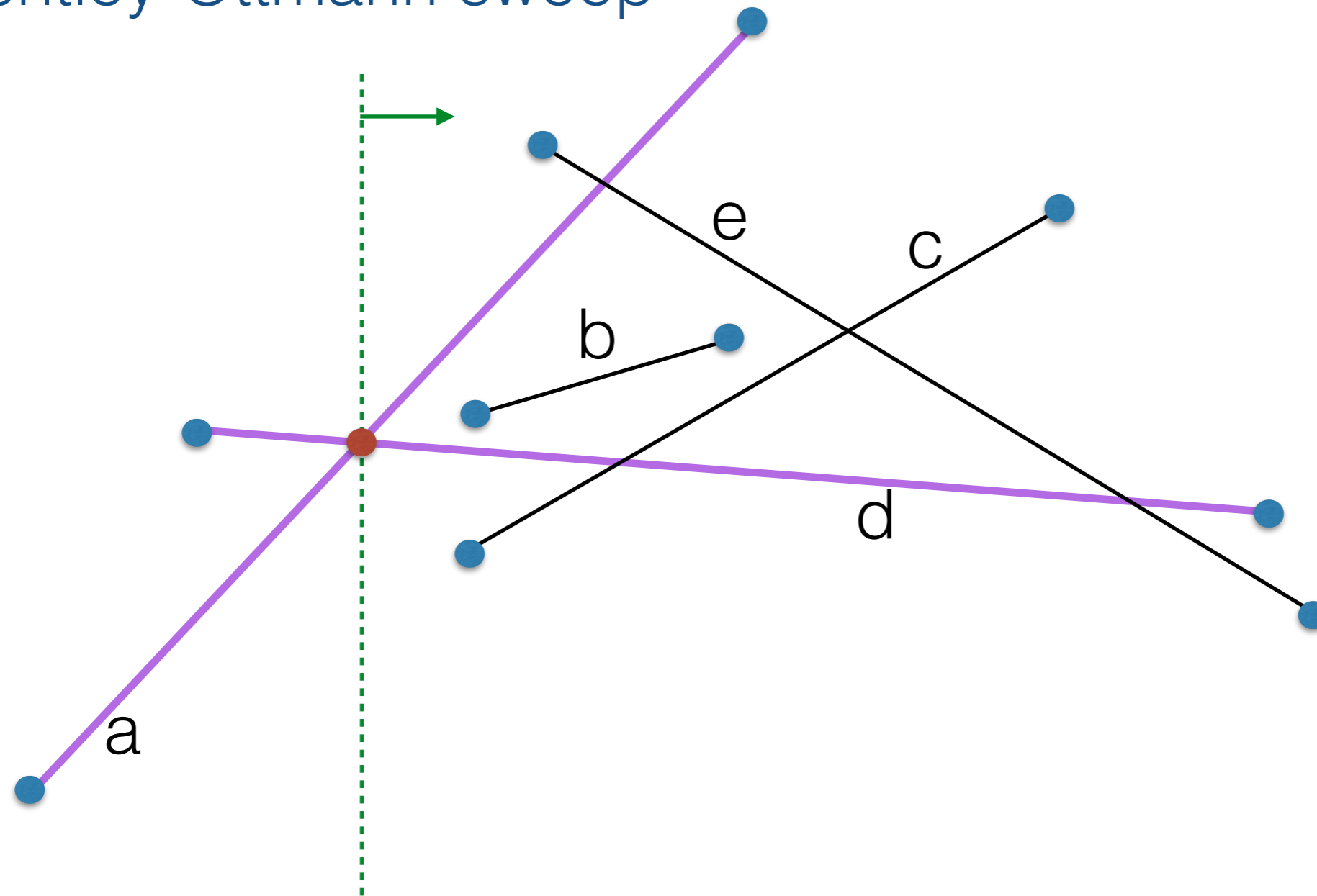


- this event is start of segment d
 - insert d in AS, d is above a ($a < d$)
 - check if (d,a) intersect to the right of the line; they do; report point and insert it in the list of future events

Bentley-Ottmann sweep

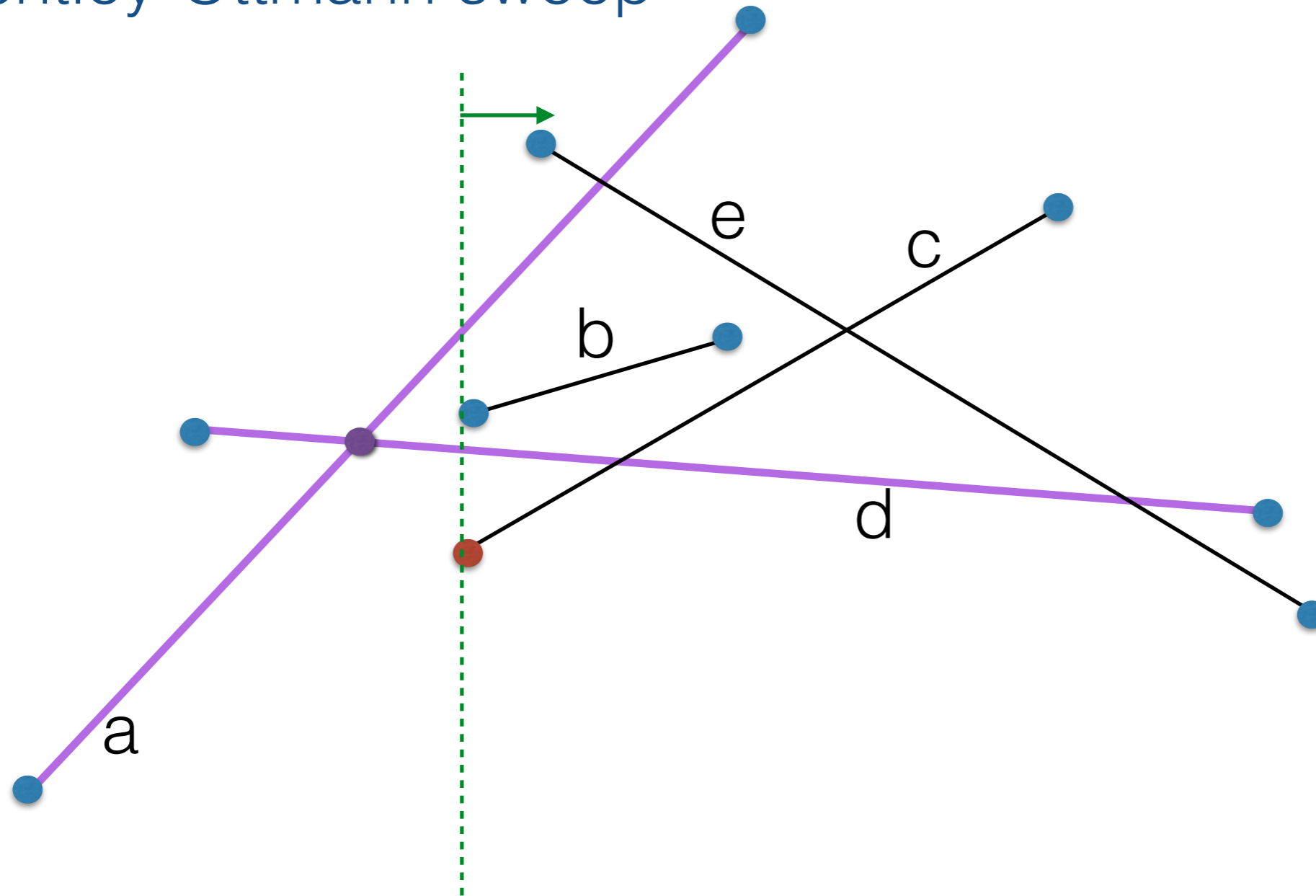


Bentley-Ottmann sweep

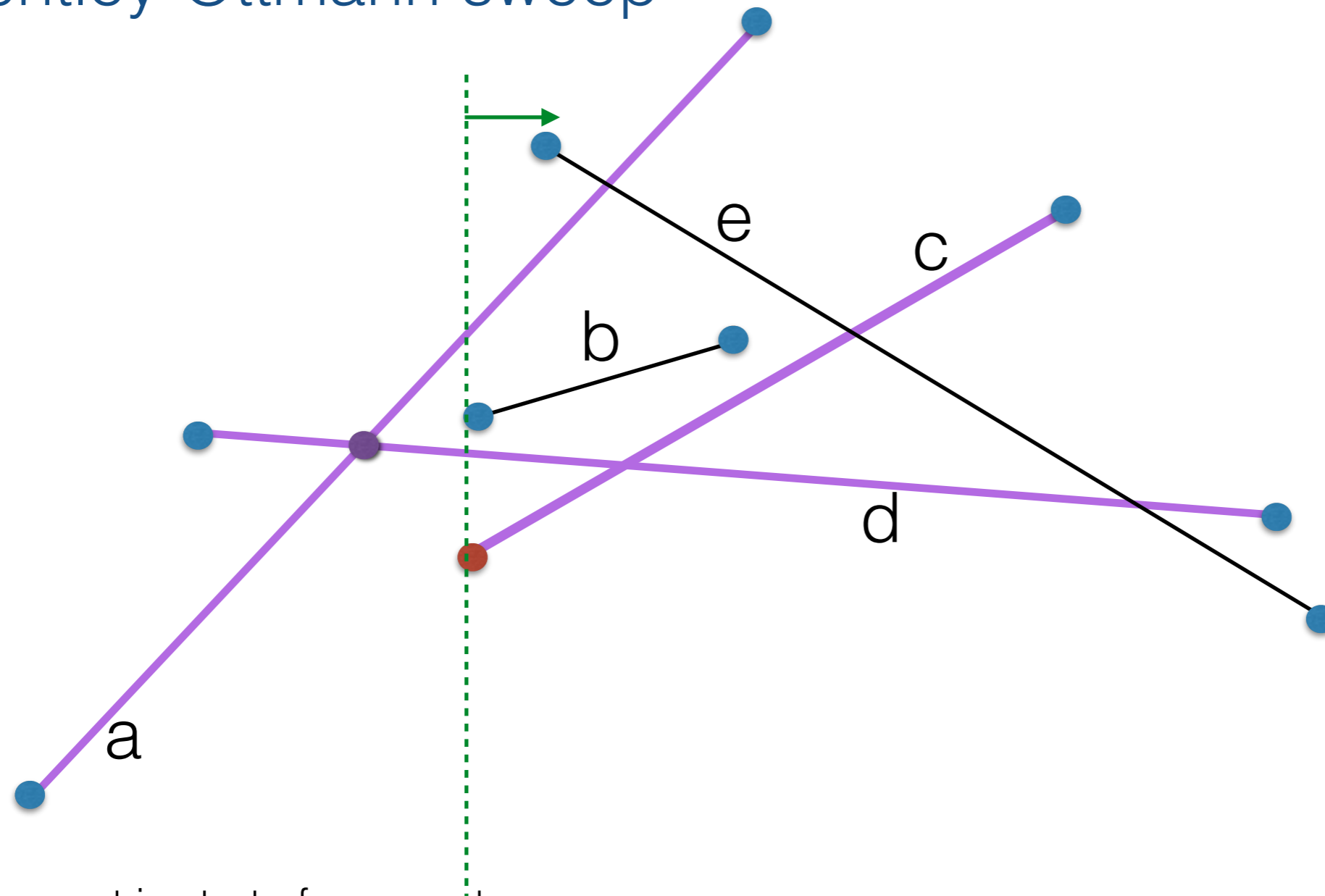


- this event is an intersection point of (a,d):
 - flip a and d in AS: a is now above d ($d < a$)

Bentley-Ottmann sweep

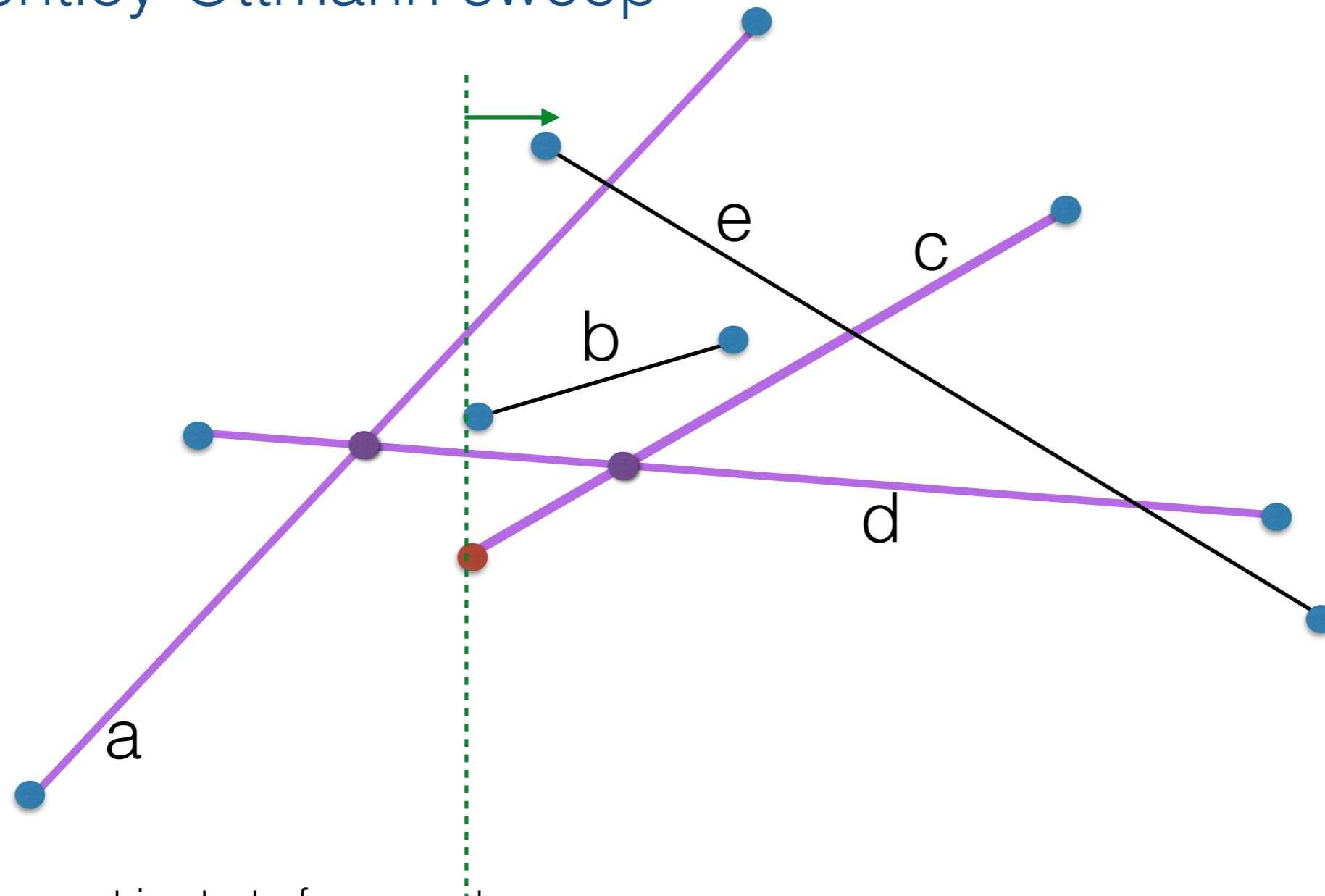


Bentley-Ottmann sweep



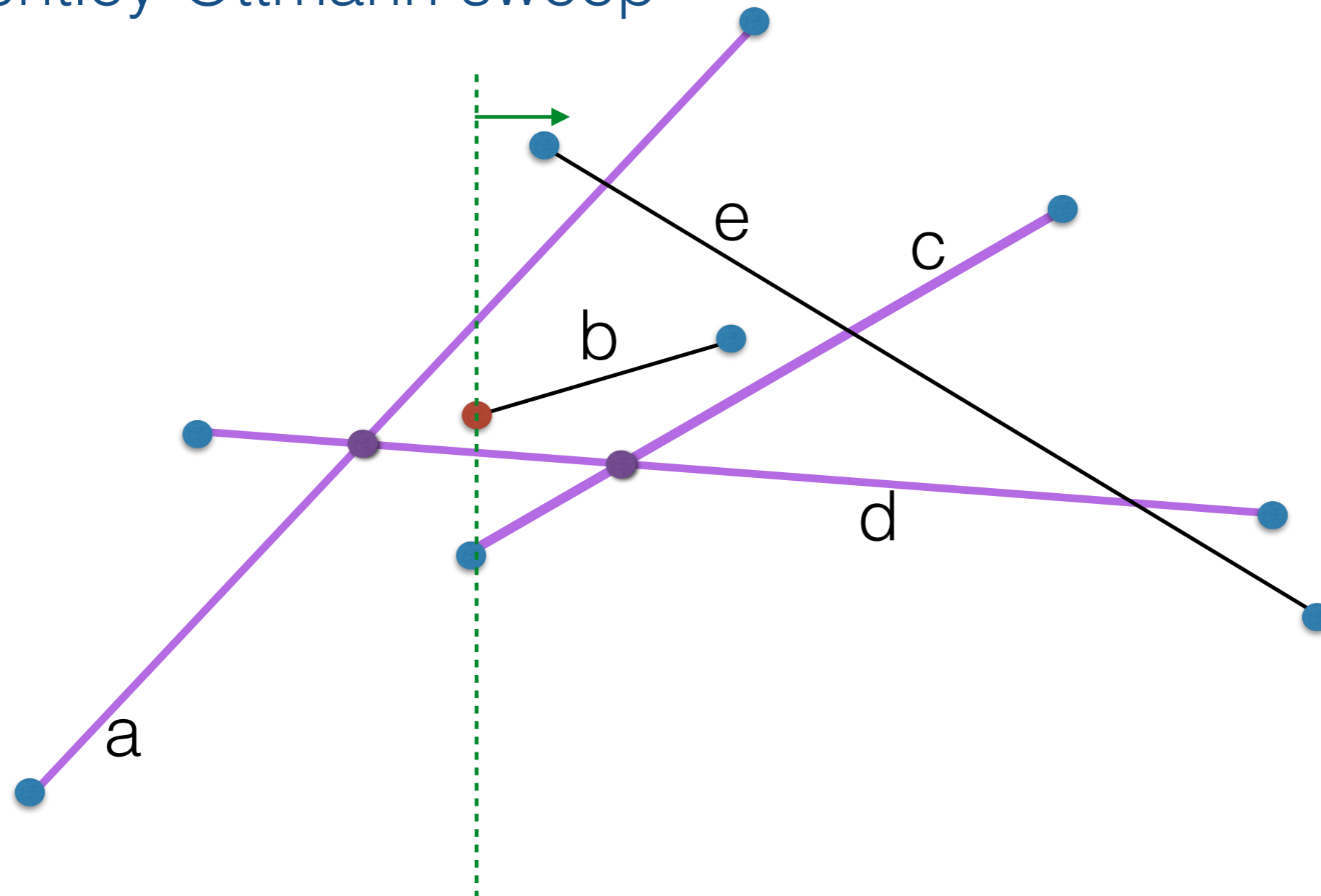
- this event is start of segment c:
 - insert c in AS; c is below d ($c < d < a$)
 - check c with its above and below neighbors for intersection to the right of the sweep line; this detects the intersection point of c and d; report it and insert it as future event

Bentley-Ottmann sweep



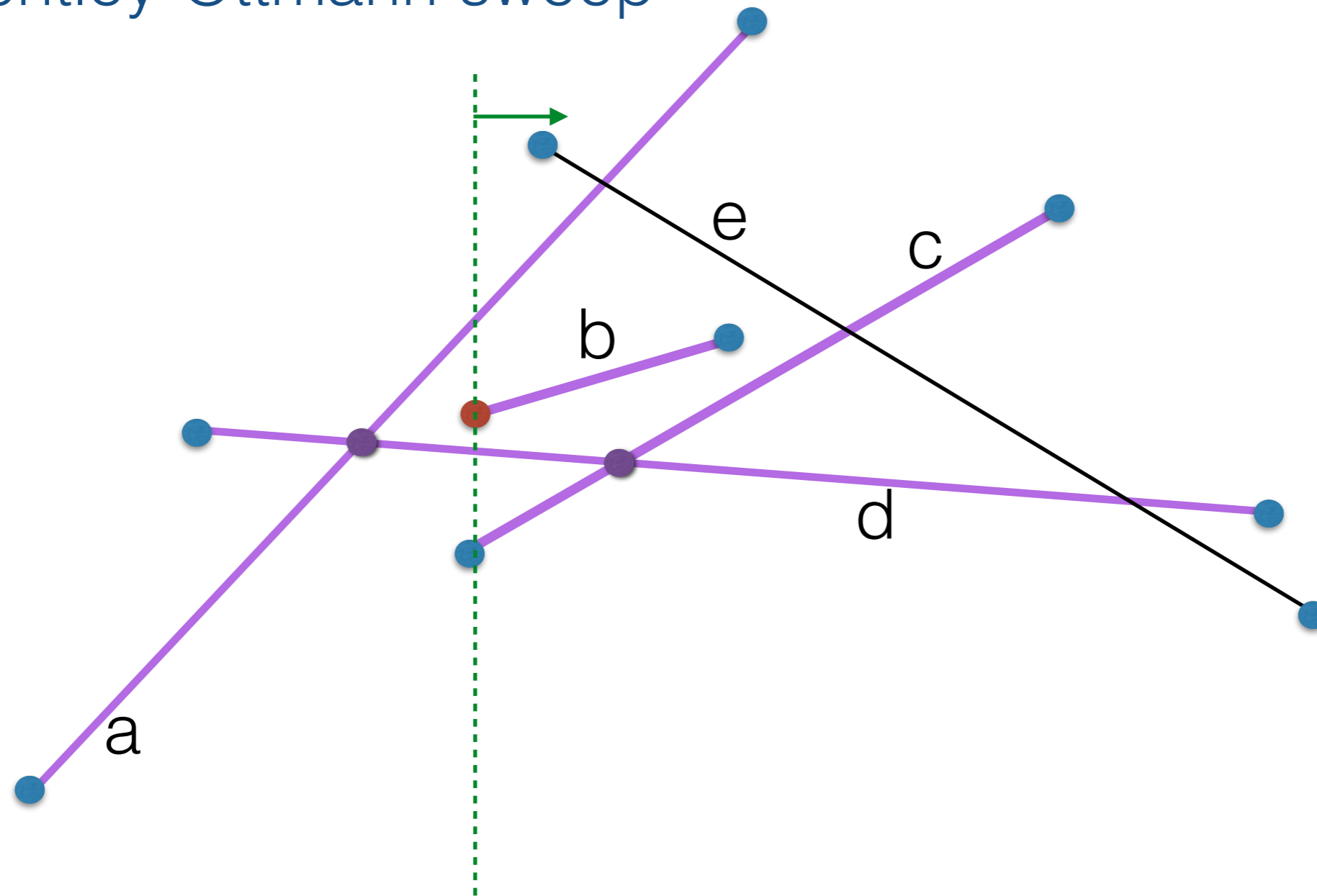
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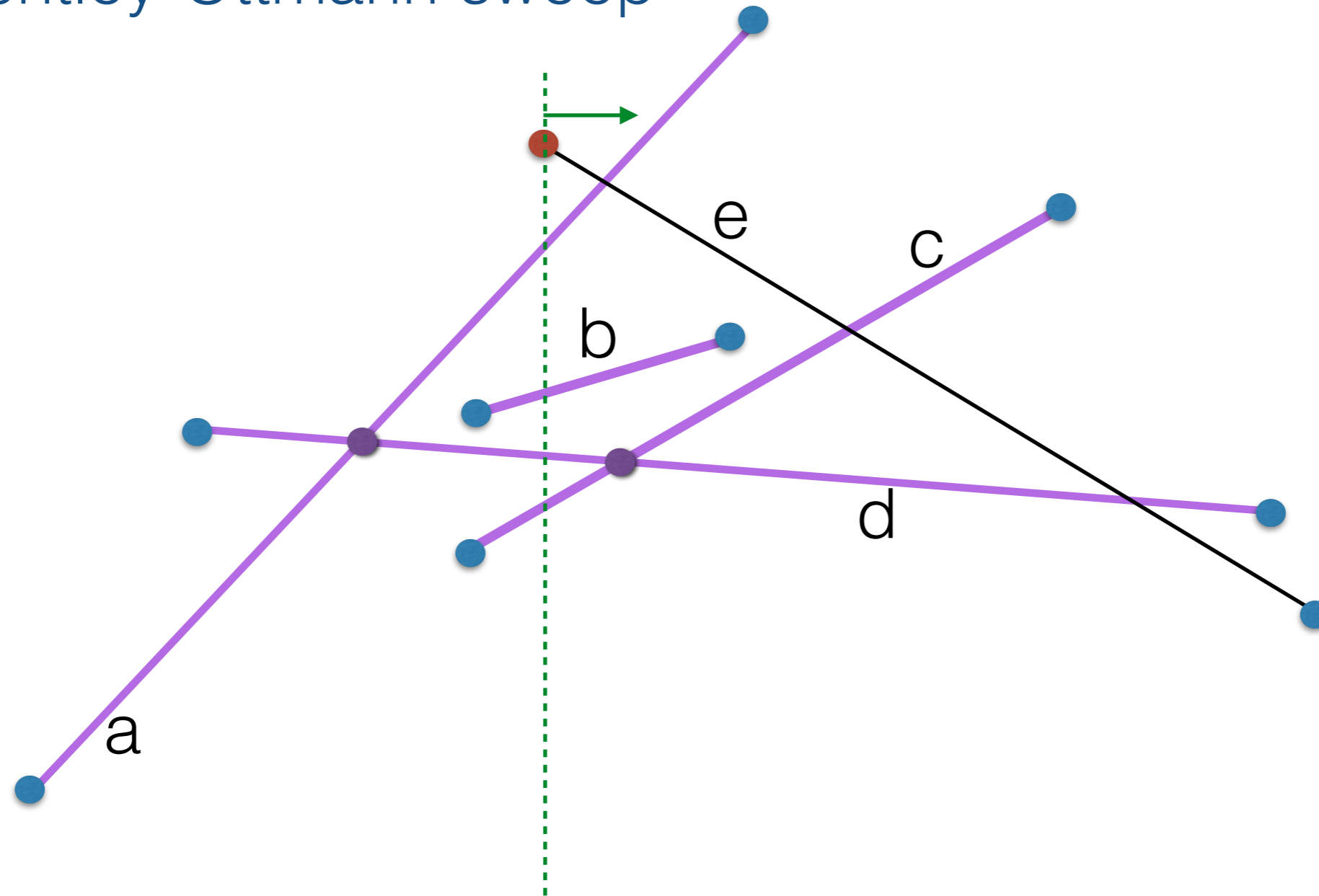
- this event is start of segment b:

Bentley-Ottmann sweep



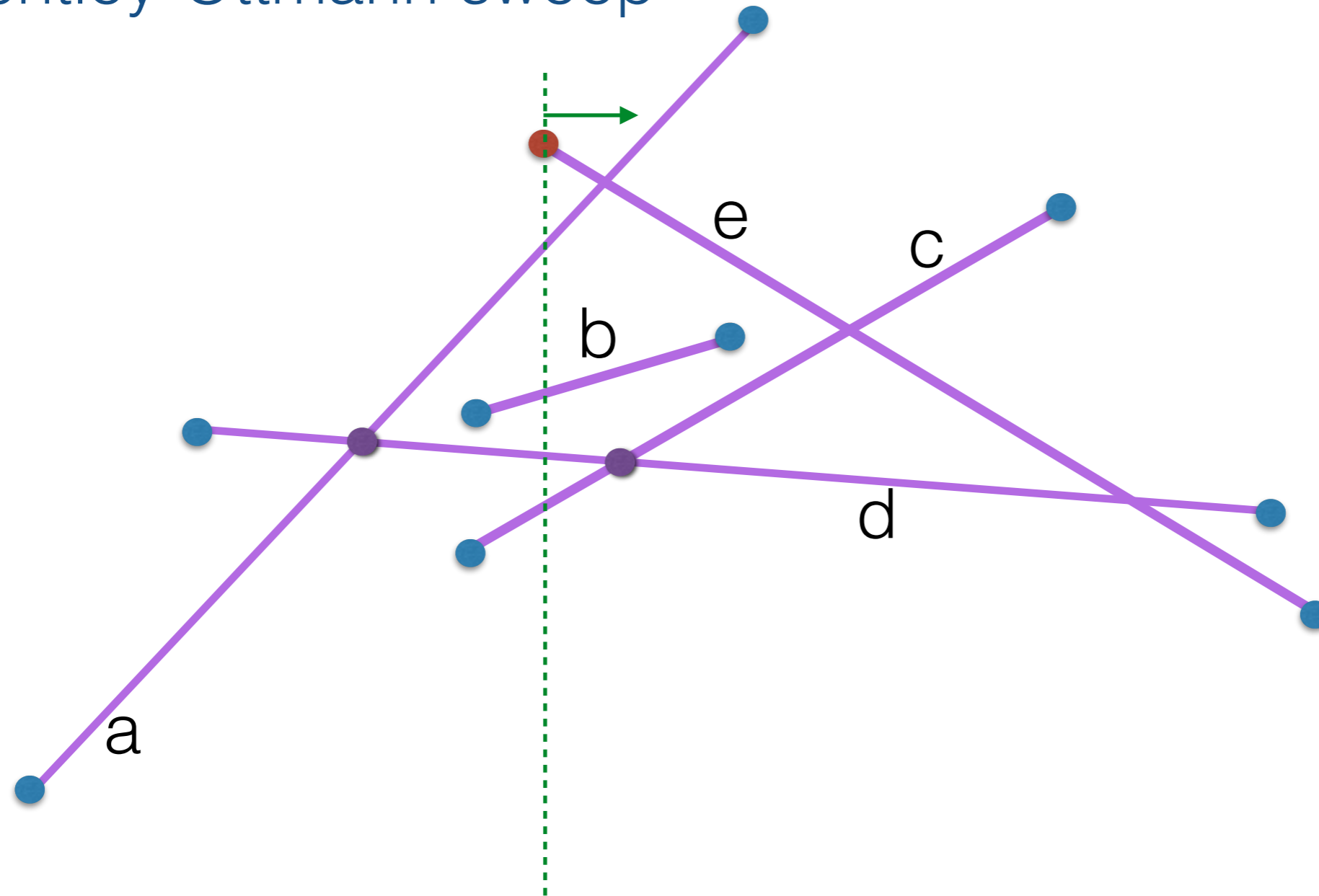
- this event is start of segment b:
 - insert b in AS; $c < d < b < a$
 - check b with its above and below neighbors for intersection to the right of the sweep line; (d,b) don't intersect; (b, a) don't intersect

Bentley-Ottmann sweep



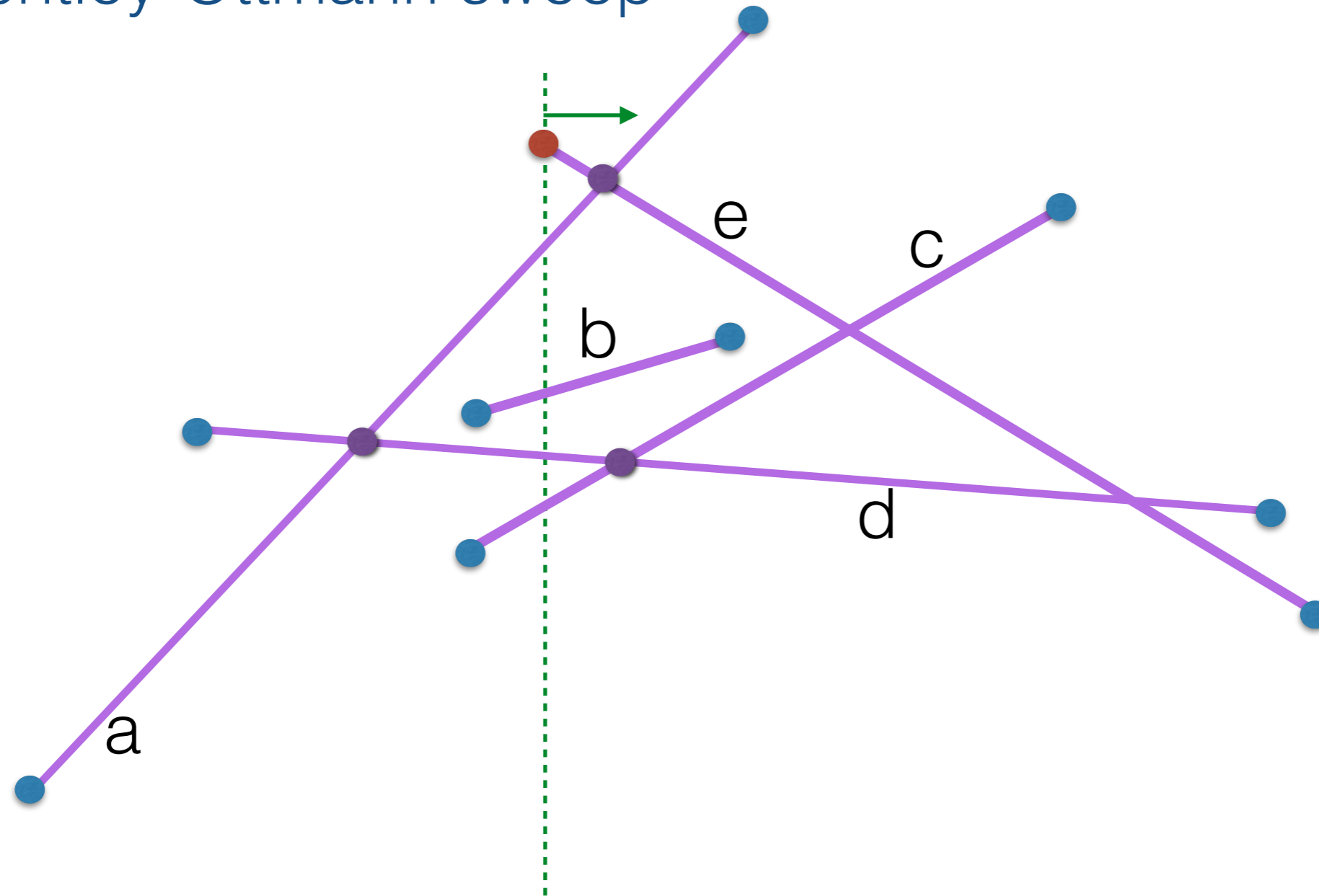
- this event is start of segment e:

Bentley-Ottmann sweep



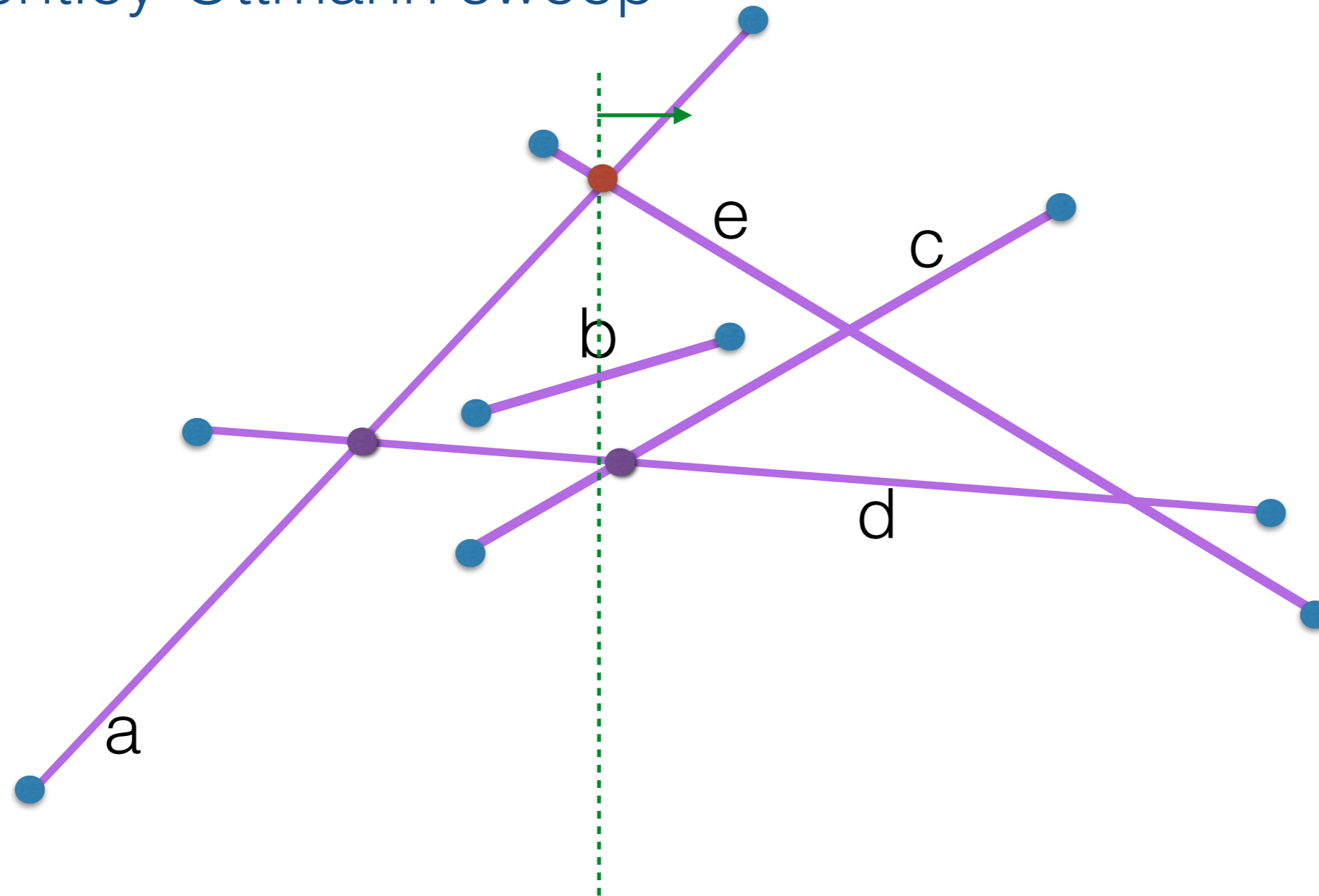
- this event is start of segment e:
 - insert e in AS: $c < d < b < a < e$
 - check e with its above and below neighbors for intersection to the right of the sweep line; this detects intersection point of (a,e); report it and insert it as future event

Bentley-Ottmann sweep



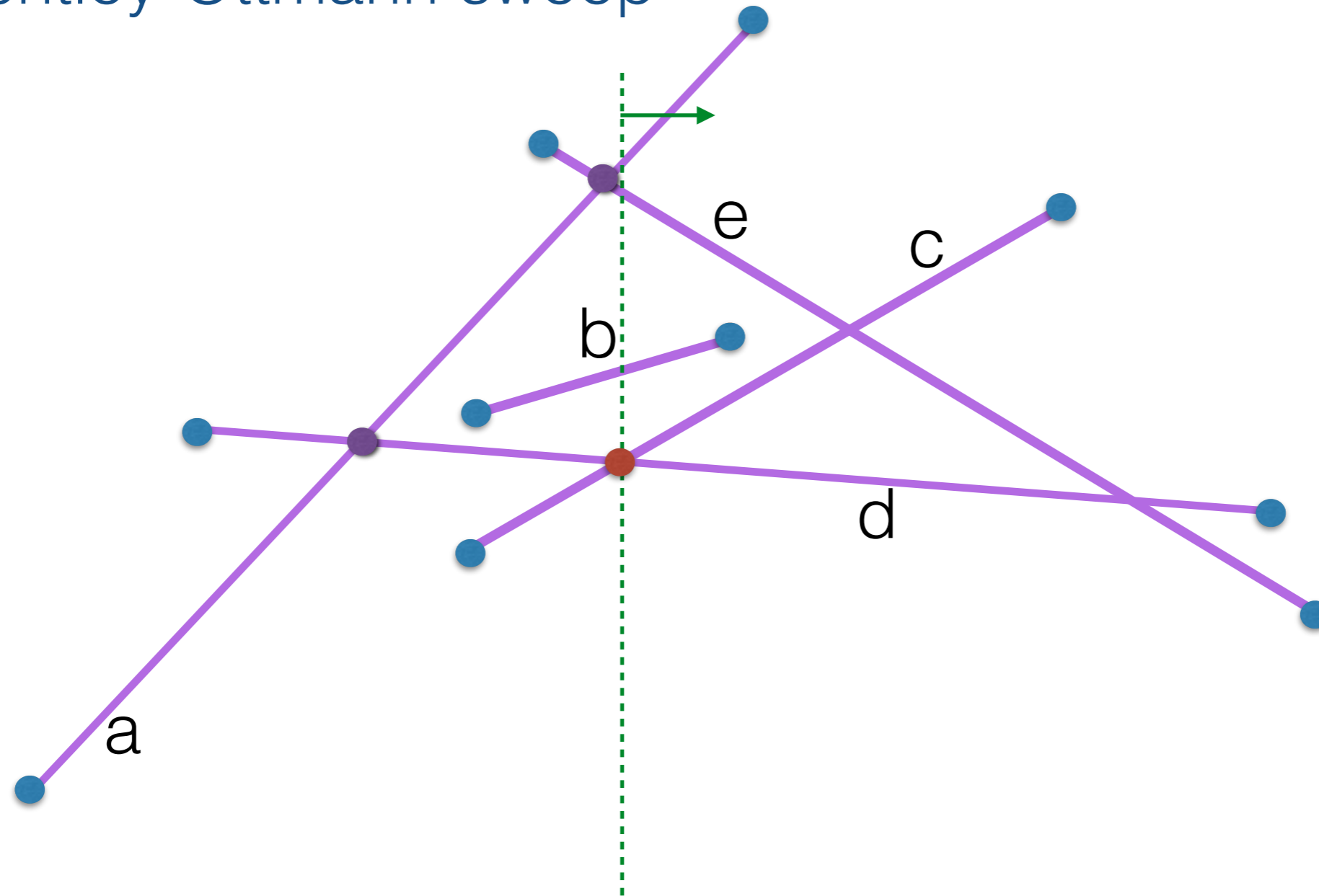
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Bentley-Ottmann sweep



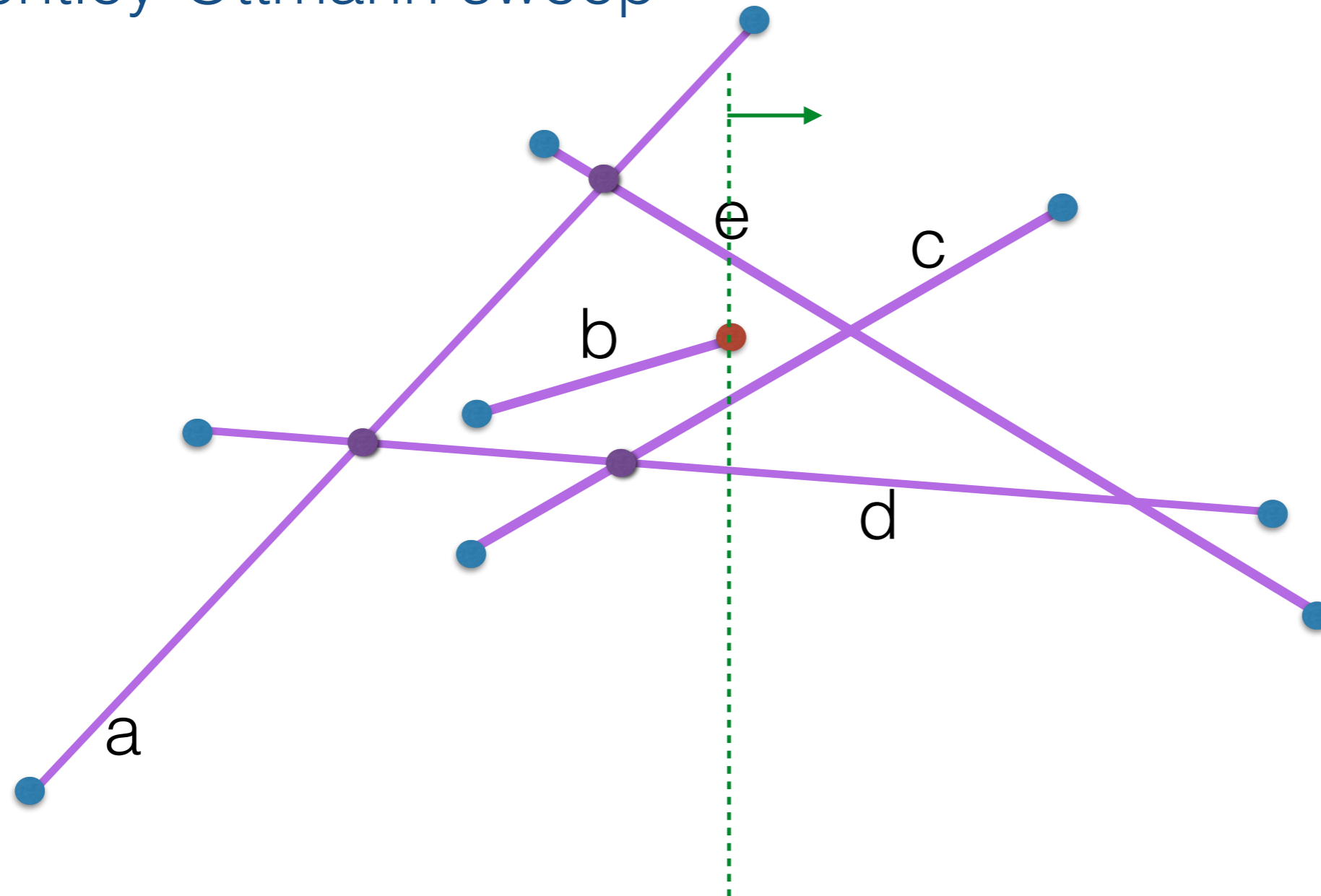
- this event is intersection of (a,e):
 - flip a and e: $c < d < b < e < a$
 - check new neighbors (e,b) for intersection to the right of the sweep line; (e,b) don't intersect

Bentley-Ottmann sweep



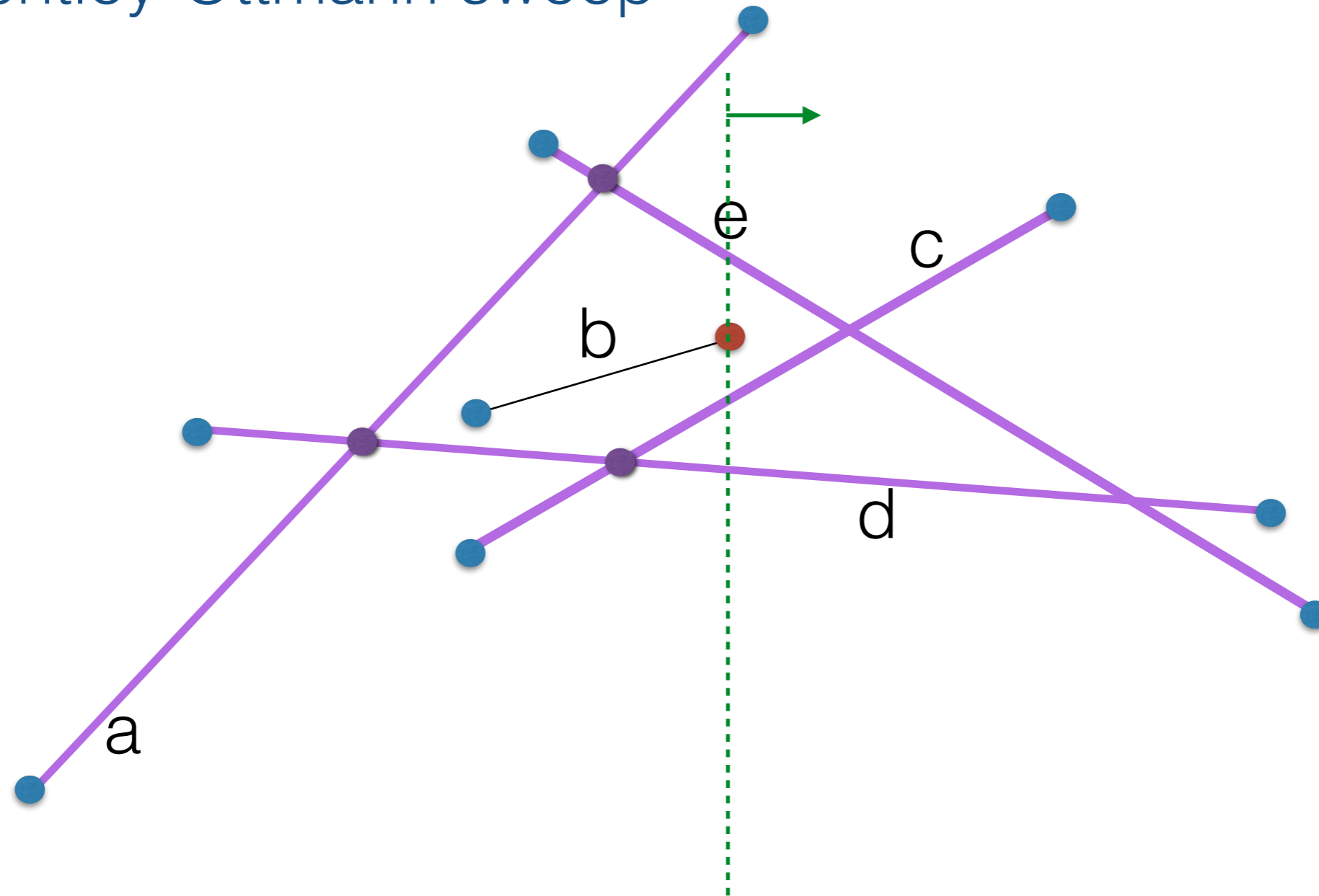
- this event is intersection of (c,d):
 - flip c and d: $d < c < b < e < a$
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Bentley-Ottmann sweep



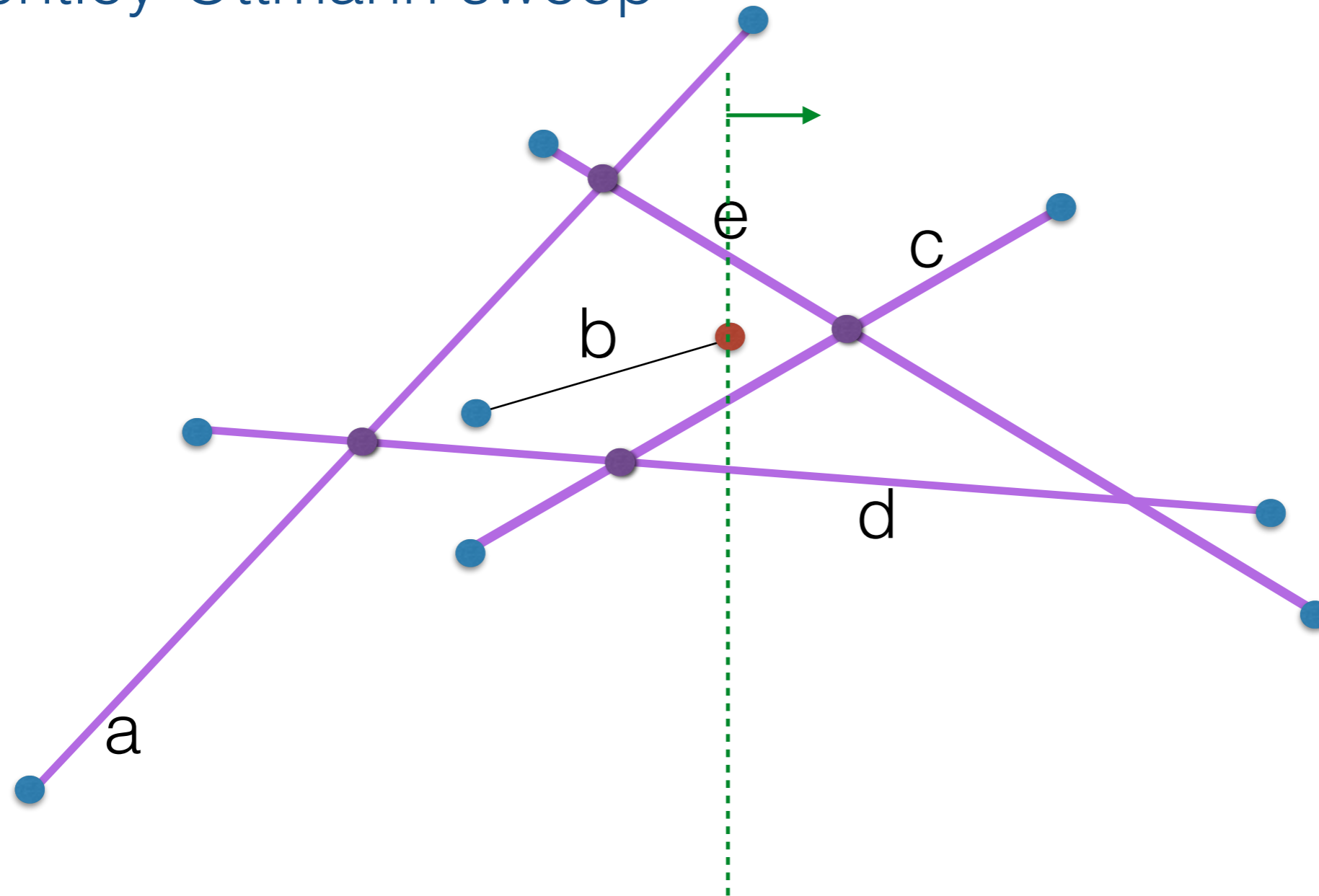
- this event is end of segment b:
 - delete b from AS: $d < c < e < a$
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Bentley-Ottmann sweep



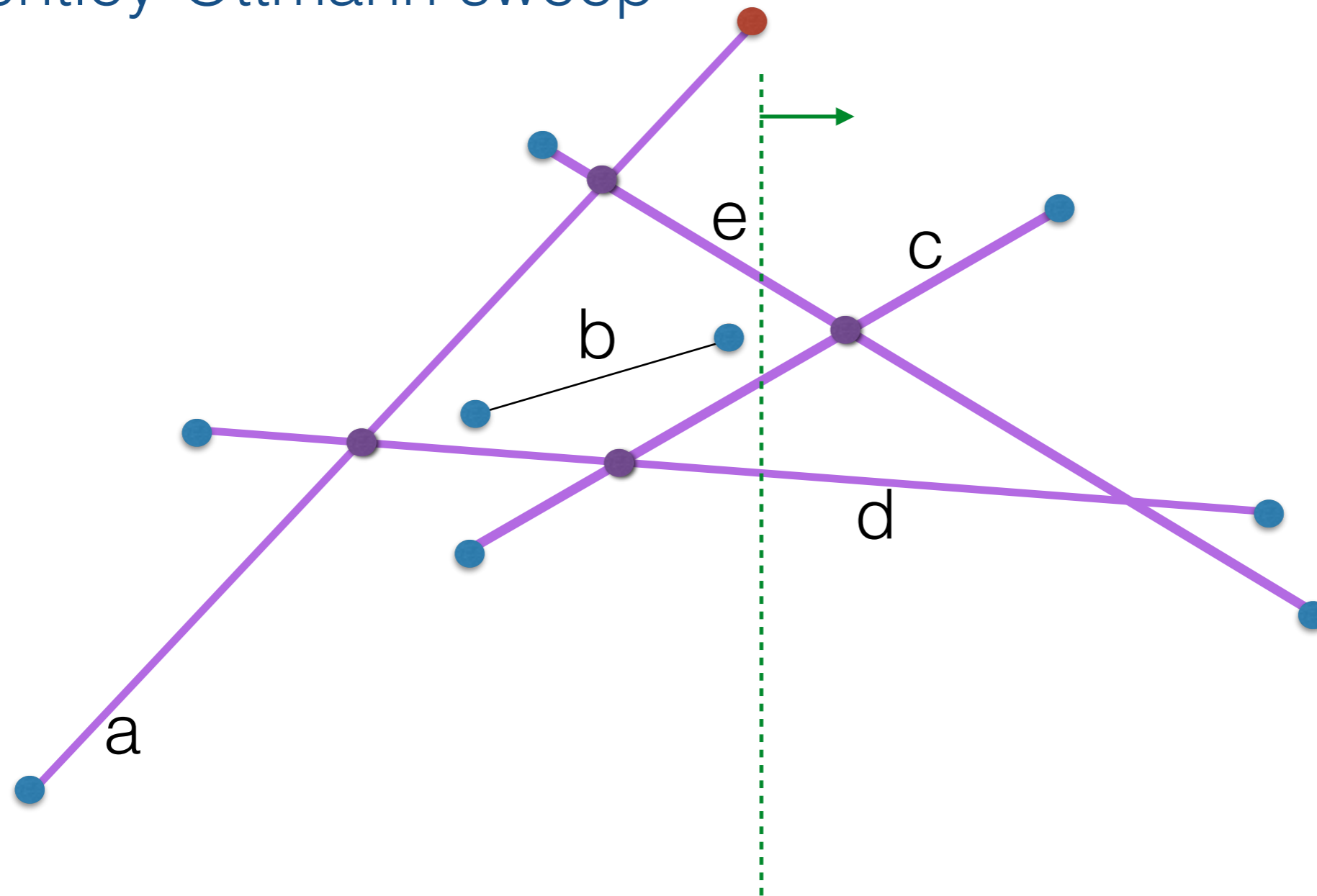
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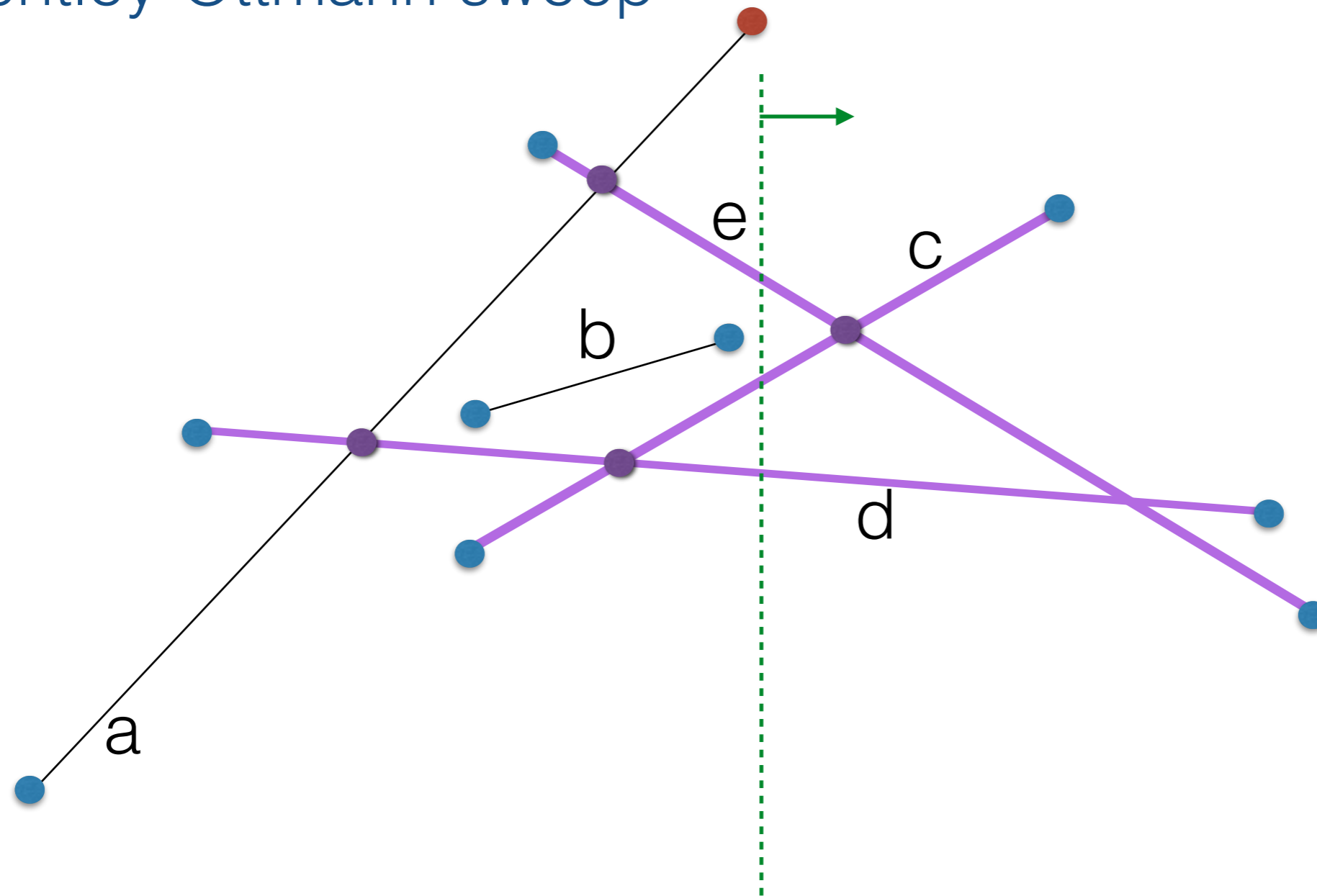
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Bentley-Ottmann sweep



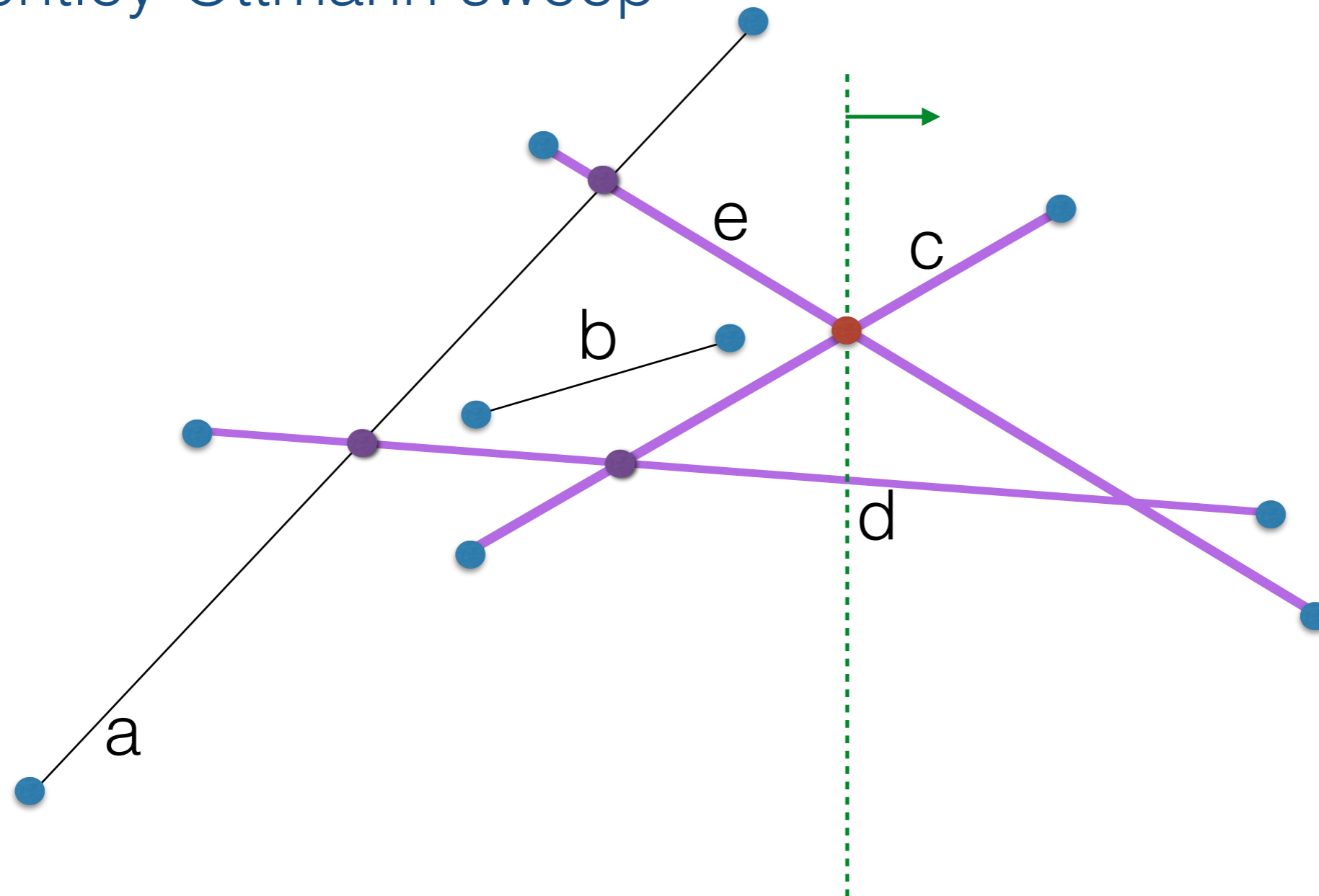
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 - delete a from AS: $d < c < e$
 - no new neighbors

Bentley-Ottmann sweep



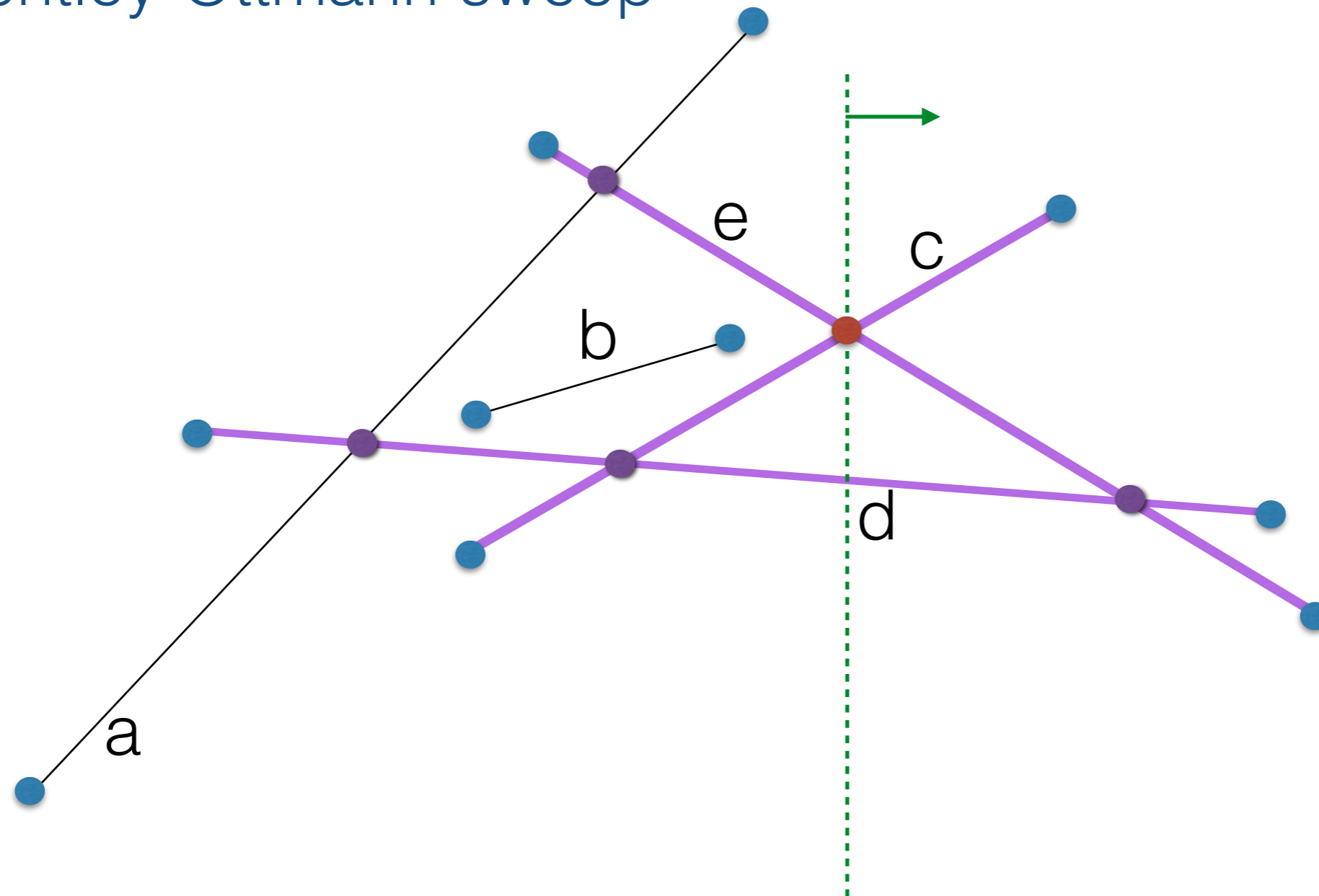
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Bentley-Ottmann sweep



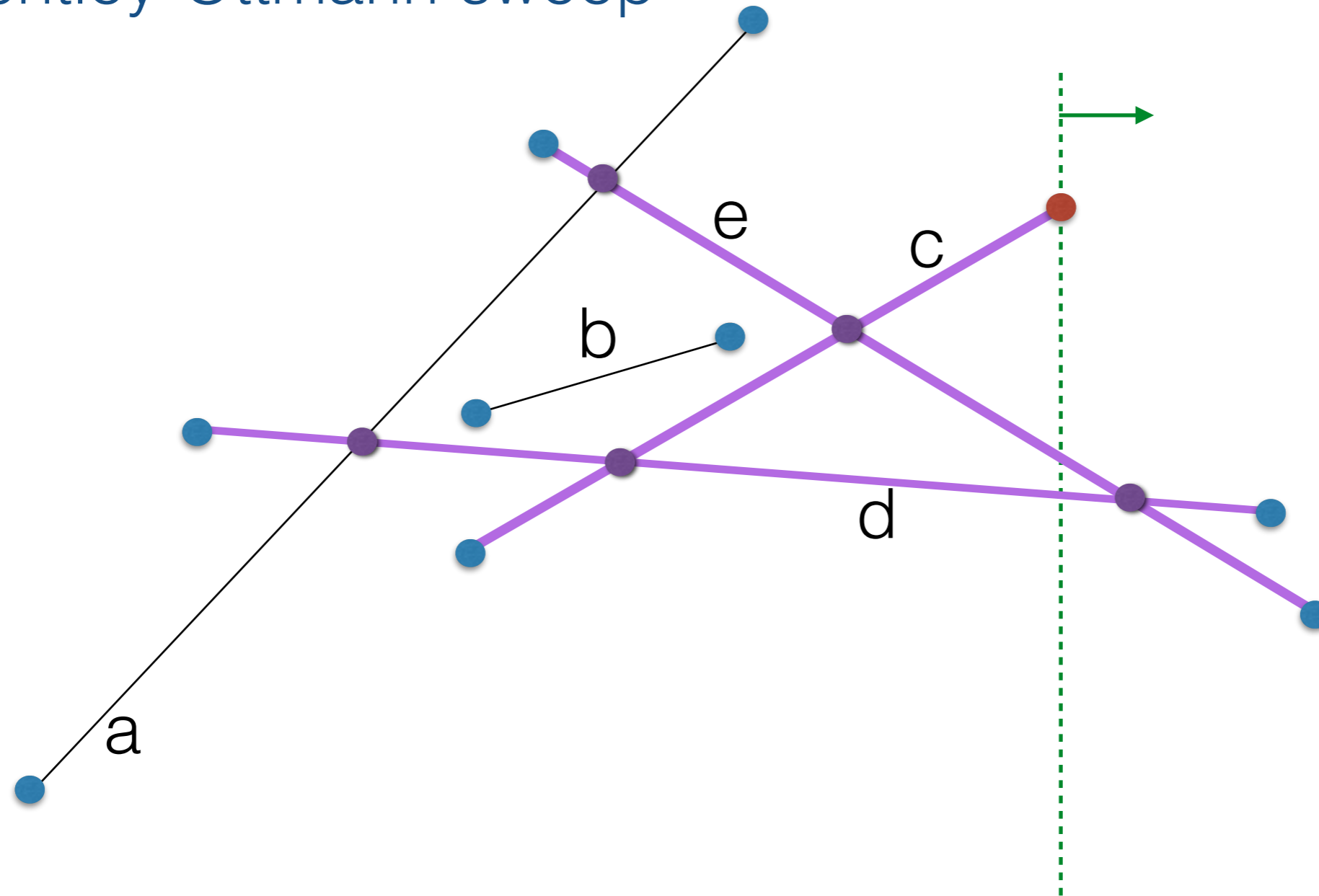
- this event is the intersection of (c,e):
 - flip c,e in AS: $d < e < c$
 - check new neighbors (d,e) for intersection to the right of the sweep line; this detects the intersection of (d,e); report it and insert it as future event

Bentley-Ottmann sweep



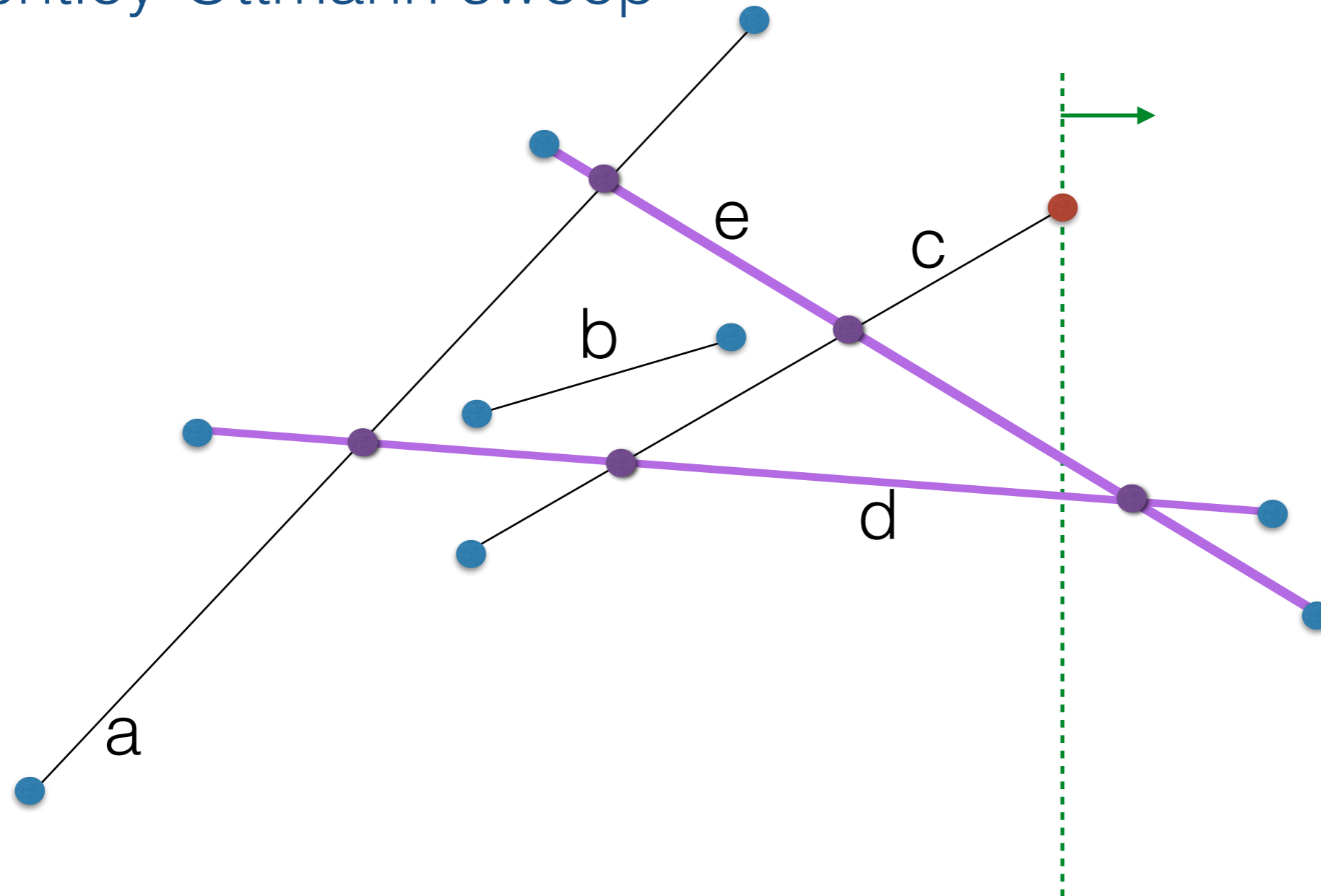
- this event is the intersection of (c,e):
 - flip c,e in AS: $d < e < c$
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Bentley-Ottmann sweep



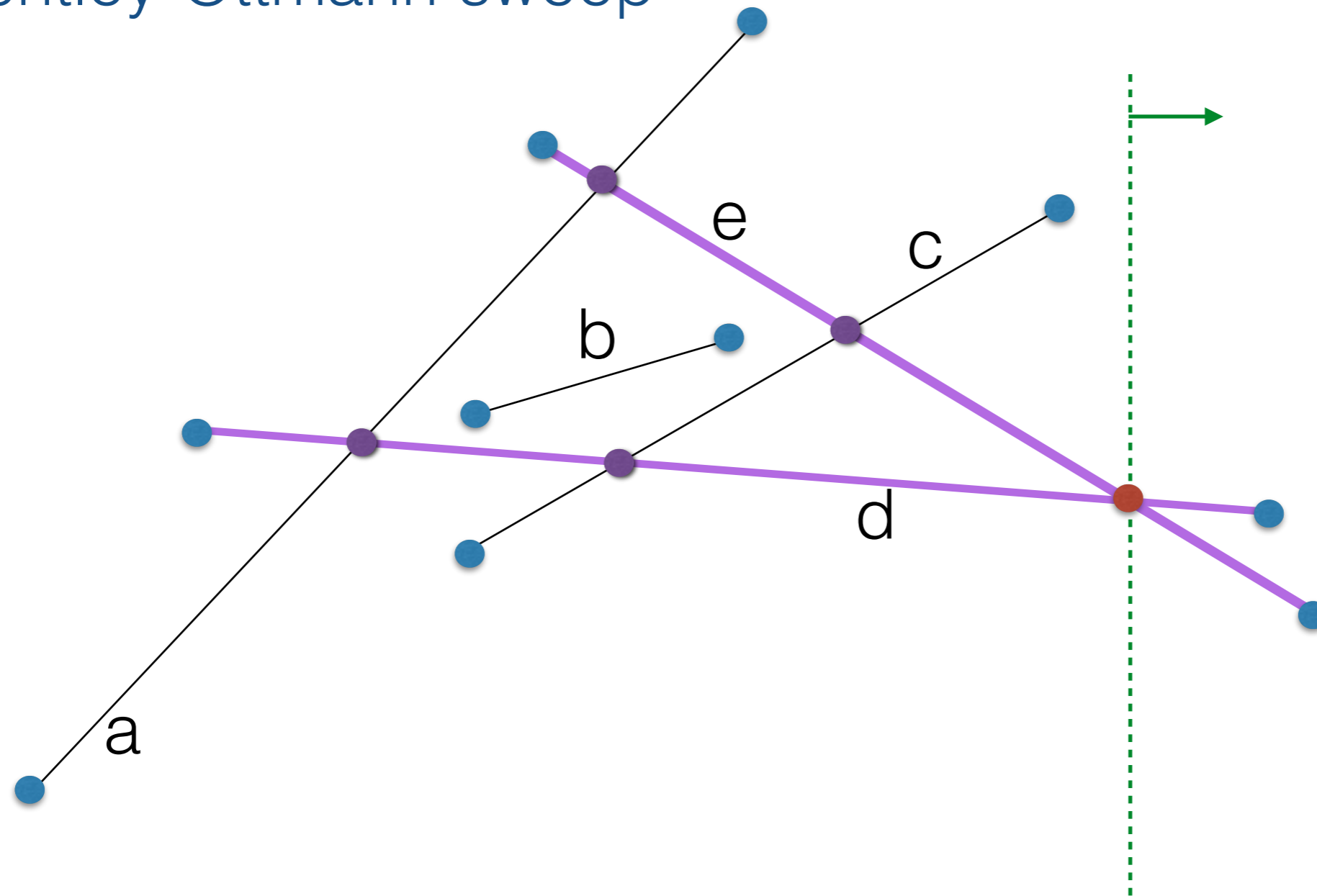
- this event is end of segment *c*:
 - delete *c* in AS: $d < e$
 - no new neighbors

Bentley-Ottmann sweep



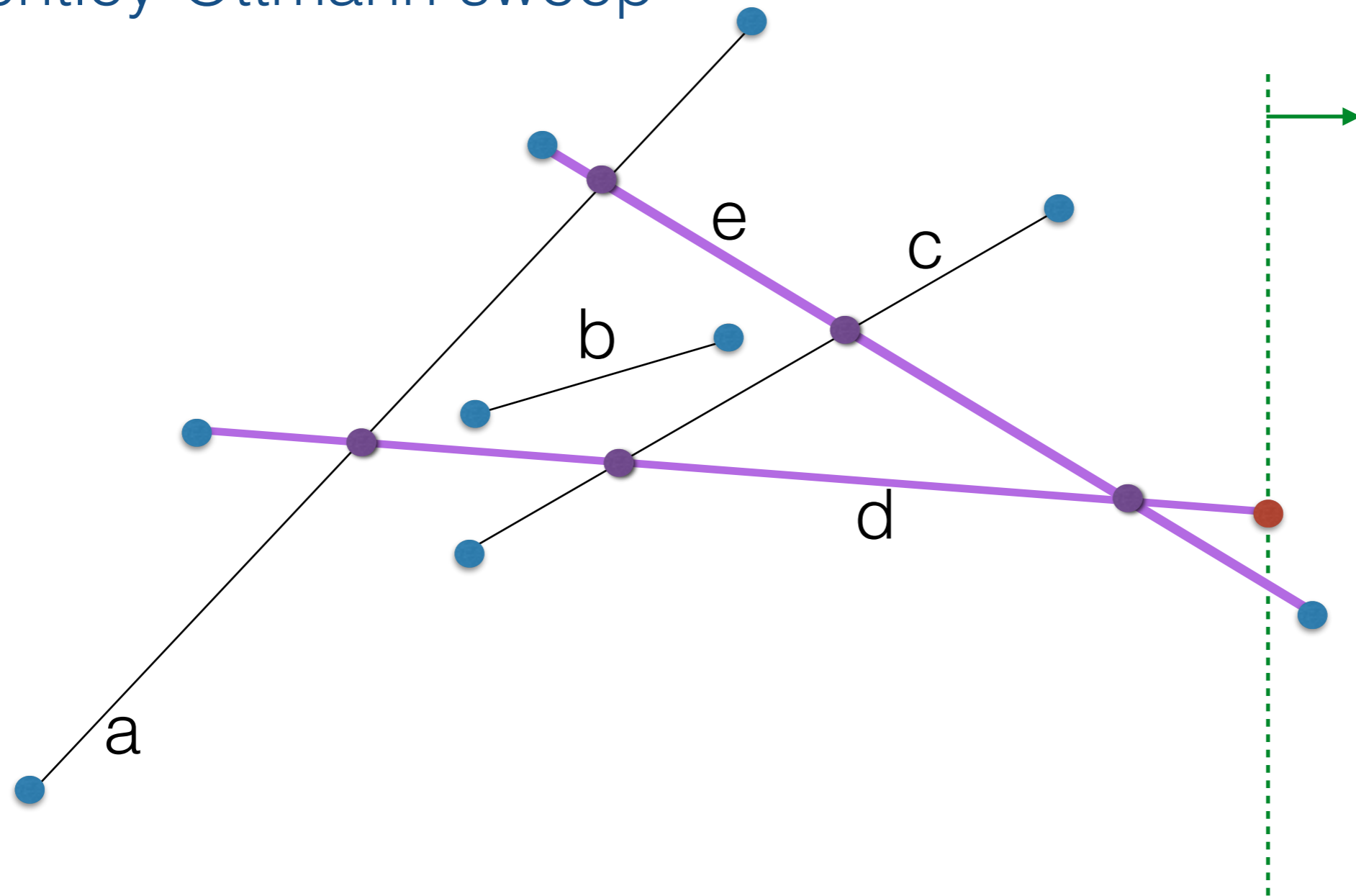
- this event is end of segment c:
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 - no new neighbors

Bentley-Ottmann sweep



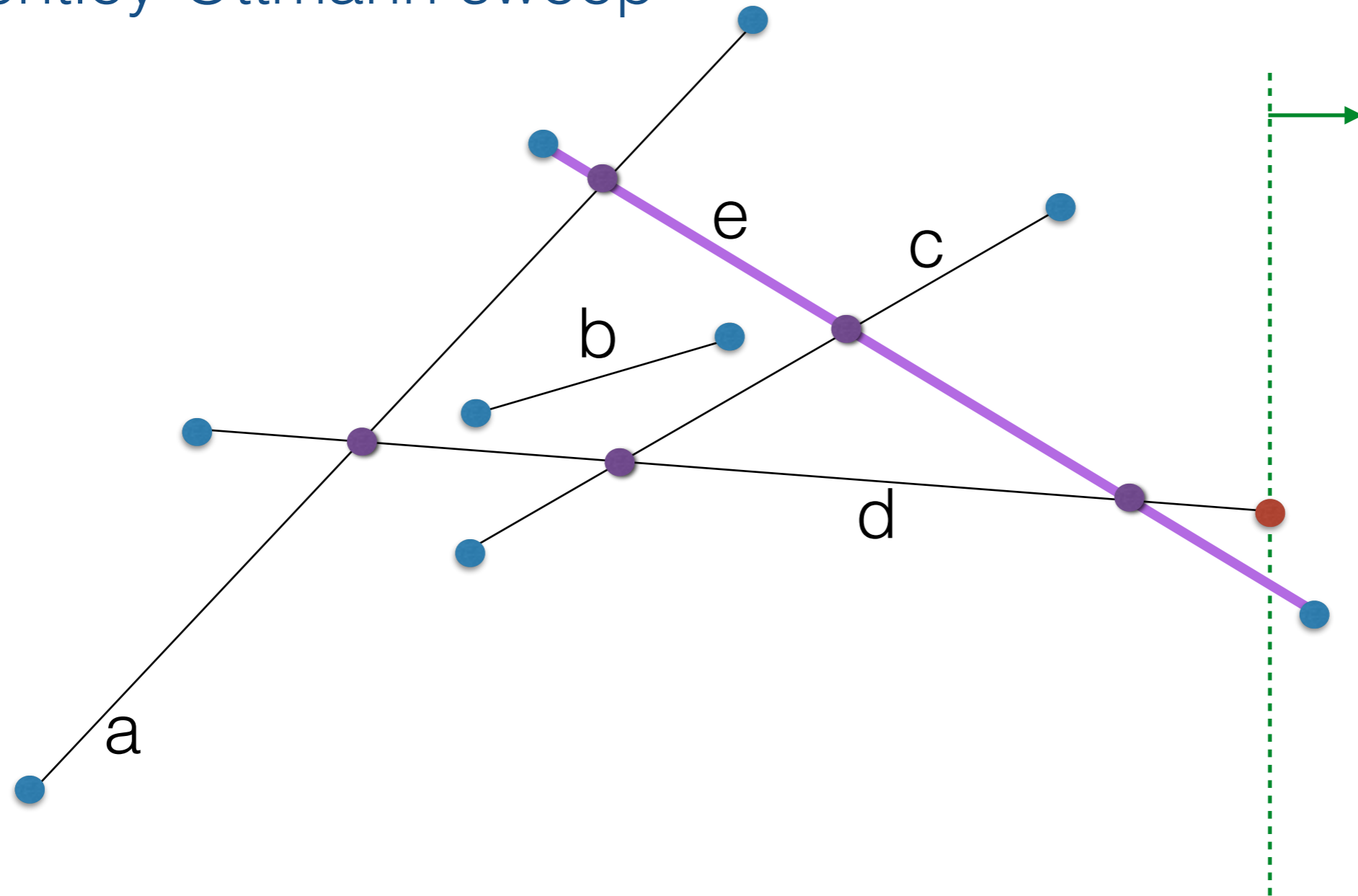
- this event is the intersection of (d,e):
 - flip d,e in AS: $e < d$
 - no new neighbors

Bentley-Ottmann sweep



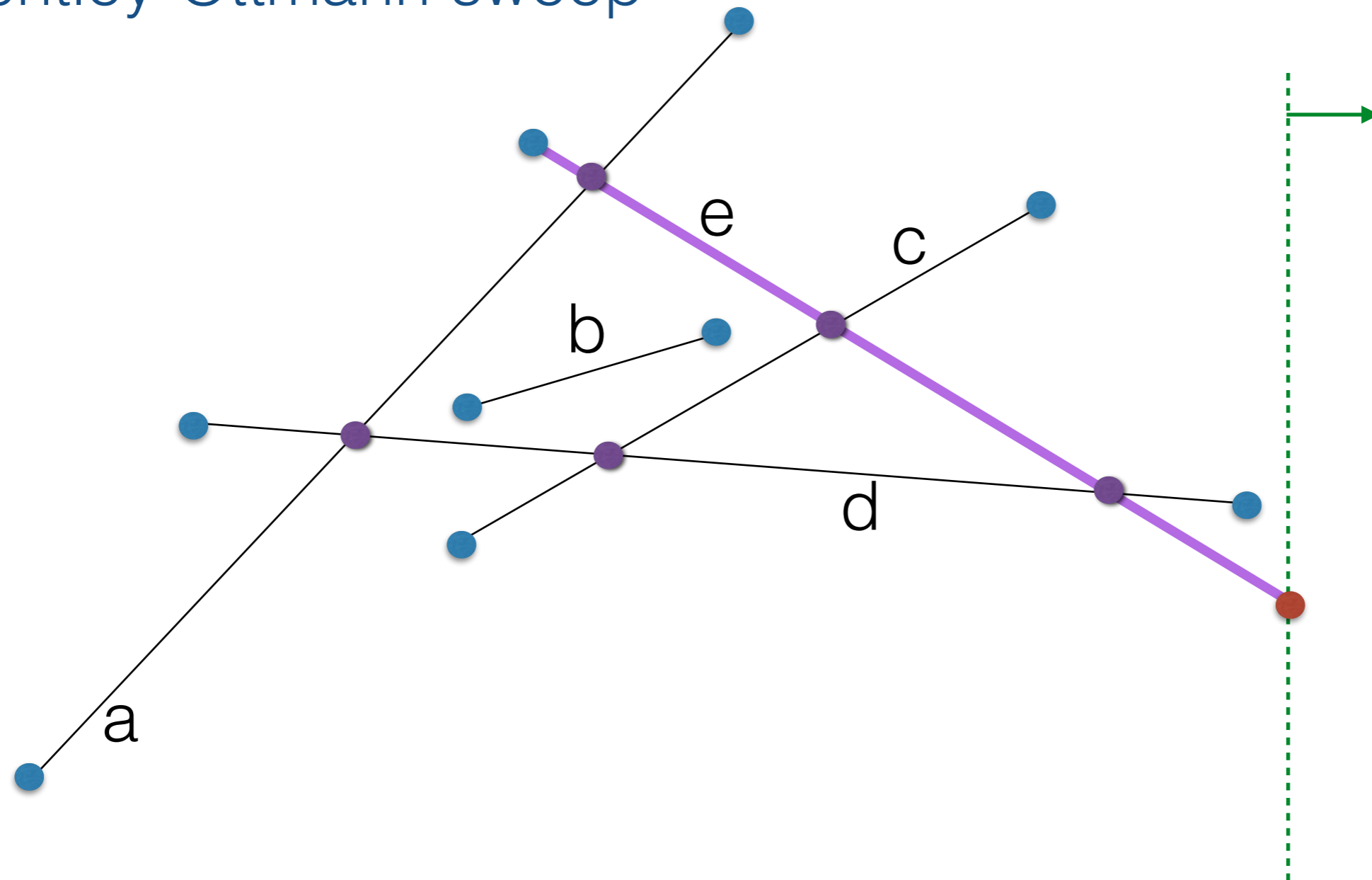
- this event is the end of d:
 - delete d in AS: e
 - no new neighbors

Bentley-Ottmann sweep



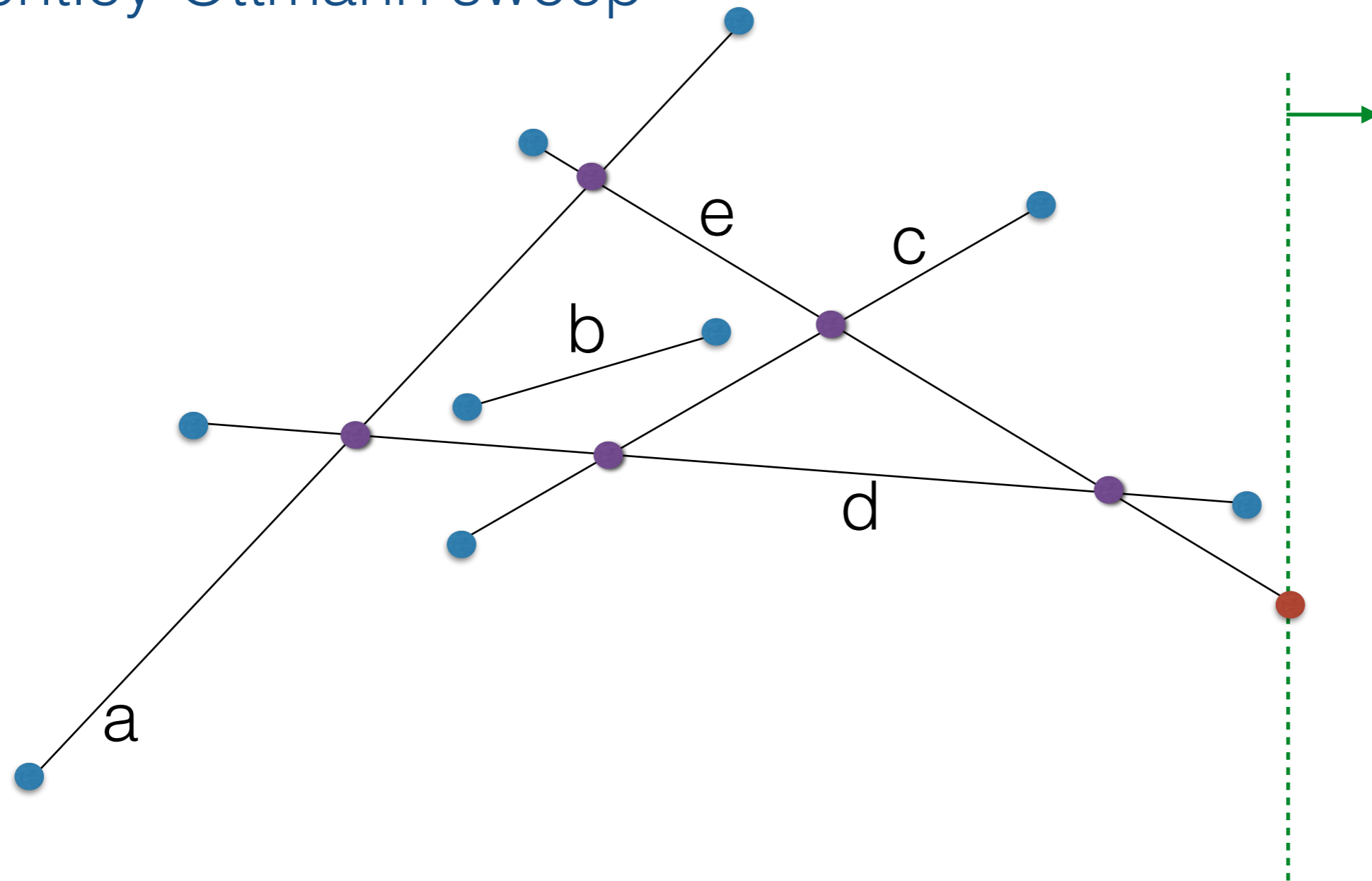
- this event is the end of d:
 - delete d in AS: e
 - no new neighbors

Bentley-Ottmann sweep



- this event is the end of e:
 - delete e in AS:
 - no new neighbors

Bentley-Ottmann sweep



- this event is the end of e:
 - delete e in AS:
 - no new neighbors

Bentley-Ottmann sweep

- Simplifying assumptions
 - no vertical segments
 - no two segments intersect at their endpoints
 - no three (or more) segments have a common intersection
 - all endpoints (of segments) and all intersection points have different x-coordinates
 - no segments overlap
- These assumptions are not realistic for real data..
- But, they don't provide insight into the plane sweep technique, so we omit them
- Real data challenges
 - dealing with degenerate cases
 - dealing with finite precision arithmetic and precision problems

Bentley-Ottmann sweep

We'll maintain the following invariants during the algorithm:

Bentley-Ottmann sweep

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- Active structure *AS*:
 - For any position of the sweep line *SL*, *AS* contains all active segments (ie segments that start before *SL* and end after *SL*)
 - *AS* is sorted by their *y*-coordinates of their intersection with *SL*

Bentley-Ottmann sweep

We'll maintain the following invariants during the algorithm:

- **Active structure AS:**
 - For any position of the sweep line SL, AS contains all active segments (ie segments that start before SL and end after SL)
 - AS is sorted by their y-coordinates of their intersection with SL
- **Event list EL:**
 - For any position of SL, EL contains segment endpoints to the right of SL, and also the intersections to the right of SL of active segments that were/are neighbors in SL
 - EL is sorted by x-coordinate

Bentley-Ottmann sweep

We'll maintain the following invariants during the algorithm:

- Active structure AS:
 - For any position of the sweep line SL, AS contains all active segments (ie segments that start before SL and end after SL)
 - AS is sorted by their y-coordinates of their intersection with SL
- Event list EL:
 - For any position of SL, EL contains segment endpoints to the right of SL, and also the intersections to the right of SL of active segments that were/are neighbors in SL
 - EL is sorted by x-coordinate
- For any position of the sweep line SL, all pairs of intersecting dead segments have been reported.

Algorithm Bentley-Ottmann (S)

//S is a set of n line segments in the plane

- initialize AS= {}
- sort $2n$ endpoints of all segments in S by x-coord and store them in EventList
- while EventList not empty
 - let e be the next event from EventList; delete it from EL
 - //sweep line moves to $x=e.x$
 - if e is left endpoint of a segment l
 - //l becomes active
 - insert l in AS in the right place
 - check if l intersects with l->prev and l->succ in AS to the right of the sweep line; if they do, insert their intersection point in the EventList
 - //optional: since l.prev and l.succ are not neighbors anymore, we check if they intersect and if they do, delete that intersection point from the EventList
 - if e is the right endpoint of a segment
 - ...
 - if e is the intersection of two segments
 - ...
- end.

Bentley-Ottmann sweep

Questions

- AS
 - What operations do we do on AS?
 - What data structure should we use for AS?
- EL
 - Note that we know a priori the $2n$ events corresponding to start and end-points of segments, but EL is not static; the events corresponding to intersection points are generated on the fly
 - What operations do we do on EL?
 - What data structure should we use for EL?

Bentley-Ottmann sweep

Running time

- **AS**
 - What is the size of AS?
 - $O(n)$
 - How many operations?
 - $O(n+k)$
 - Overall time?
 - $O((n+k)\lg n)$
- **EventList**
 - What is the size of EventList?
 - $O(n+k)$
 - How many operations?
 - $O(n+k)$
 - Overall time?
 - $O((n+k)\lg n)$

Bentley-Ottmann sweep

Running time

- **AS**
 - What is the size of AS?
 - $O(n)$
 - How many operations?
 - $O(n+k)$
 - Overall time?
 - $O((n+k)\lg n)$
- **EventList**
 - What is the size of EventList?
 - $O(n+k)$
 - How many operations?
 - $O(n+k)$
 - Overall time?
 - $O((n+k)\lg n)$

Bentley-Ottmann sweep runs in $O((n+k)\lg n)$ time.