Computational Geometry
[csci 3250]

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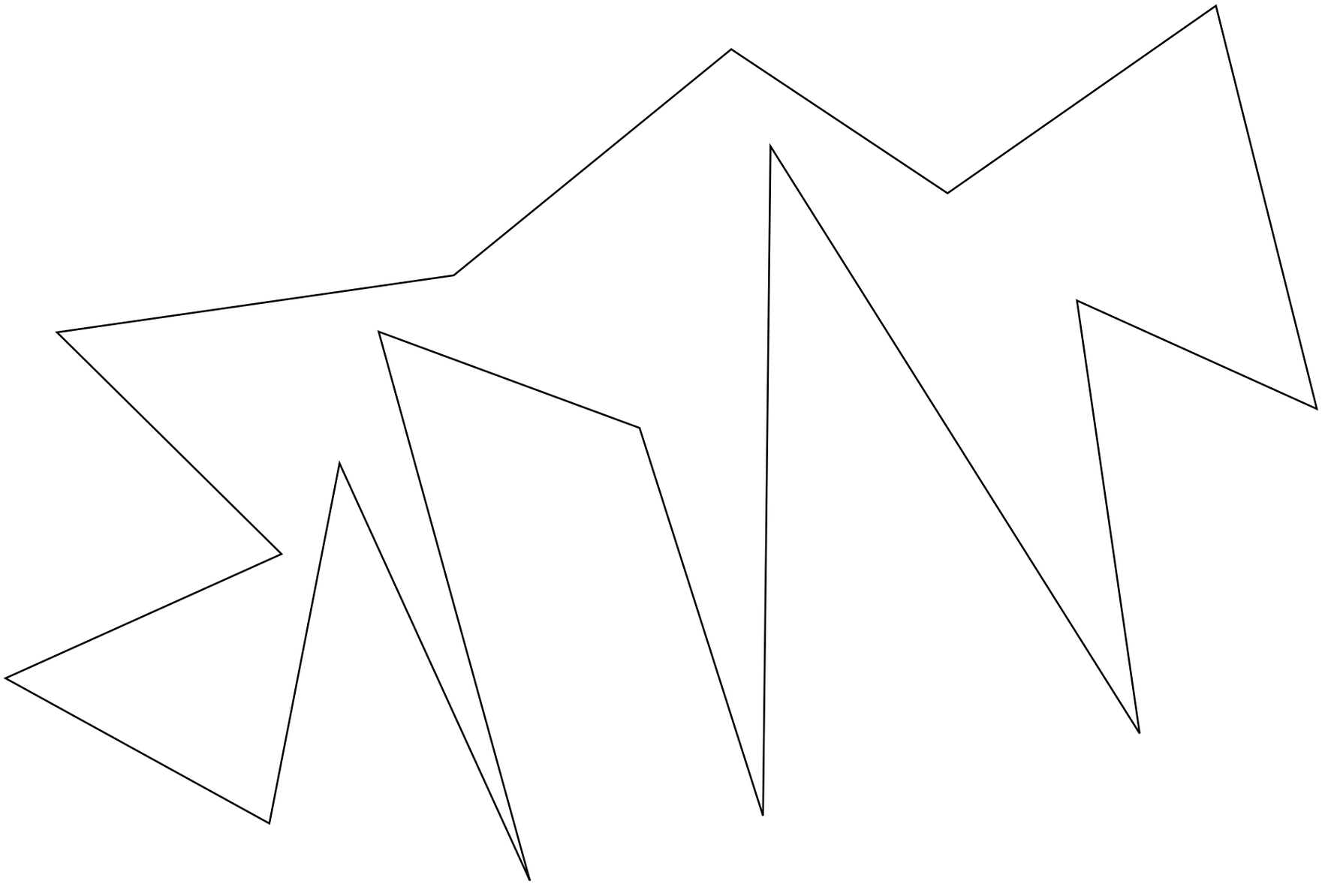
Bowdoin College

Polygon Triangulation

Polygon Triangulation

The problem: Triangulate a given polygon.

(output a set of diagonals that partition the polygon into triangles).



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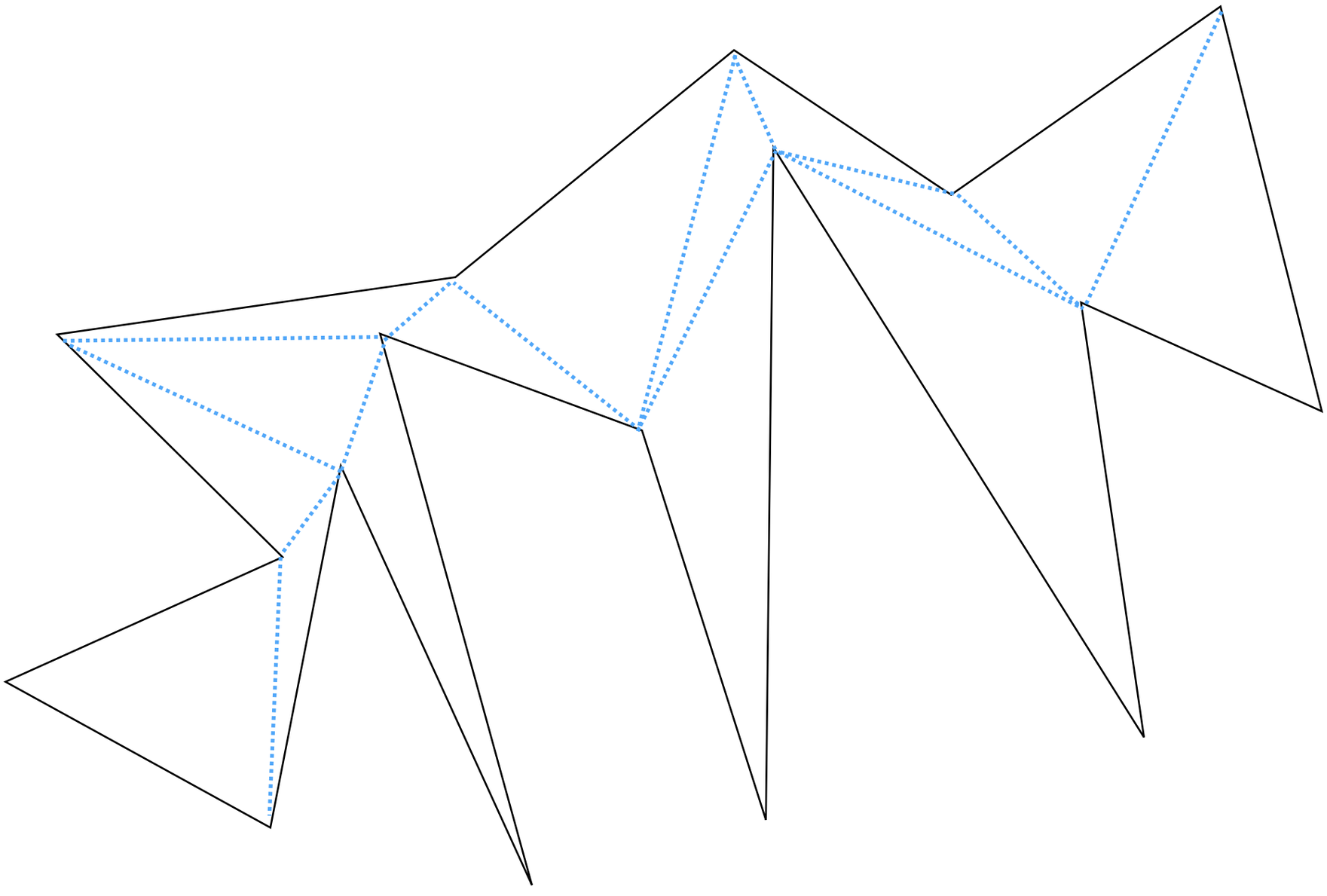

Definitions

Given a simple polygon P

- A **diagonal** is a segment between 2 non-adjacent vertices that lies entirely within the interior of the polygon.

- A **ear** with tip v_i is a set of 3 consecutive vertices v_{i-1}, v_i, v_{i+1} if $v_{i-1}v_{i+1}$ is a diagonal.
  - Put differently, v_i is an ear tip if the vertex right before it and the vertex right after it are *visible* to each other

Known Results

- Theorem: Any simple polygon must have a convex vertex.

- Theorem: Any simple polygon with n>3 vertices contains (at least) a diagonal.

- Theorem: Any polygon can be triangulated by adding diagonals.

- Theorem: Any simple polygon has at least two ears.

- All triangulations of a polygon of n vertices have n-2 triangles and n-3 diagonals.

Algorithms

- Is $p_i p_j$ a diagonal of P?
- Find a diagonal of P
- Is v_i the tip of an ear?
- Find all ears

==> Triangulate by finding ears

Polygon triangulation: First steps

- **Algorithm 1:** Triangulation by identifying ears
  - Idea: Find an ear, output the diagonal, delete the ear tip, repeat.
  - Analysis:
    - checking whether a vertex is ear tip or not: O(n)
    - checking all vertices: $O(n^2)$
    - overall $O(n^3)$

- **Algorithm 2:** Triangulation by finding diagonals
  - Idea: Find a diagonal, output it, recurse.
  - A diagonal can be found in O(n) time (using the proof that a diagonal exists)
  - $O(n^2)$

Polygon triangulation: First steps

- **Algorithm 3**: Triangulation by identifying ears in $O(n^2)$
  - Find an ear, output the diagonal, delete the ear tip, repeat.
  - Avoid recomputing ear status for all vertices every time
    - When you remove a ear tip from the polygon, which vertices might change their ear status?

History of Polygon Triangulation

- Early algorithms: $O(n^4)$, $O(n^3)$, $O(n^2)$
- Several O(n lg n) algorithms known ← practical
- …
- Many papers with improved bounds ← not practical
- …
- 1991: Bernard Chazelle (Princeton) gave an O(n) algorithm
  - https://www.cs.princeton.edu/~chazelle/pubs/polygon-triang.pdf
  - Ridiculously complicated, not practical
  - O(1) people actually understand it (and I'm not one of them)

- No algorithm is known that is practical enough to run faster than the O(n lg n) algorithms
- OPEN problem
  - A practical algorithm that's better than O(n lg n).

An O(n lg n) Polygon Triangulation Algorithm

- Ingredients
  - Consider the special case of triangulating a **monotone/unimonotone** polygon
  - Convert an arbitrary polygon into monotone/unimonotone polygons

Monotone chains

A polygonal chain is **x-monotone** if any line perpendicular to x-axis intersects it in one point (one connected component).

Monotone chains

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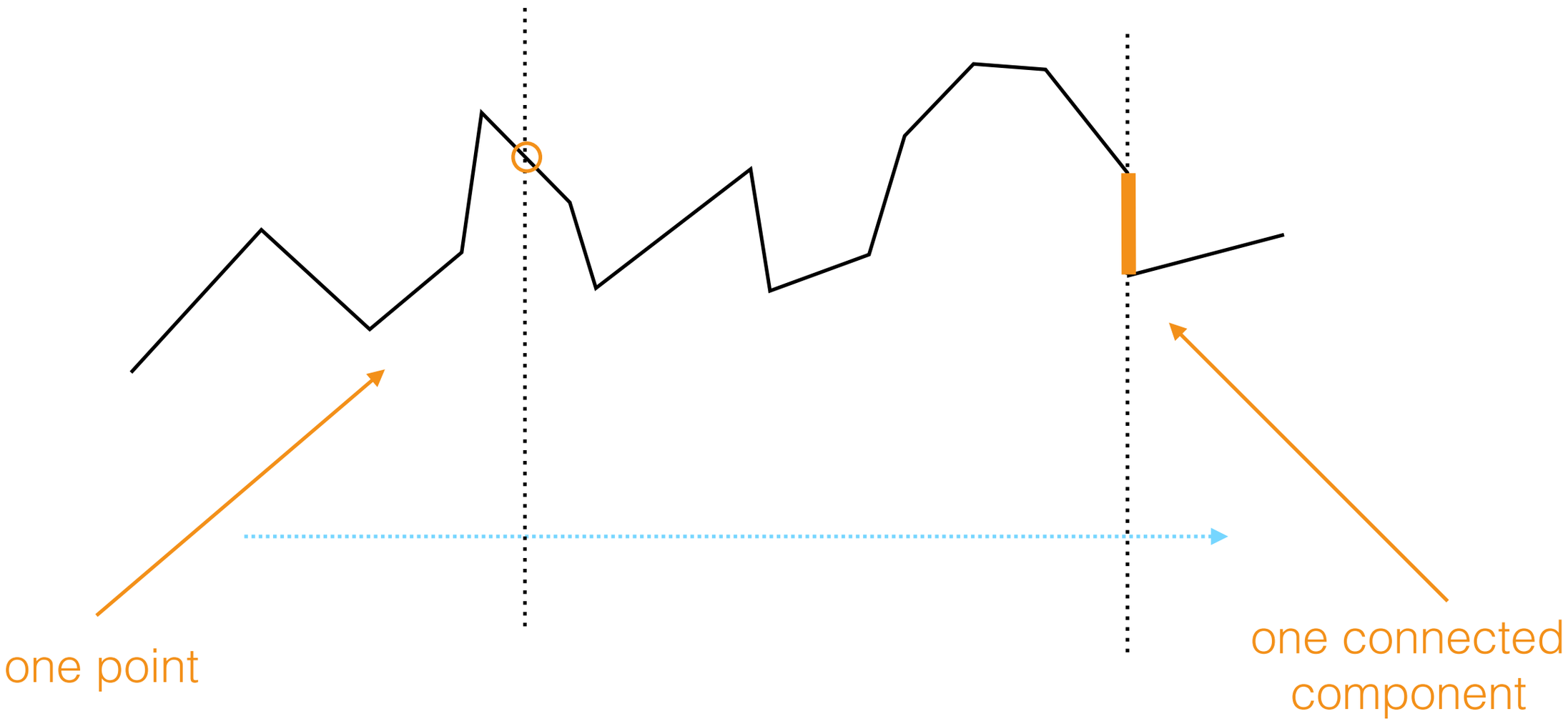
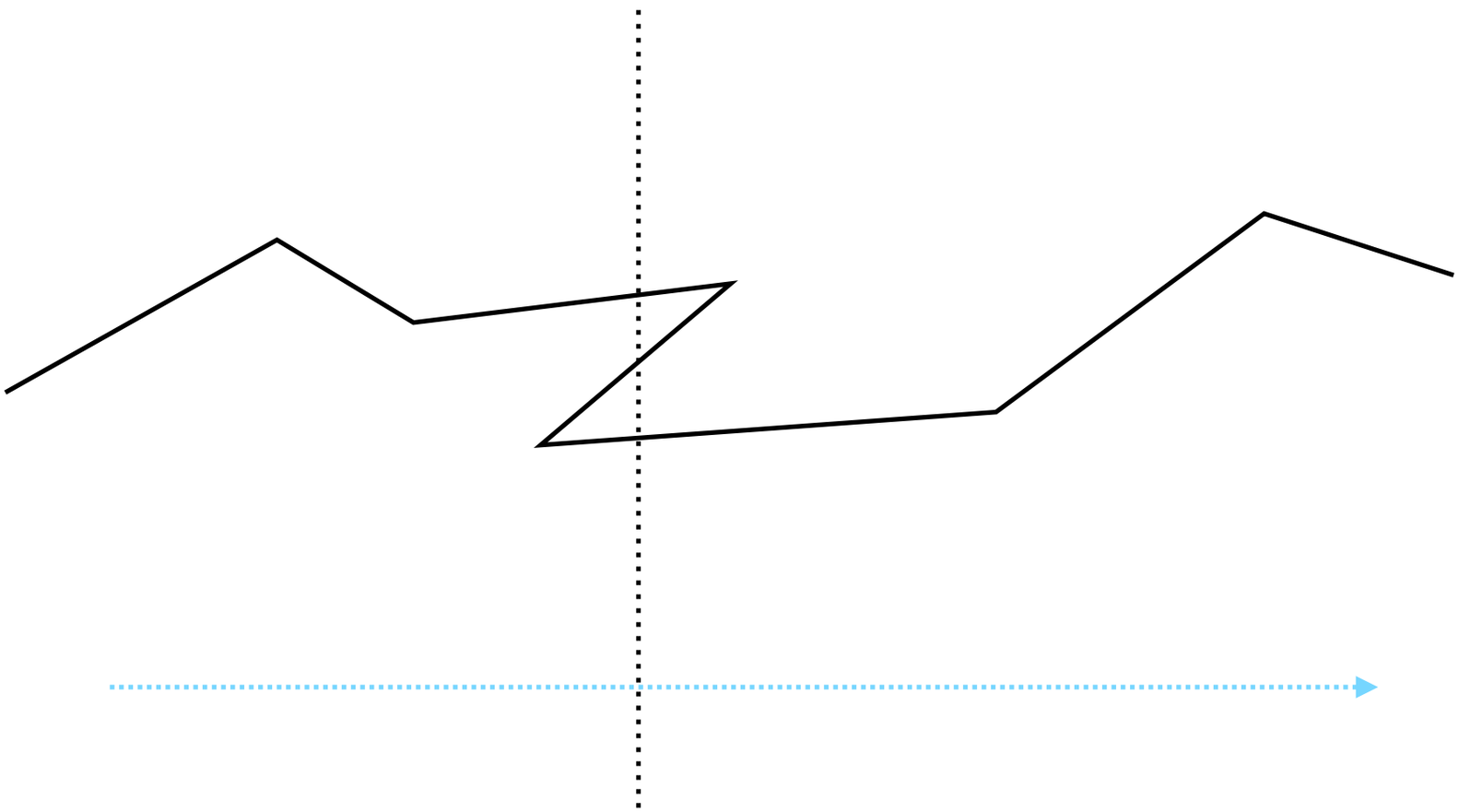


Monotone chains



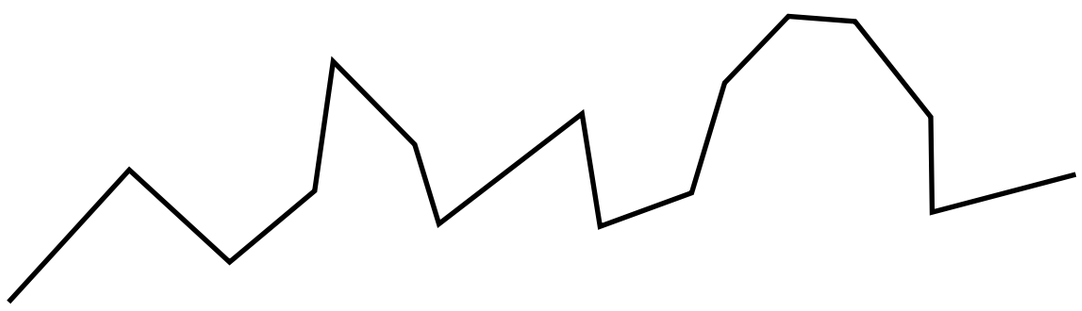
Not x-monotone

Monotone chains

- Claim: Let u and v be the points on the chain with min/max x-coordinate. The vertices on the boundary of an x-monotone chain, going from u to v, are in x-order.

x-monotone

Monotone chains



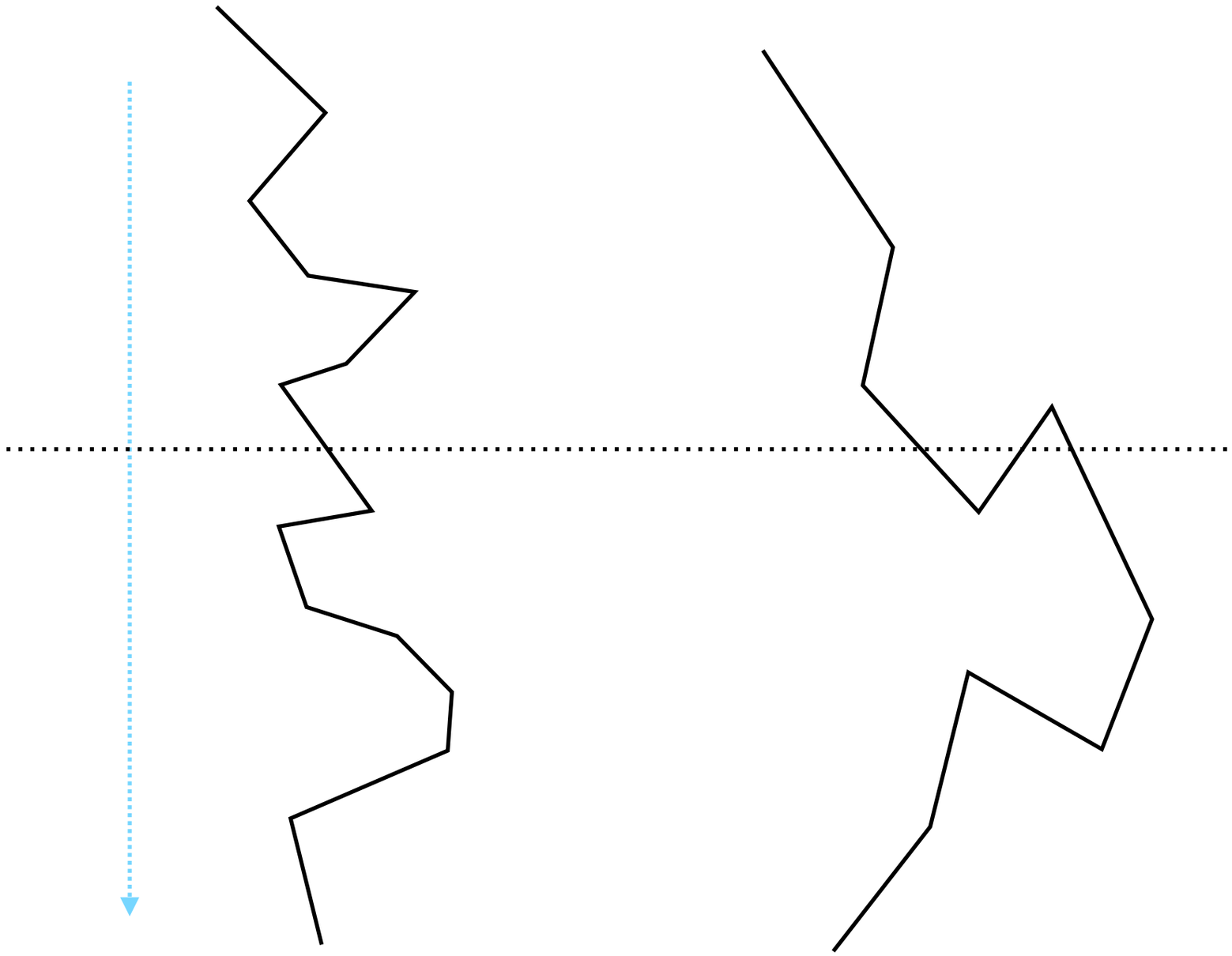
x-monotone

not x-monotone

As you travel along this chain, your x-coordinate is staying the same or increasing

Monotone chains

A polygonal chain is **y-monotone** if any line perpendicular to **y-axis** intersects it in one point (one connected component).

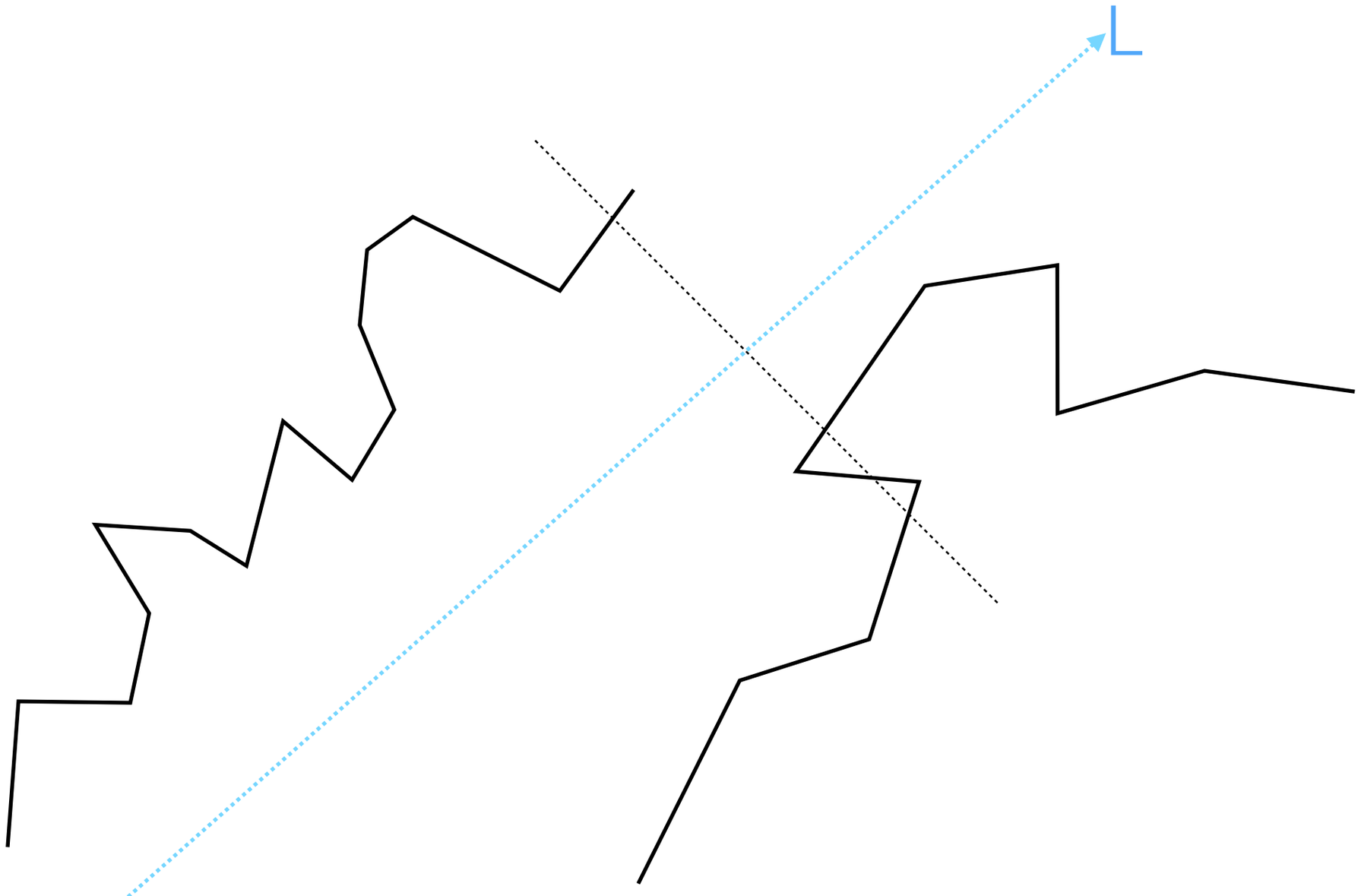


y-monotone

not y-monotone

Monotone chains

A polygonal chain is **L-monotone** if any line perpendicular to **line L** intersects it in one point (one connected component).

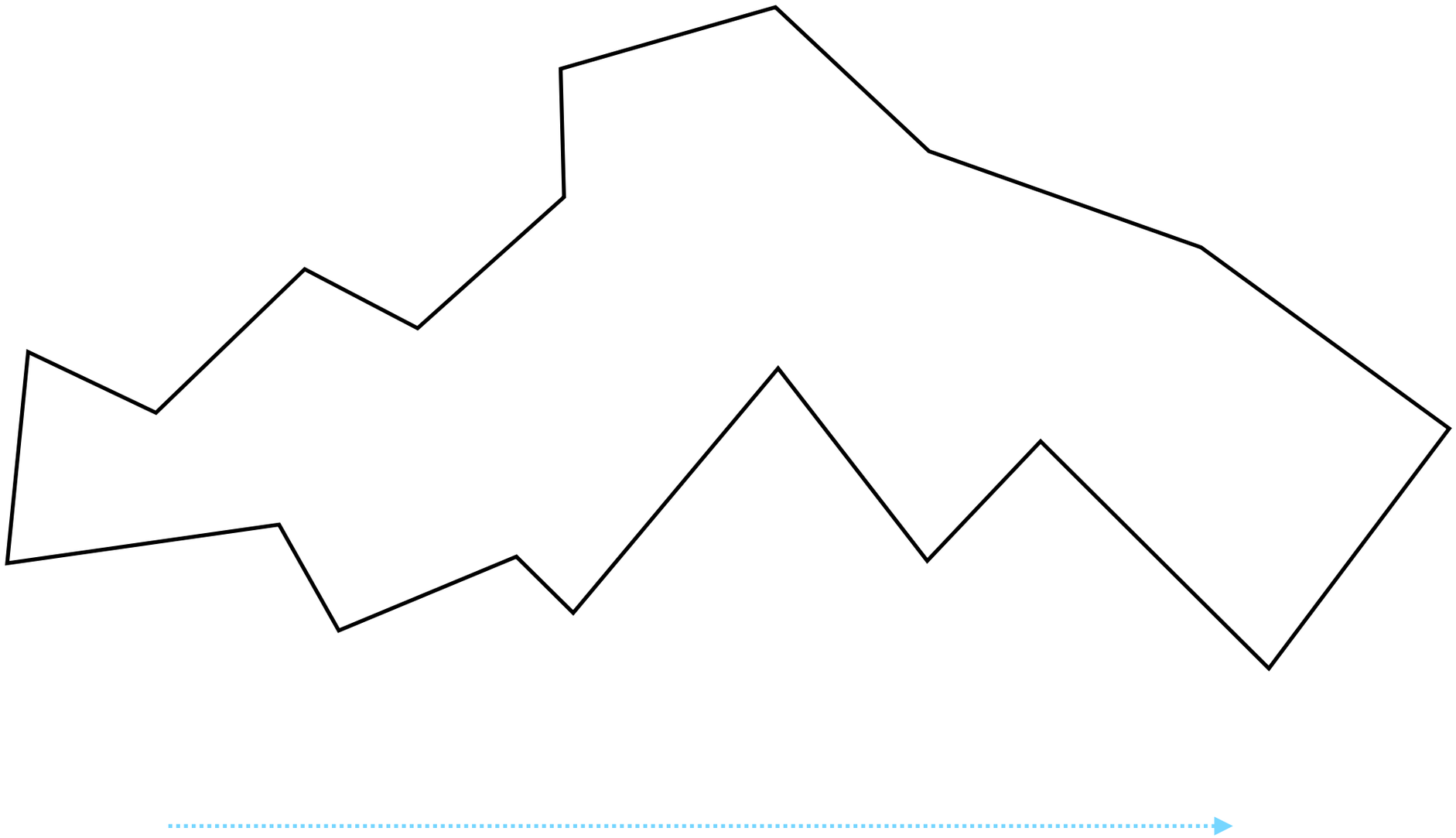


y-monotone

not L-monotone

Monotone polygons

A polygon is **x-monotone** if its boundary can be split into two x-monotone chains.



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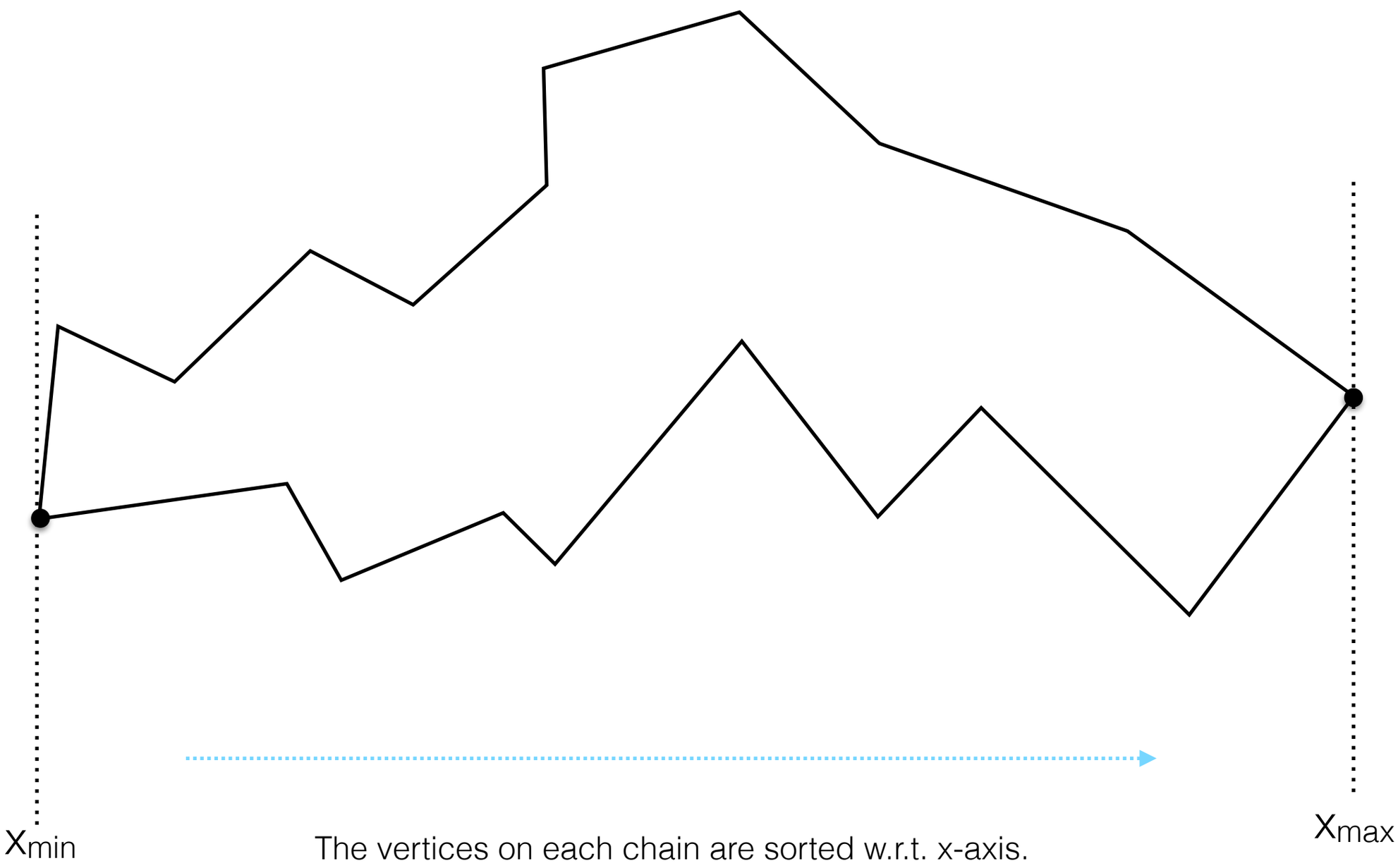

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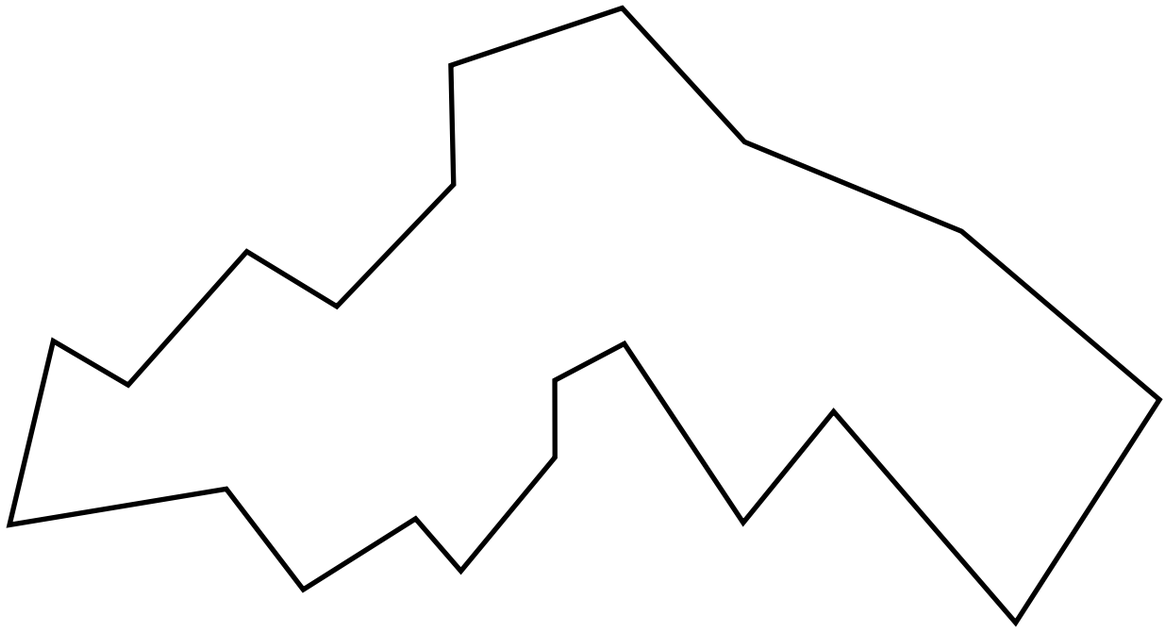
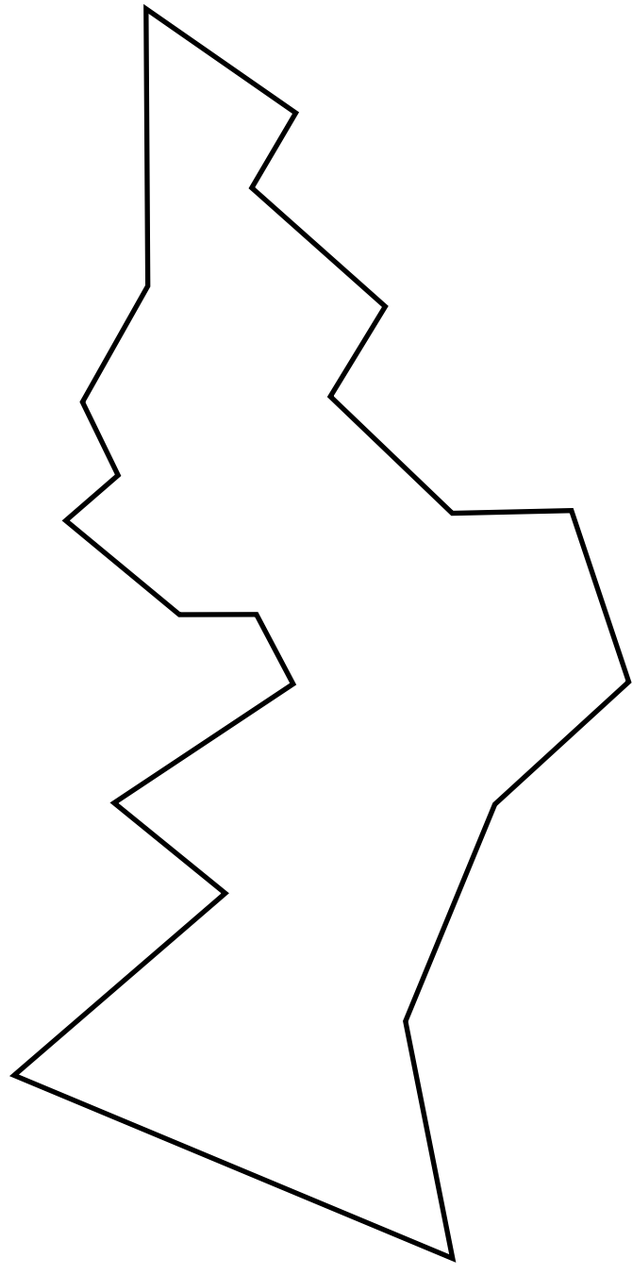


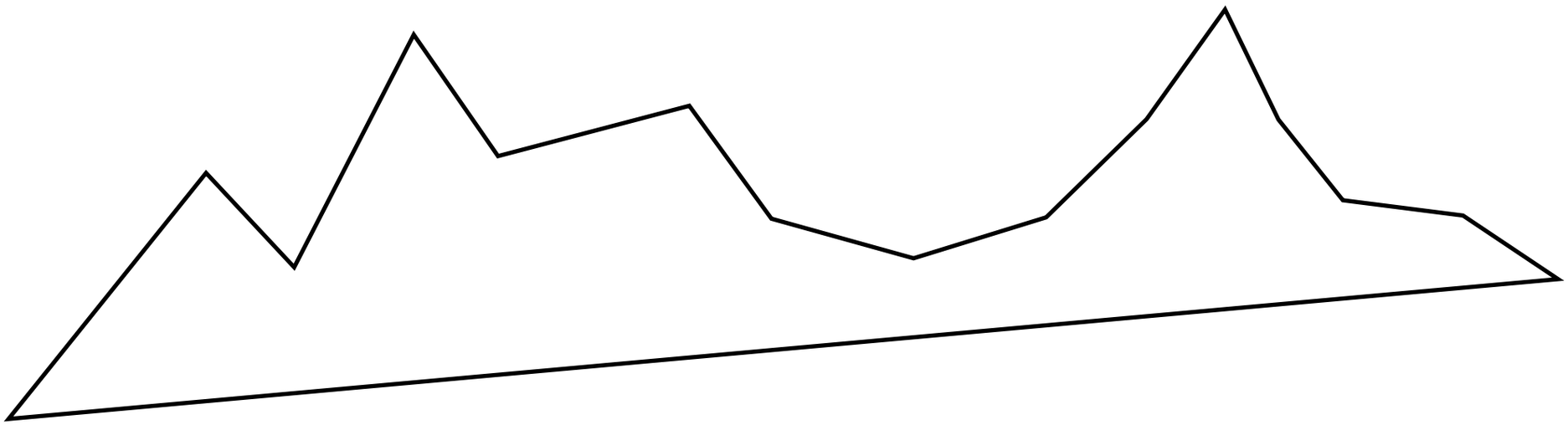
x-monotone



y-monotone

Monotone Mountains

A polygon is an **x-monotone mountain** if it is monotone and one of the two chains is a single segment.



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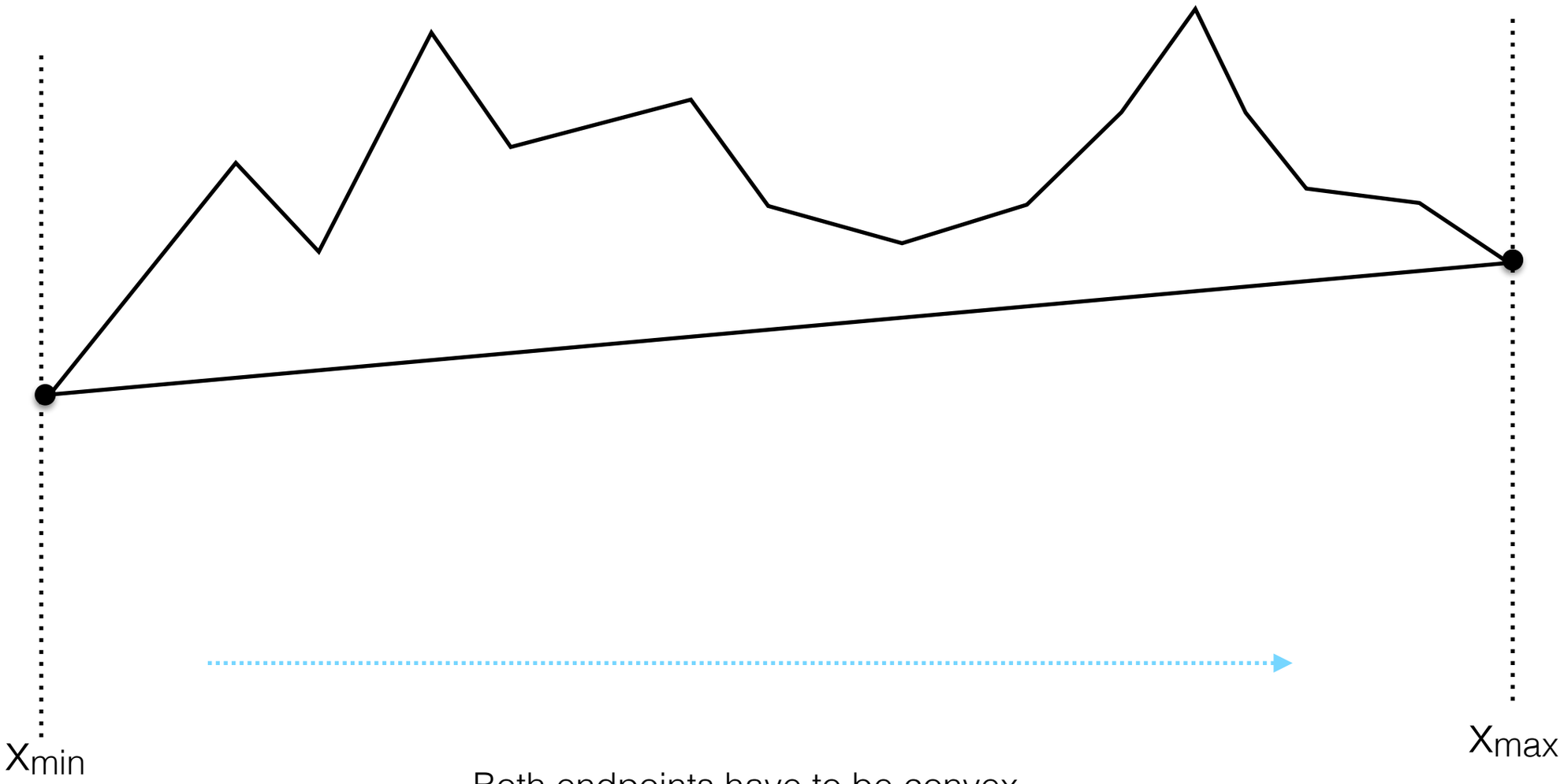
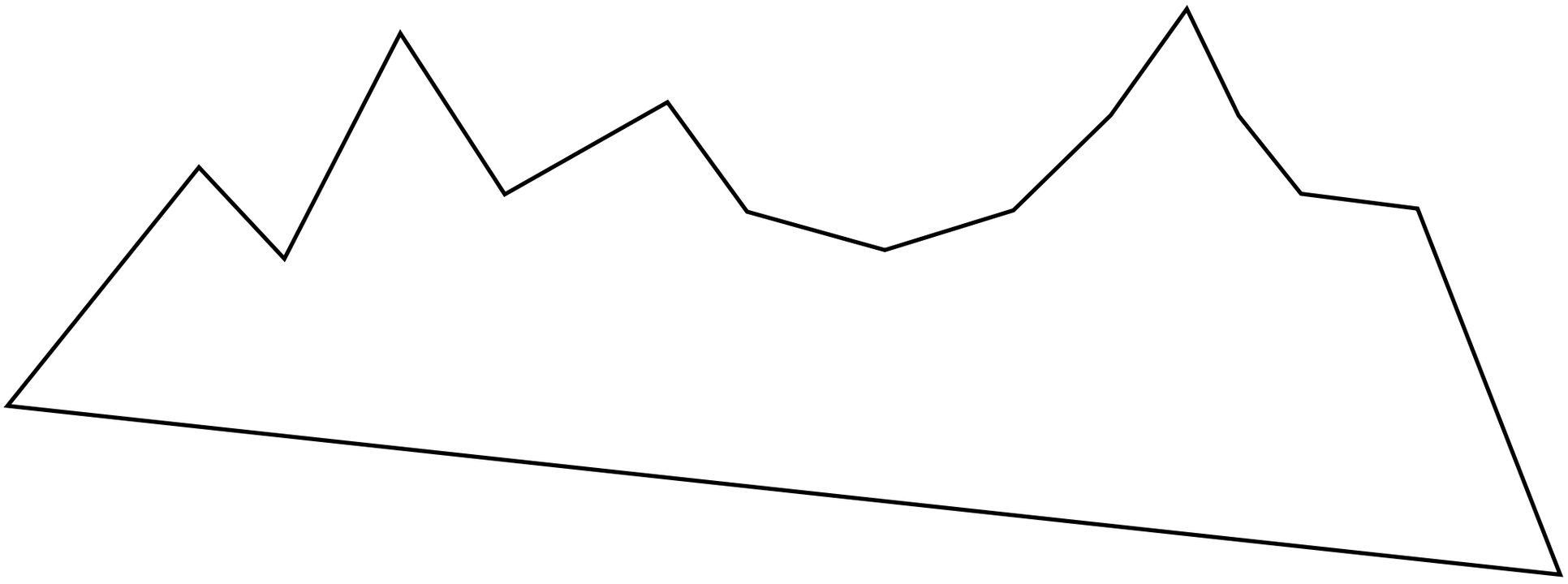


Both endpoints have to be convex.

Monotone mountains are easy to triangulate!



Class work: come up with an algorithm and analyze it.

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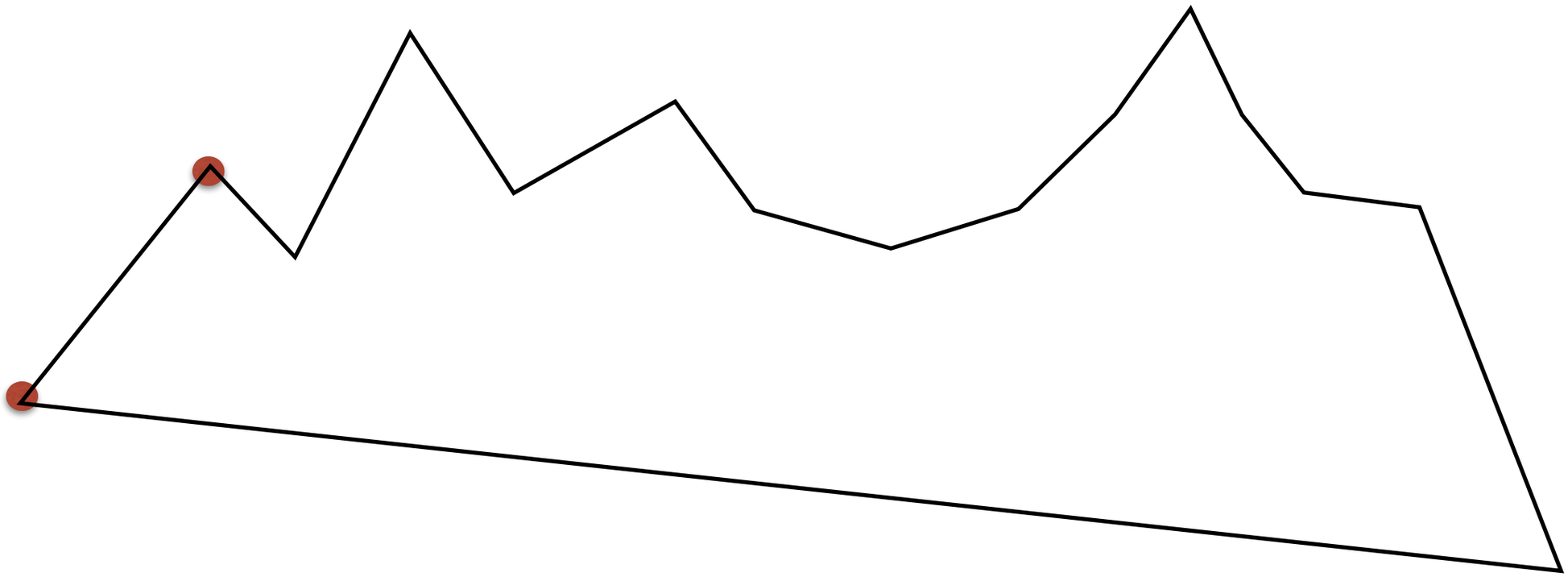

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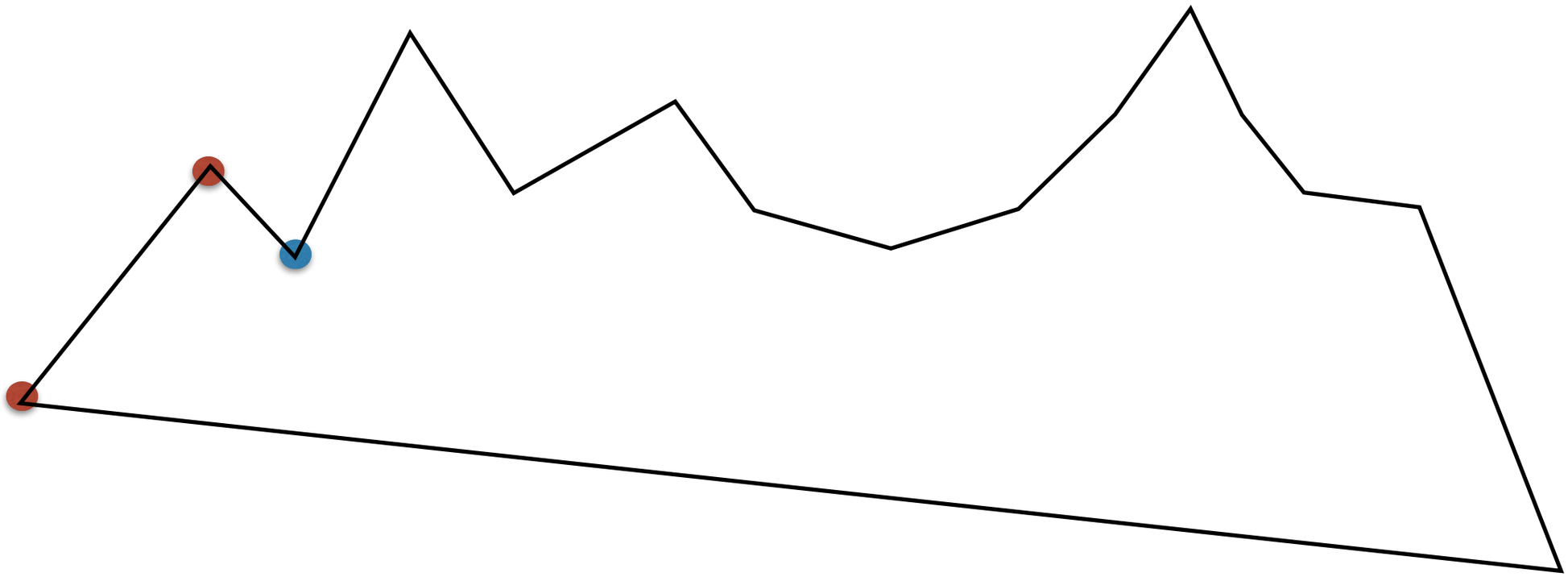

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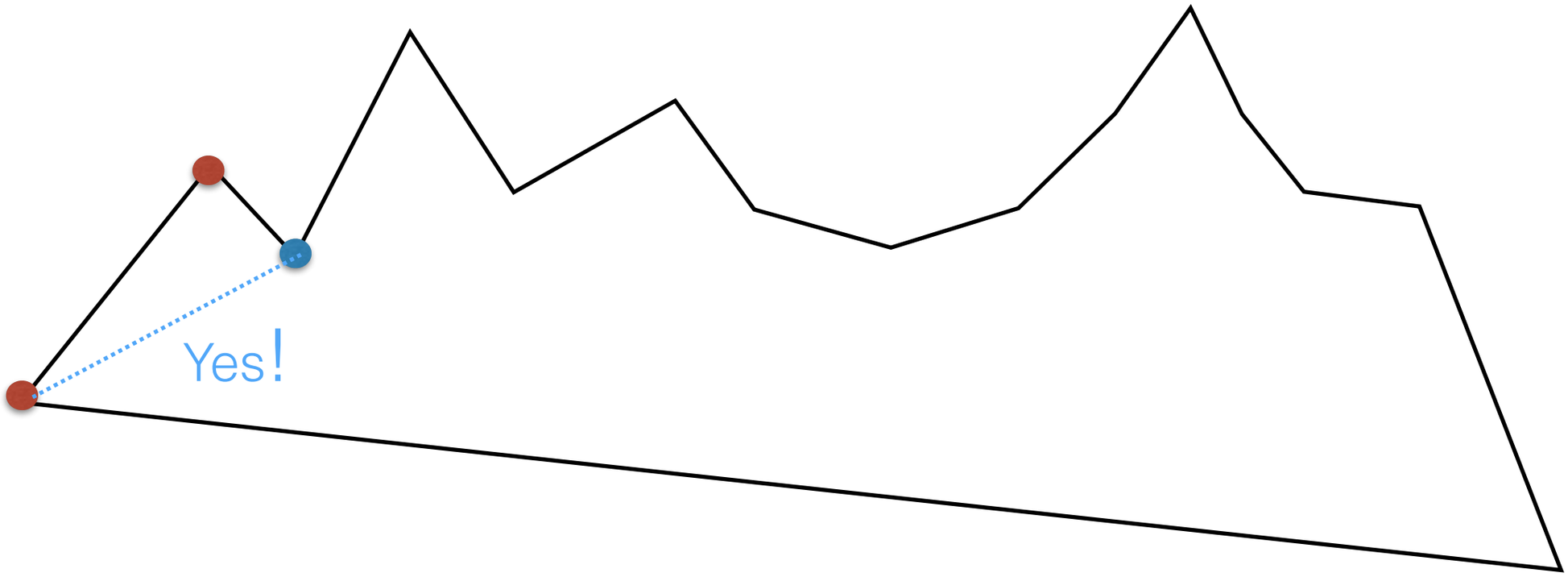

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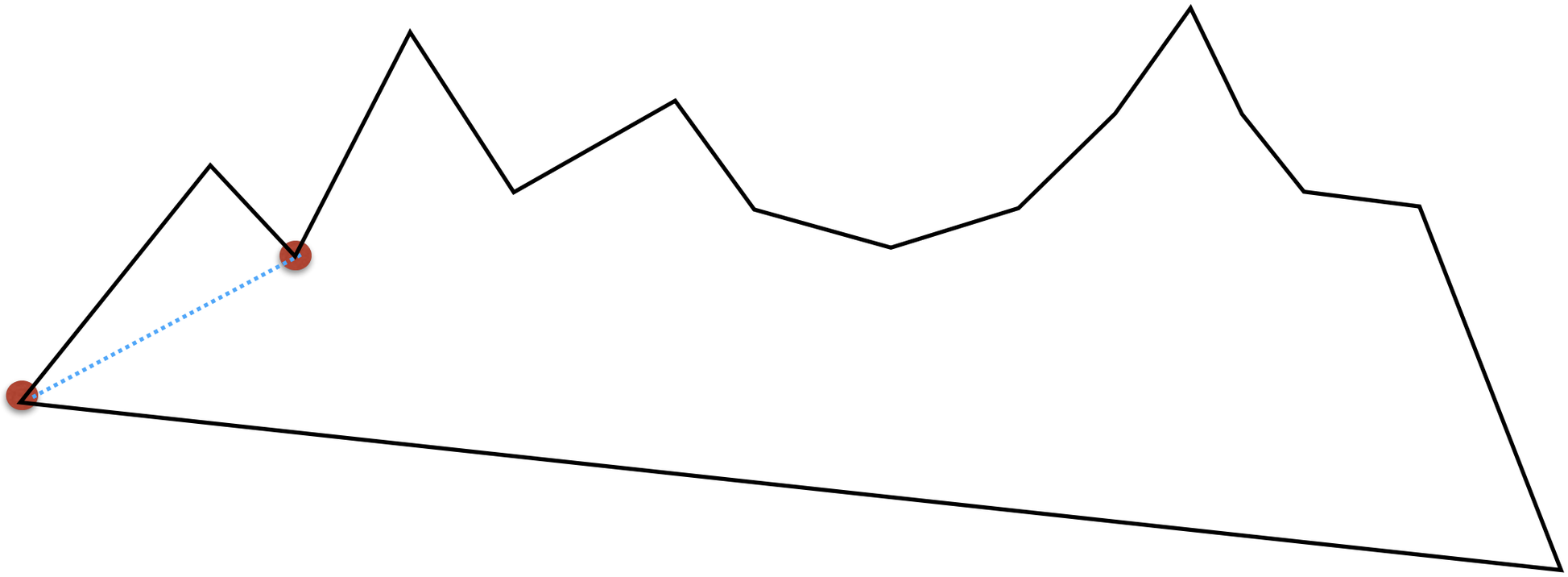

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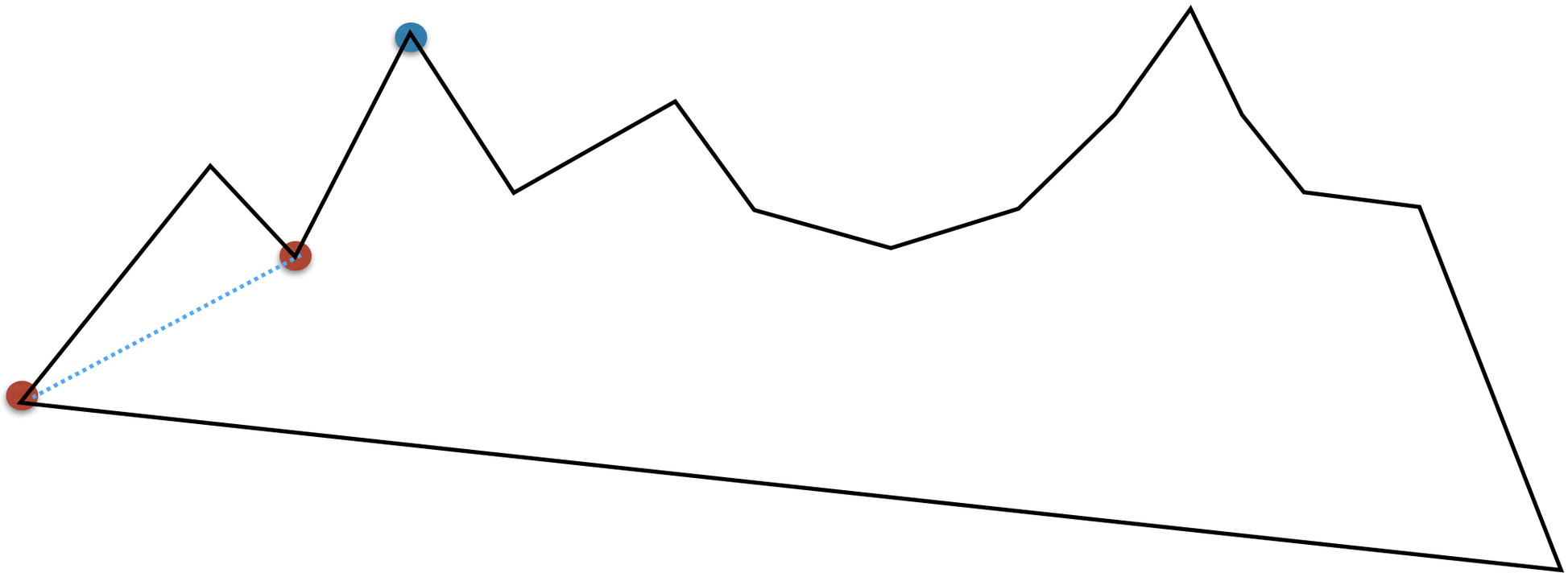

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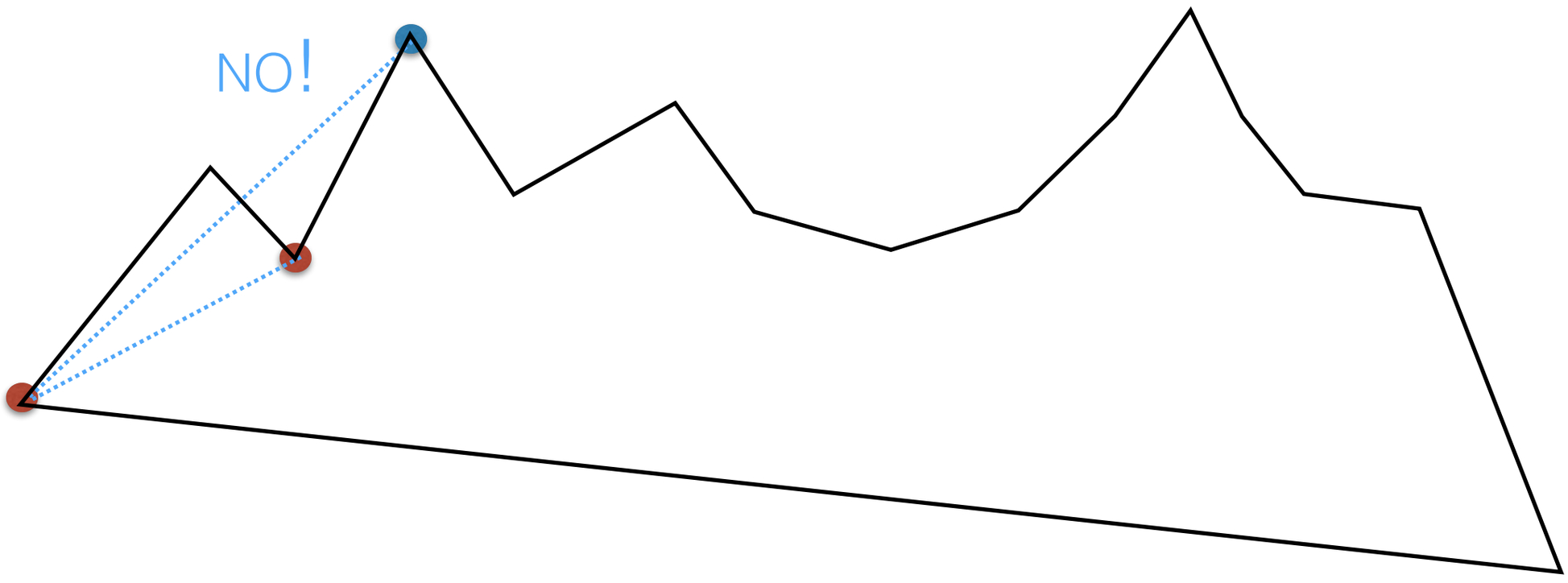

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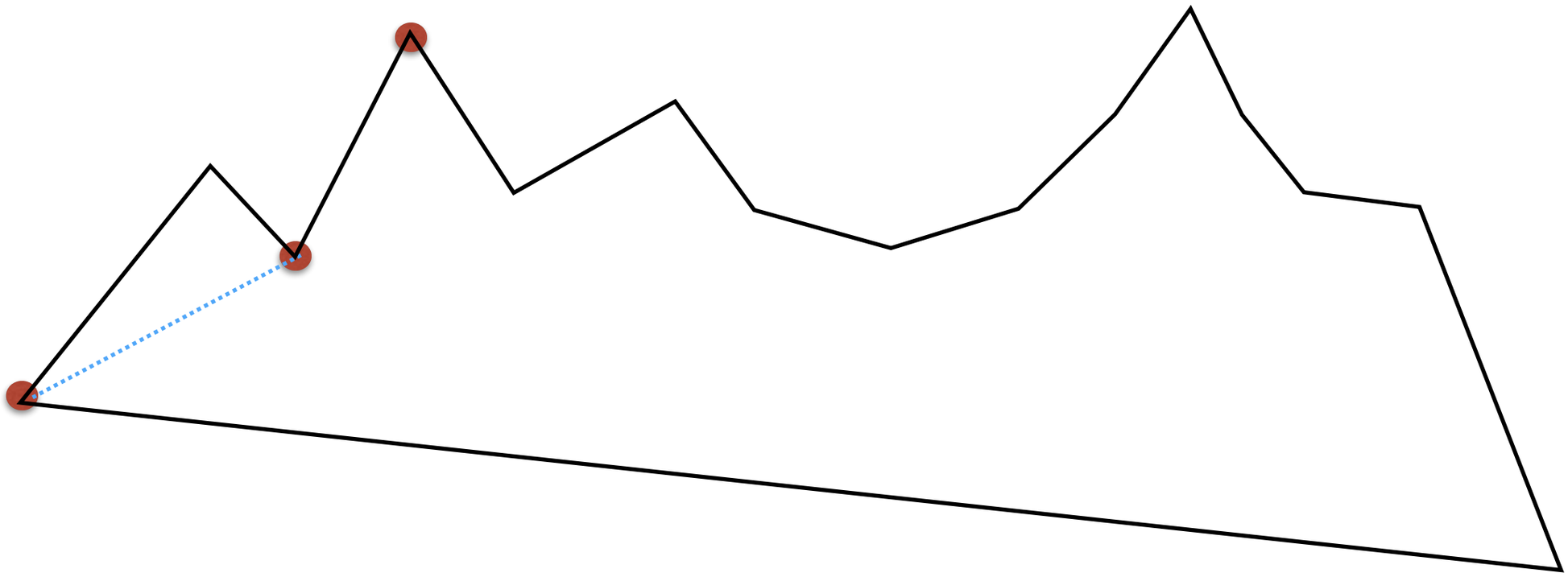

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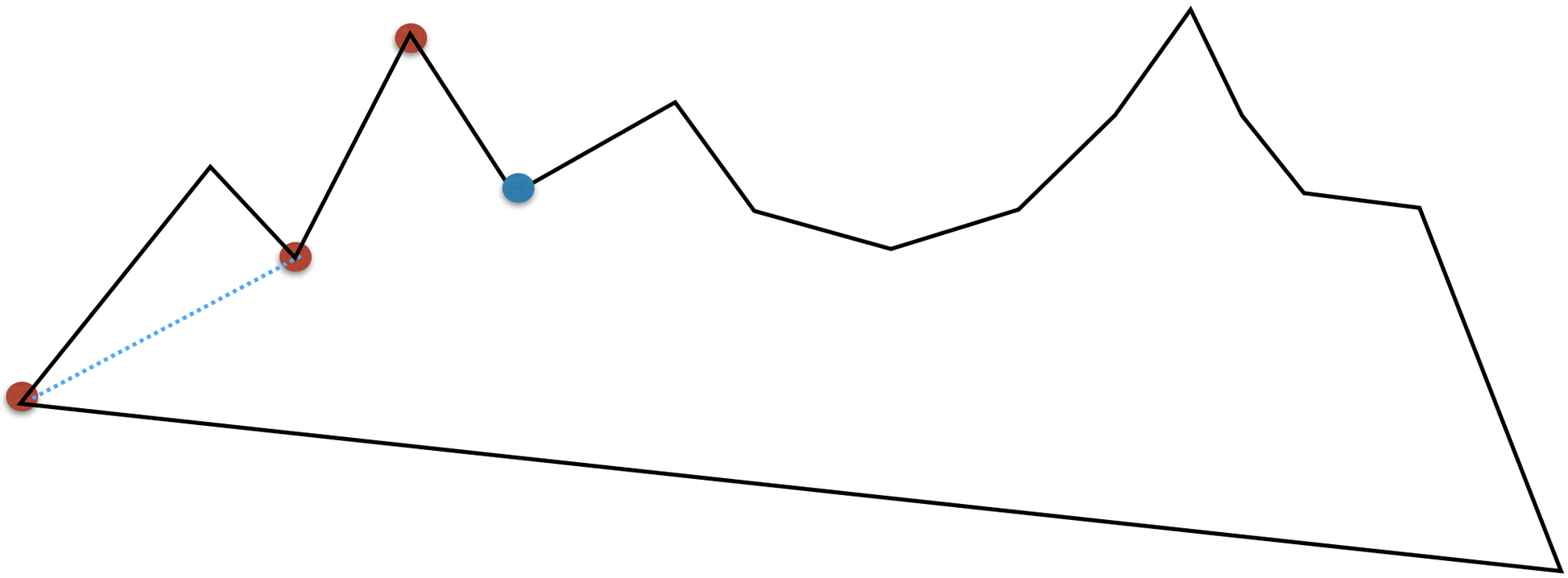

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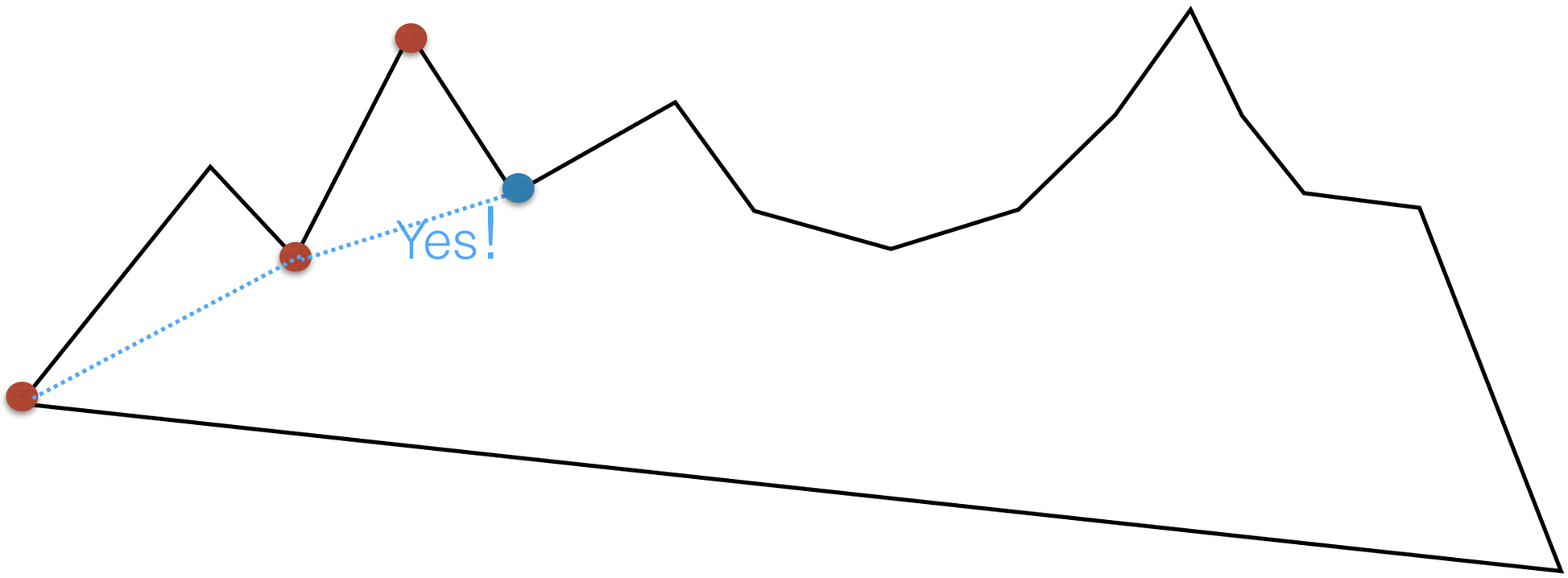

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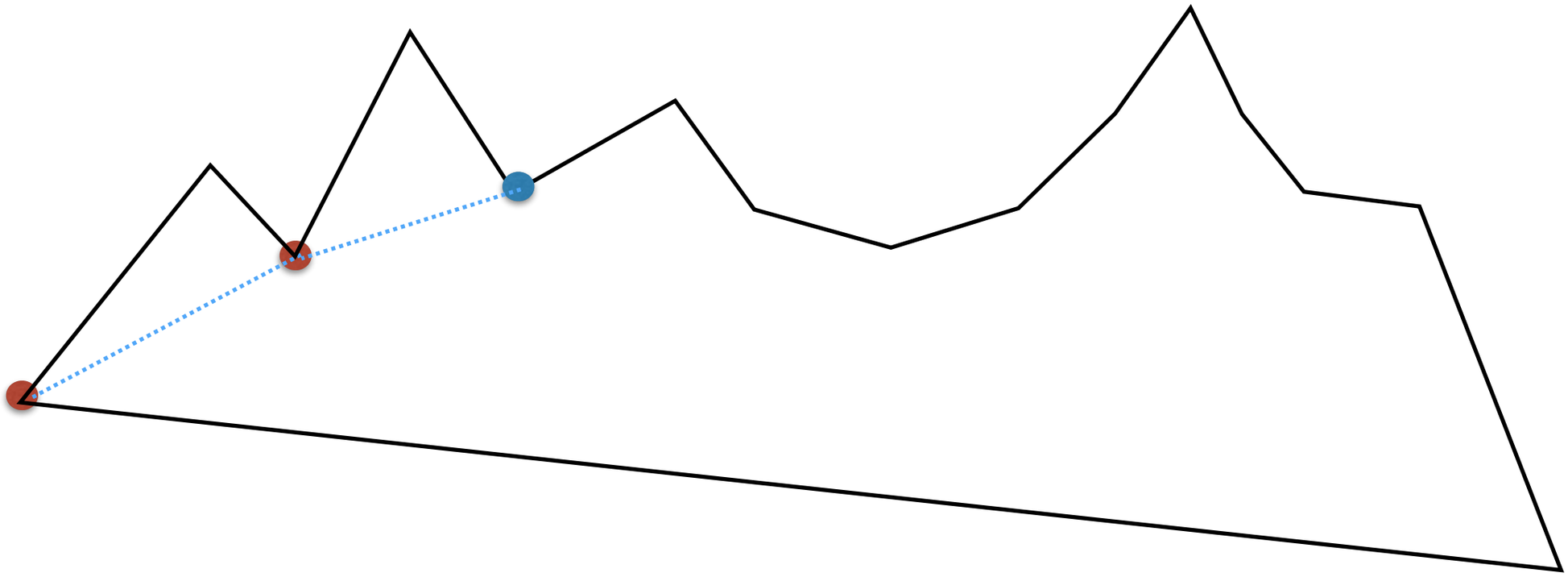

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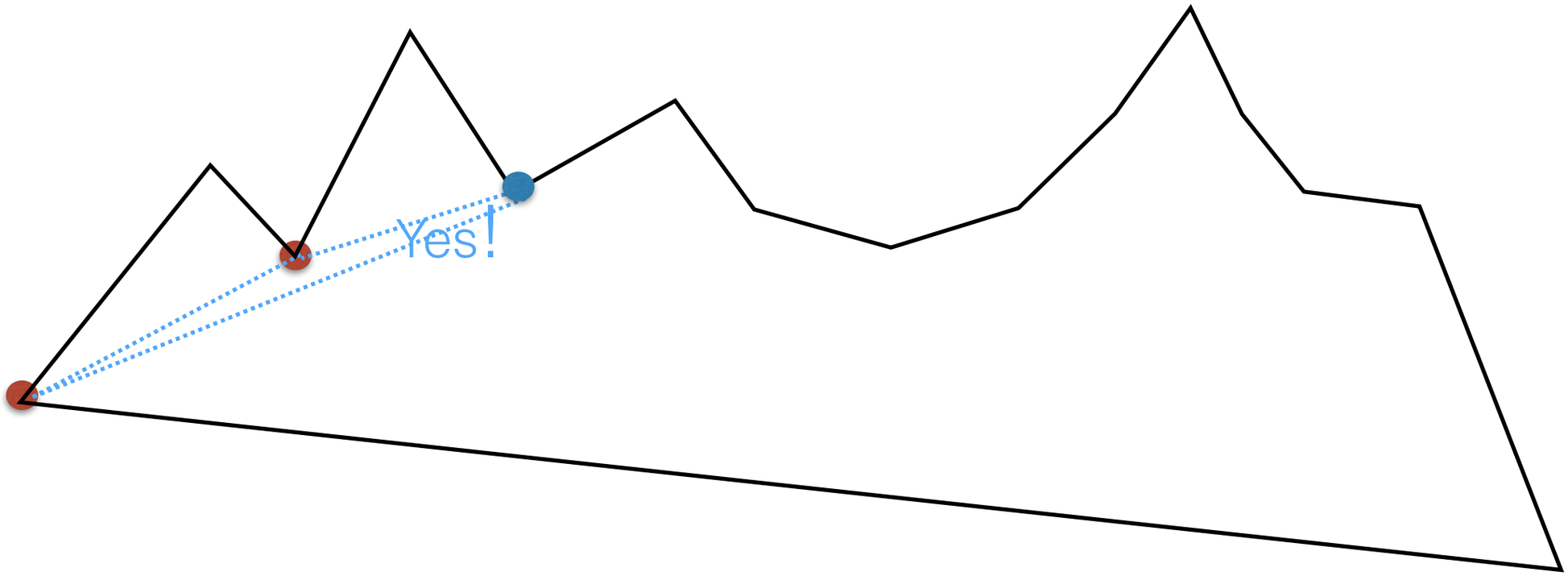

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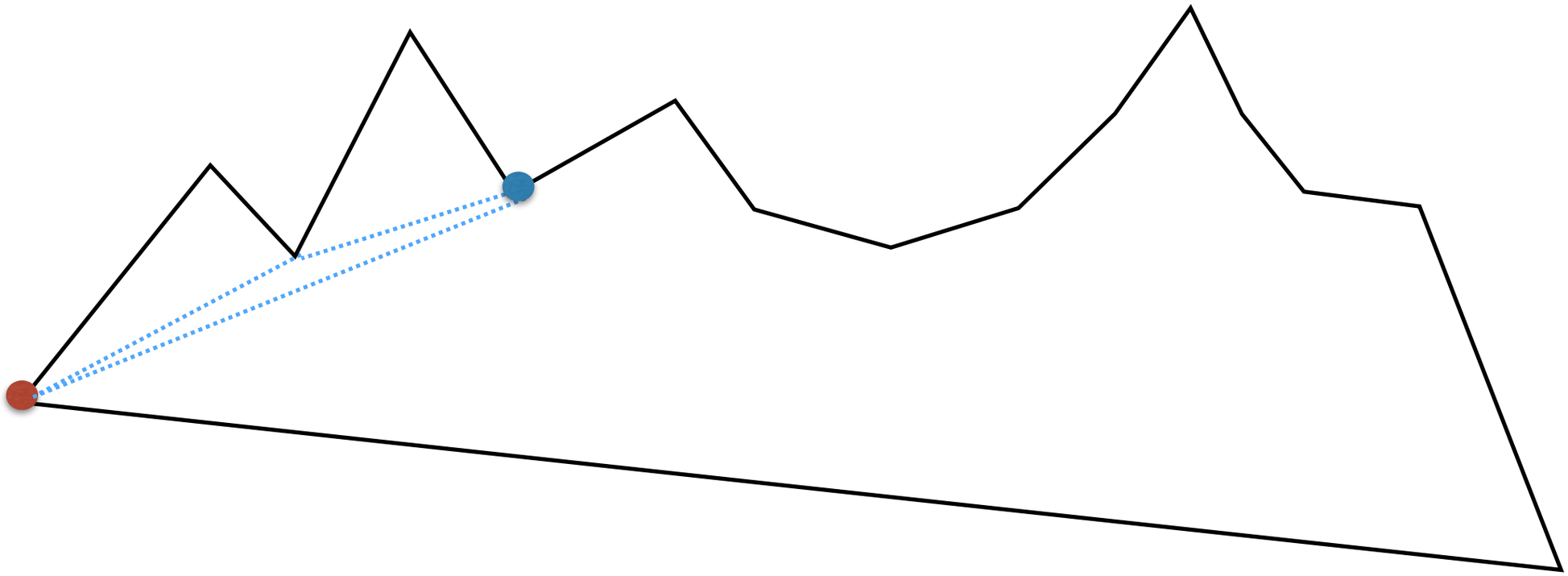

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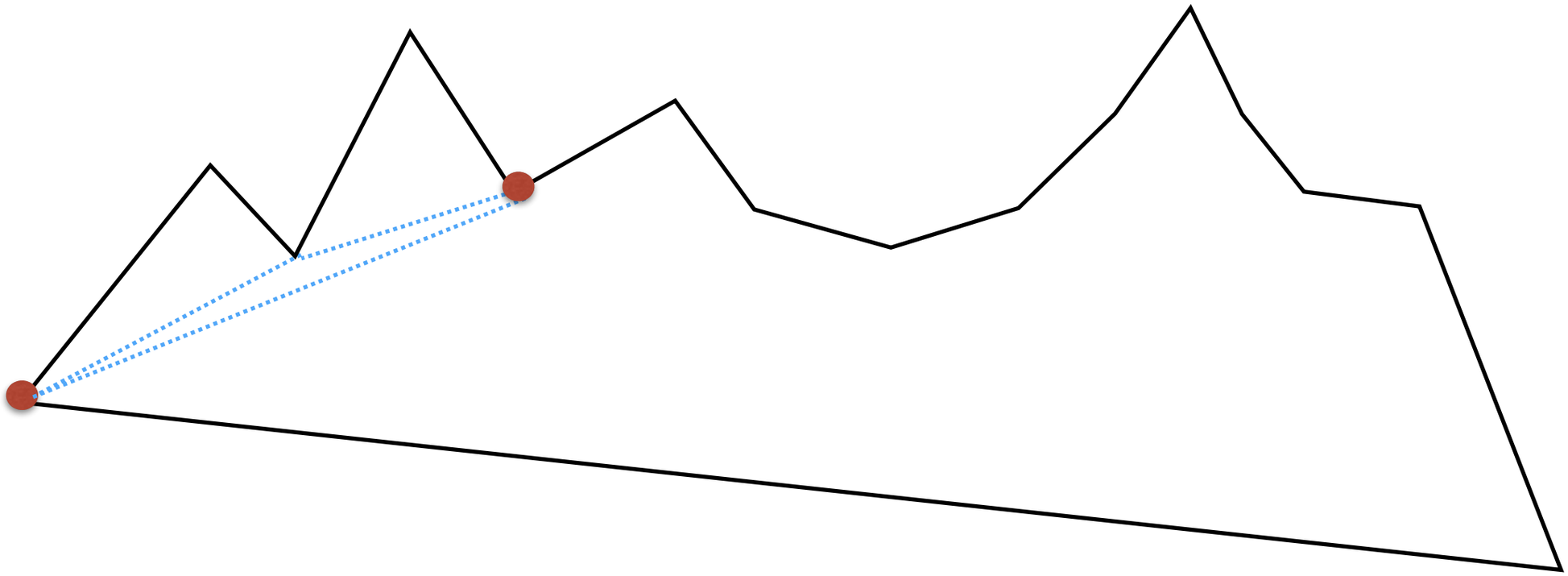

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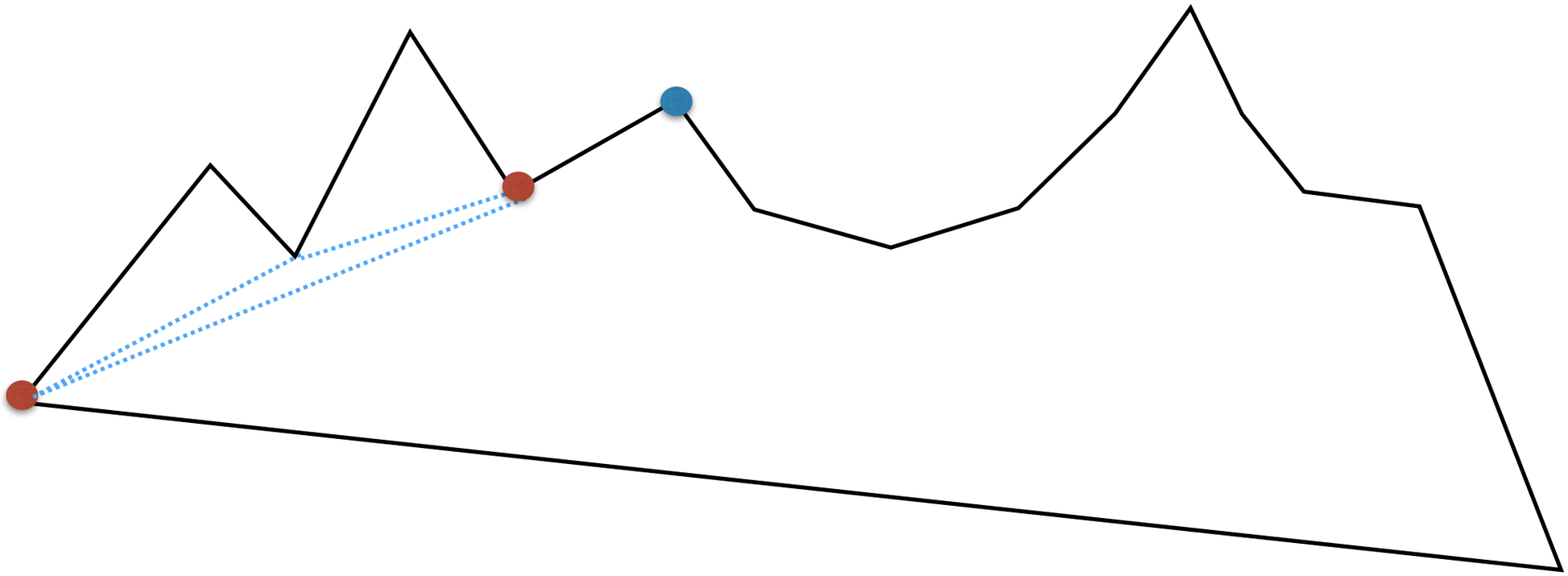

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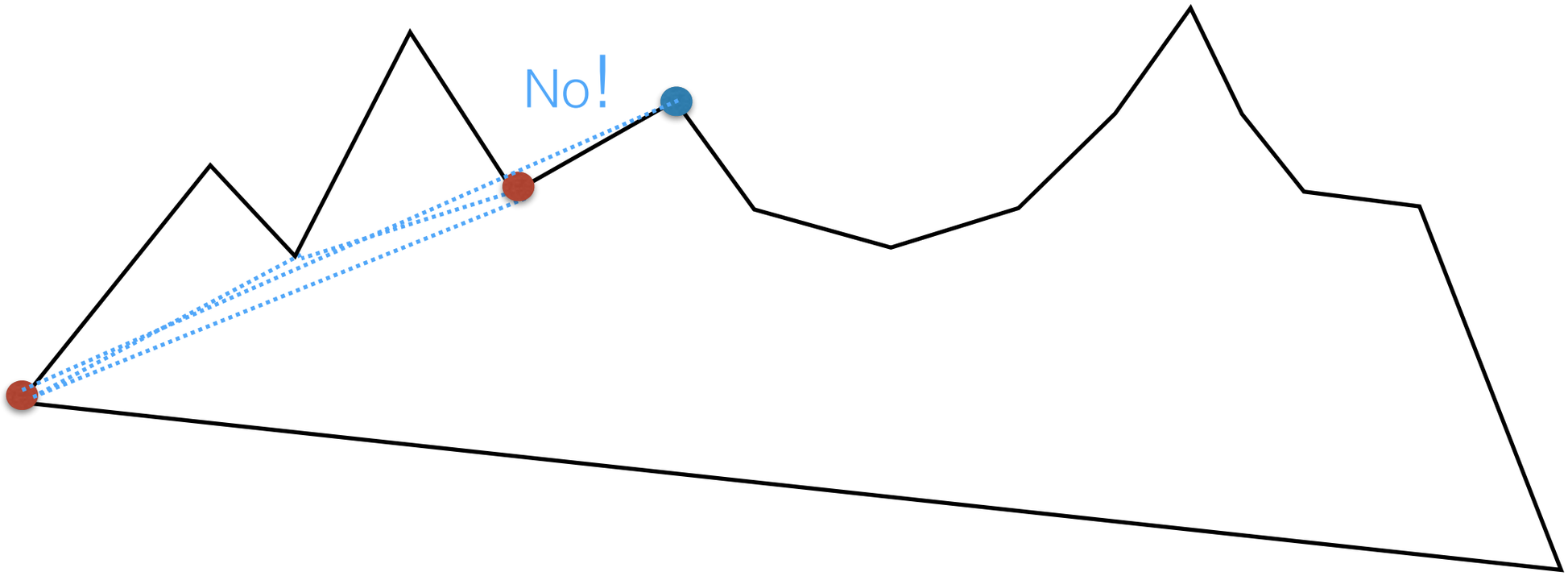

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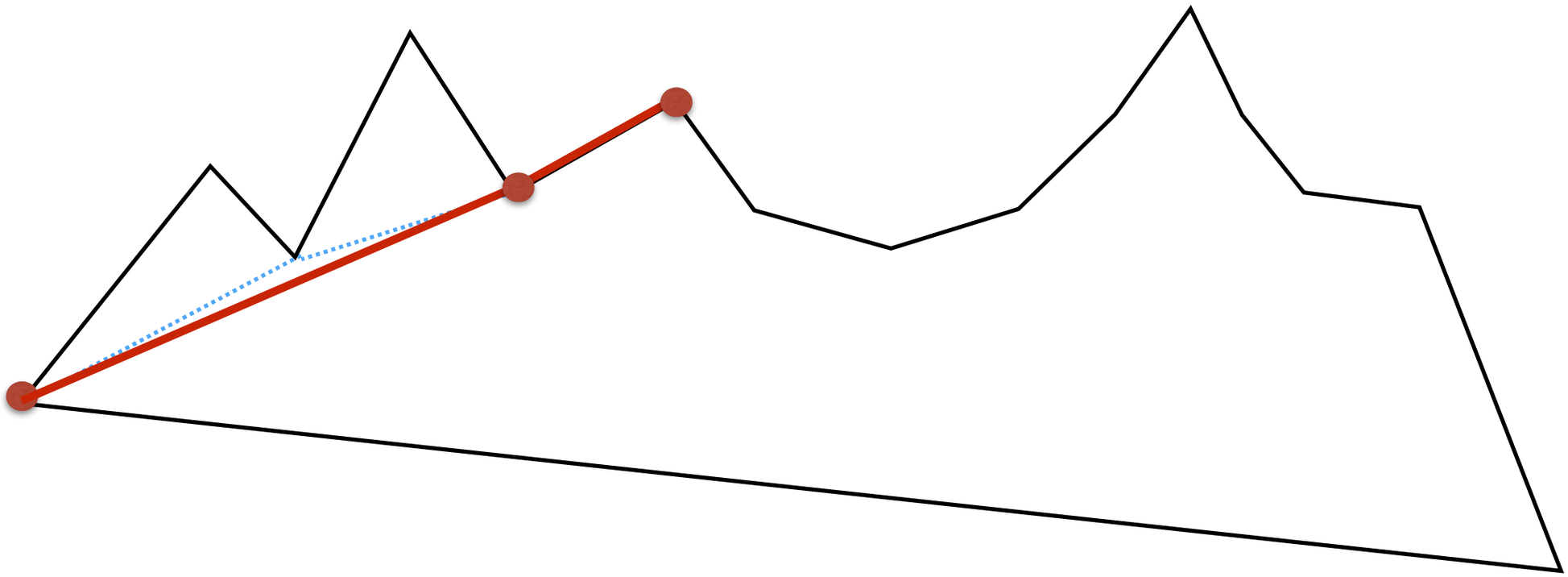

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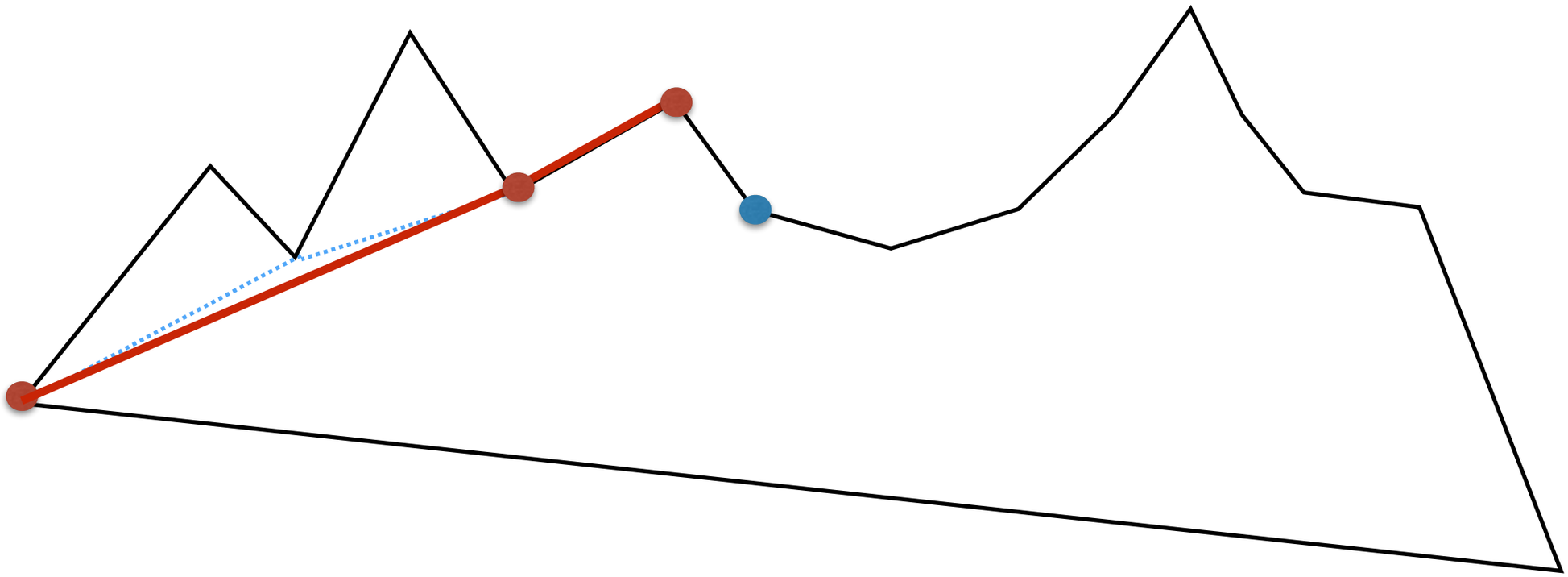

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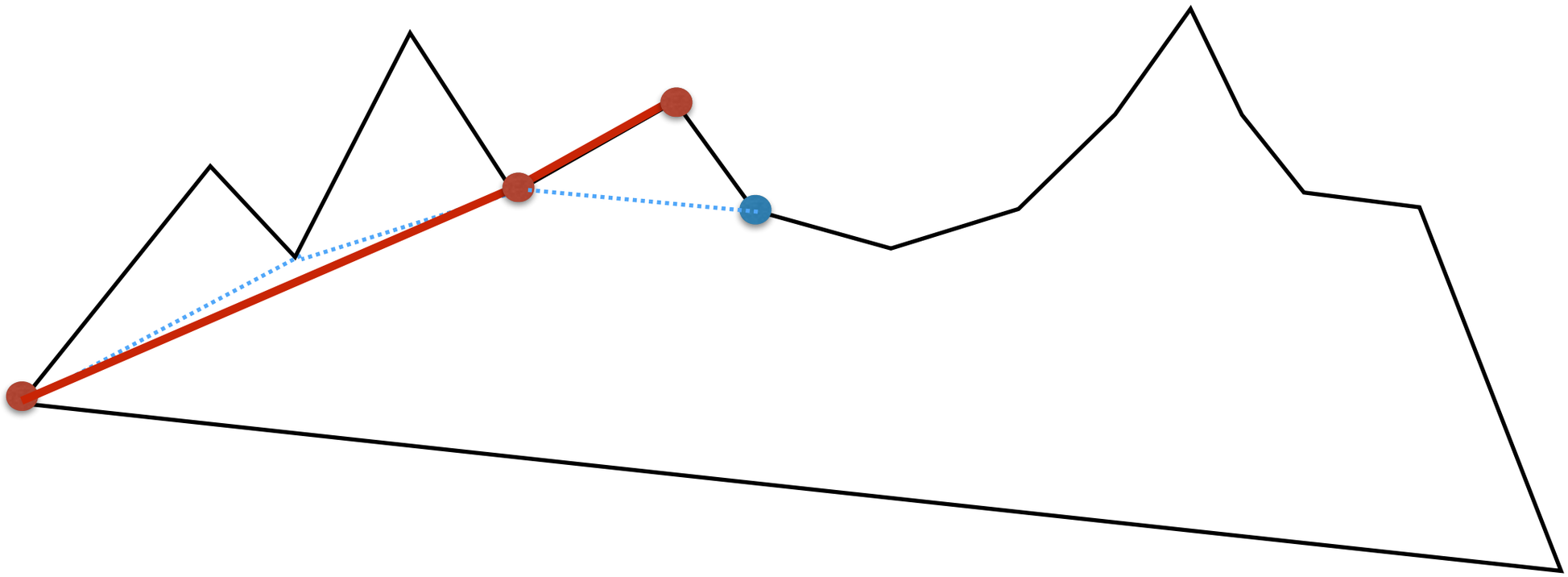

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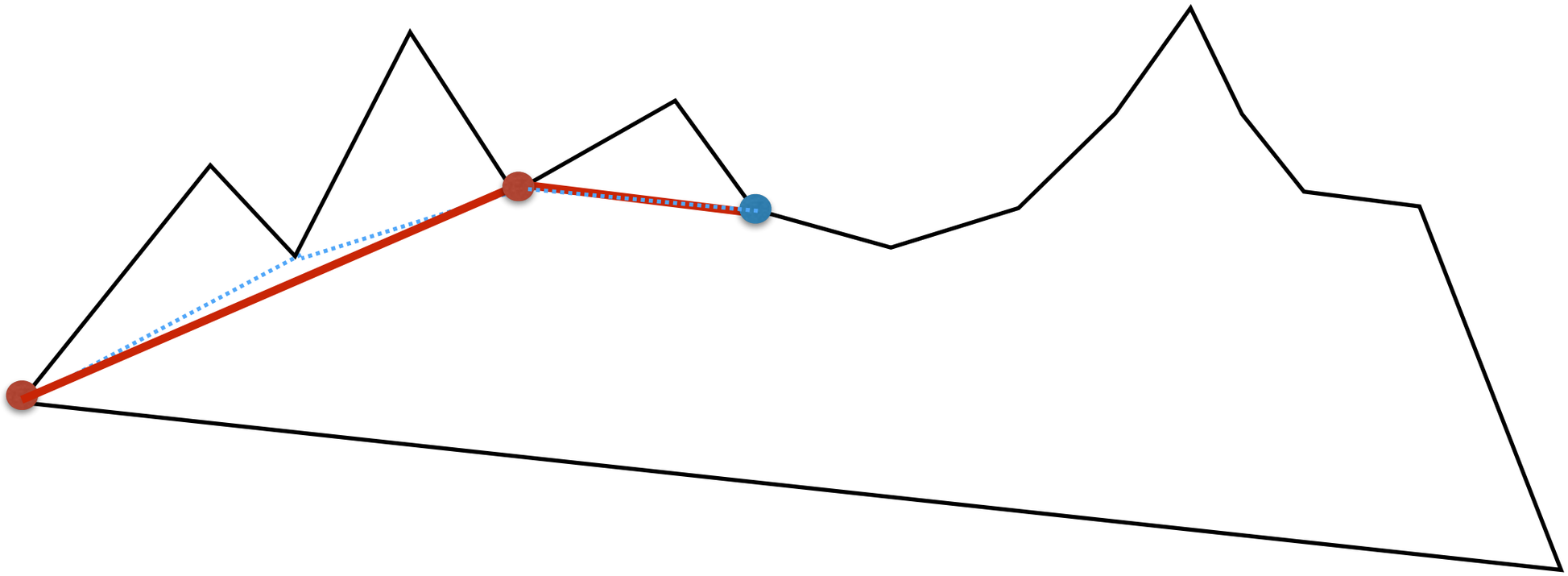

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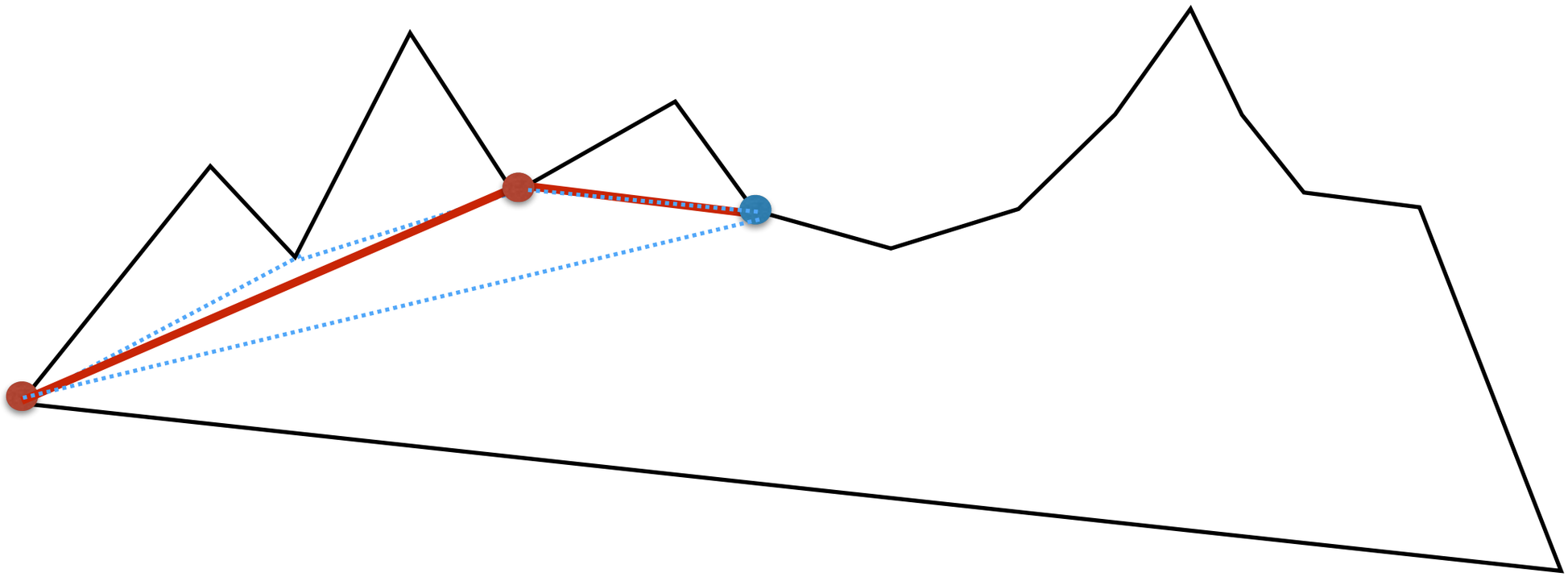

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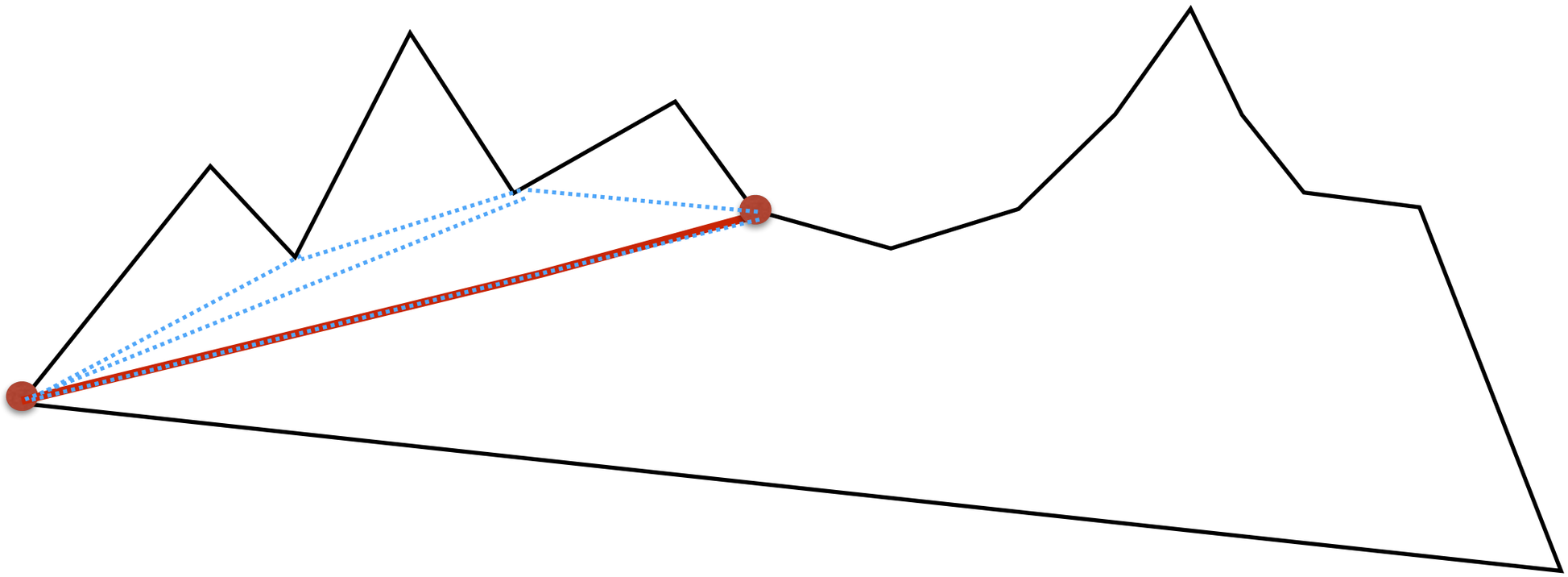

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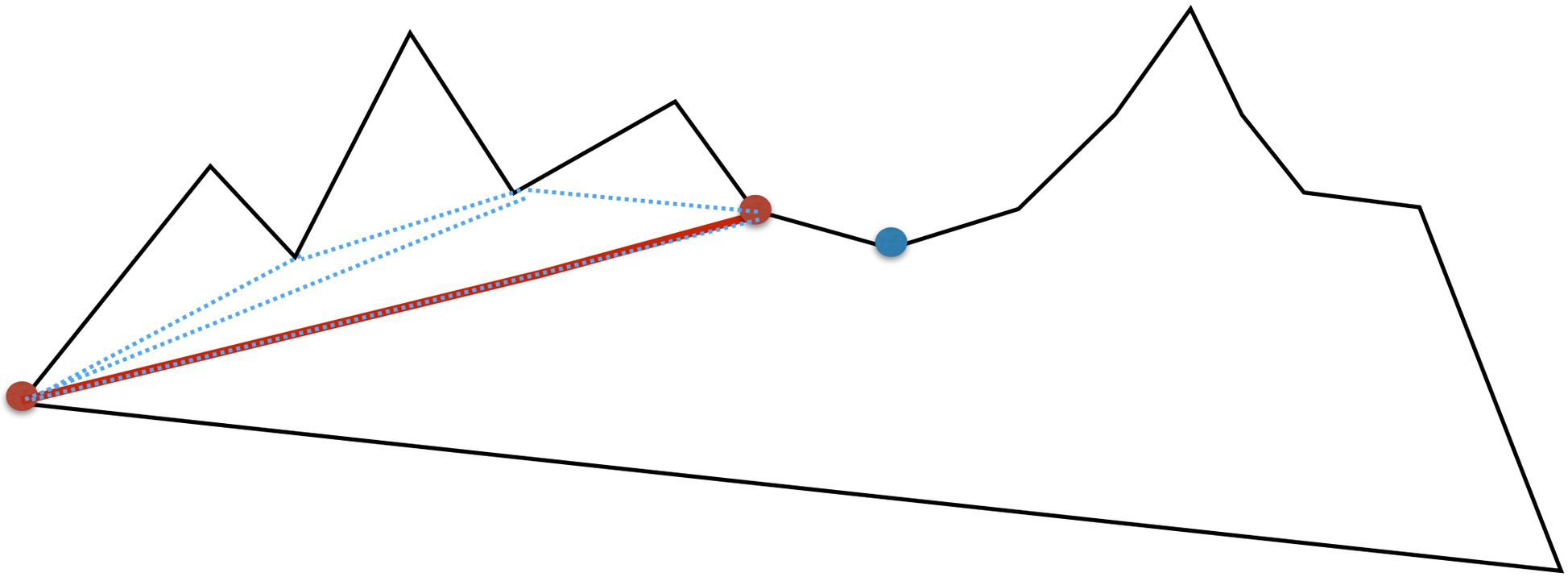

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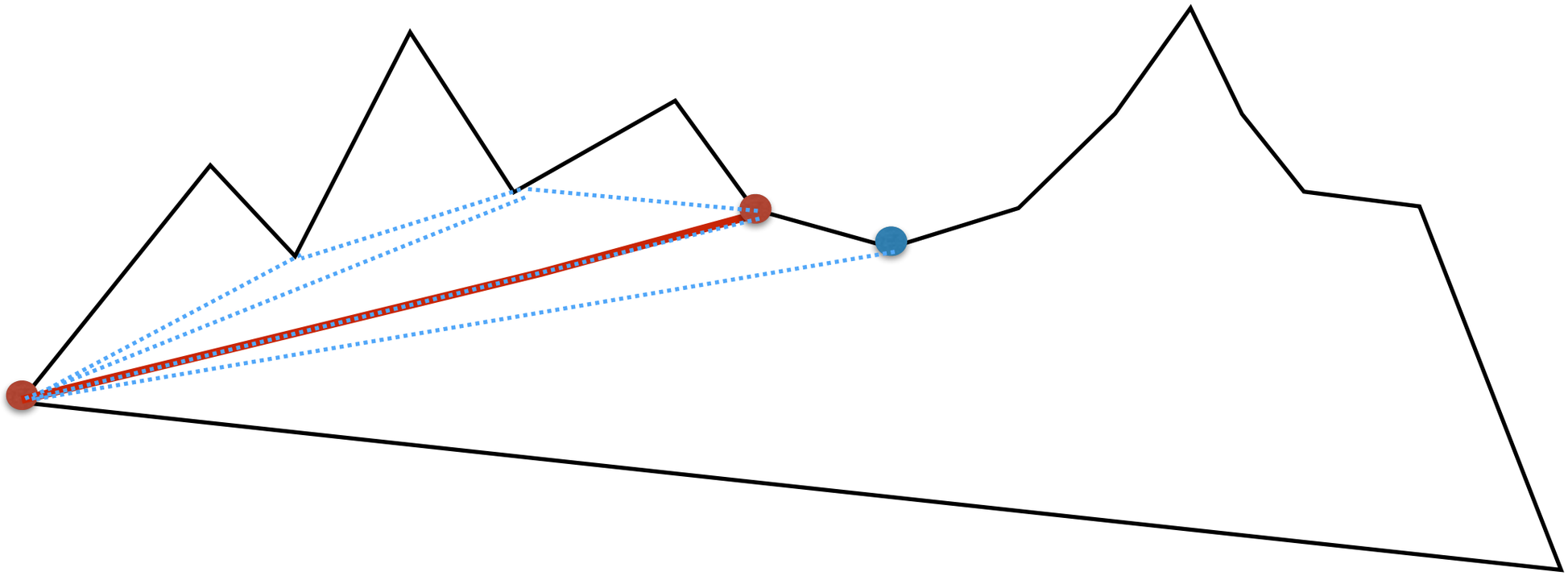

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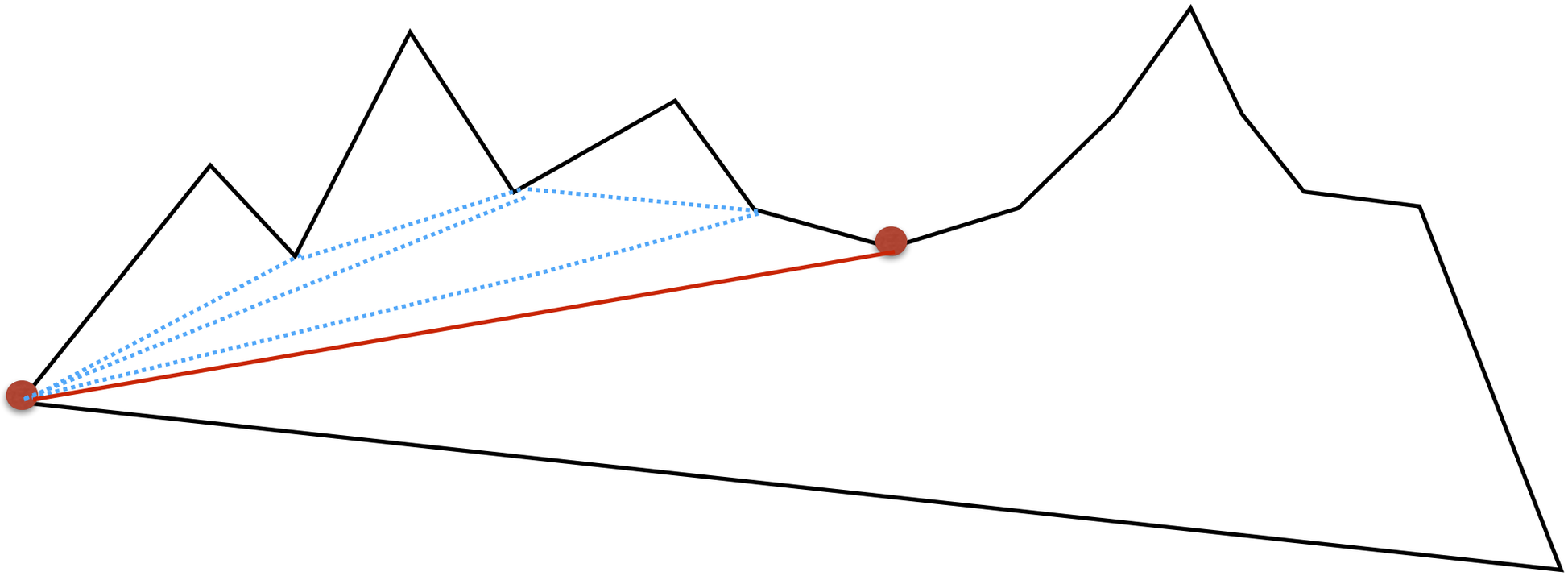

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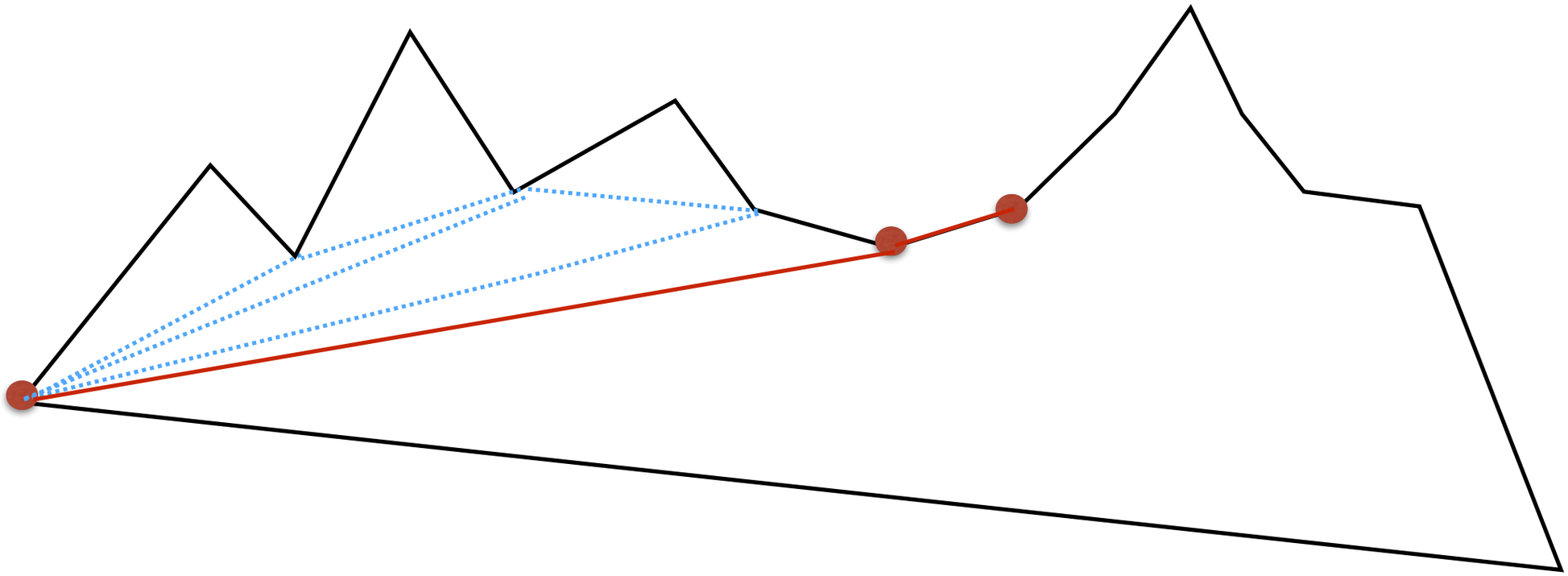

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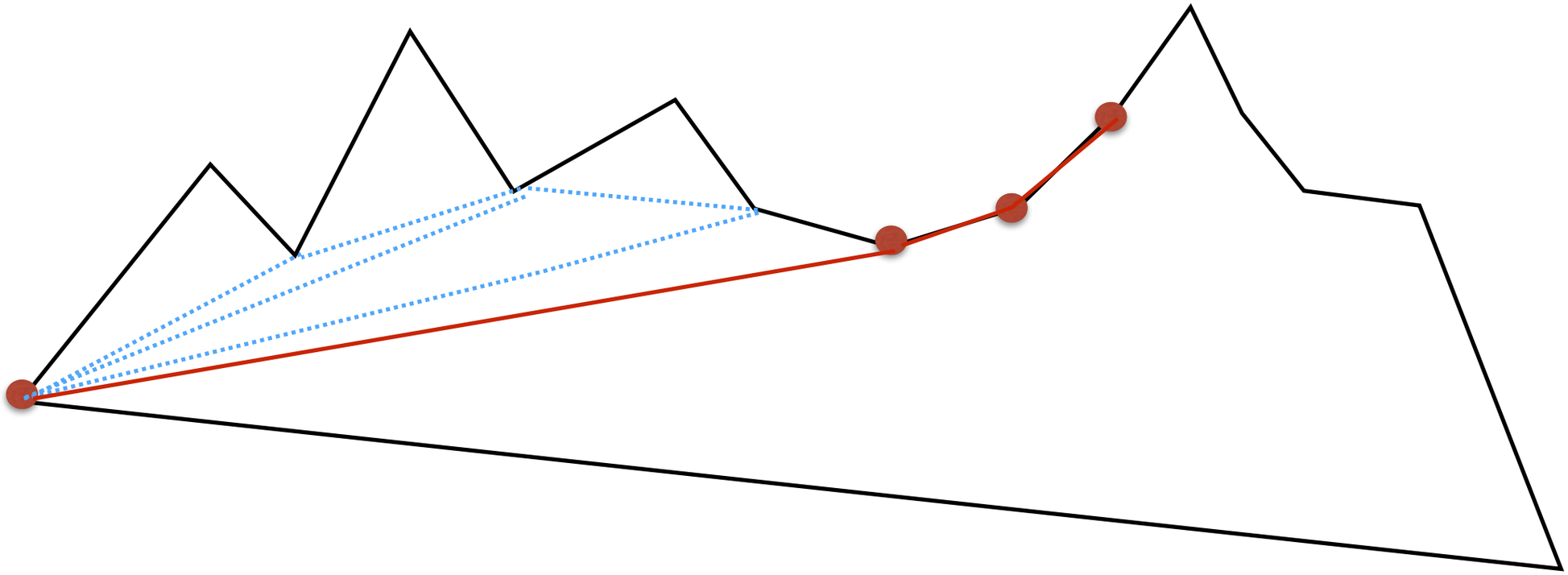

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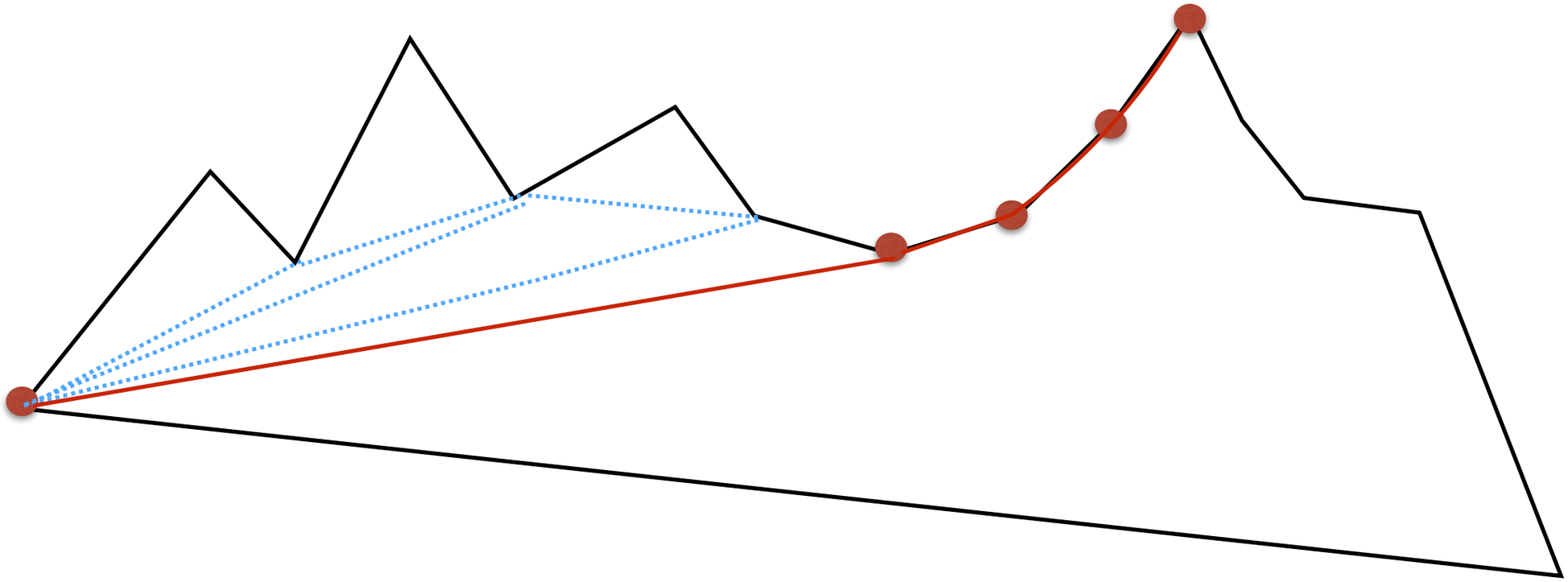

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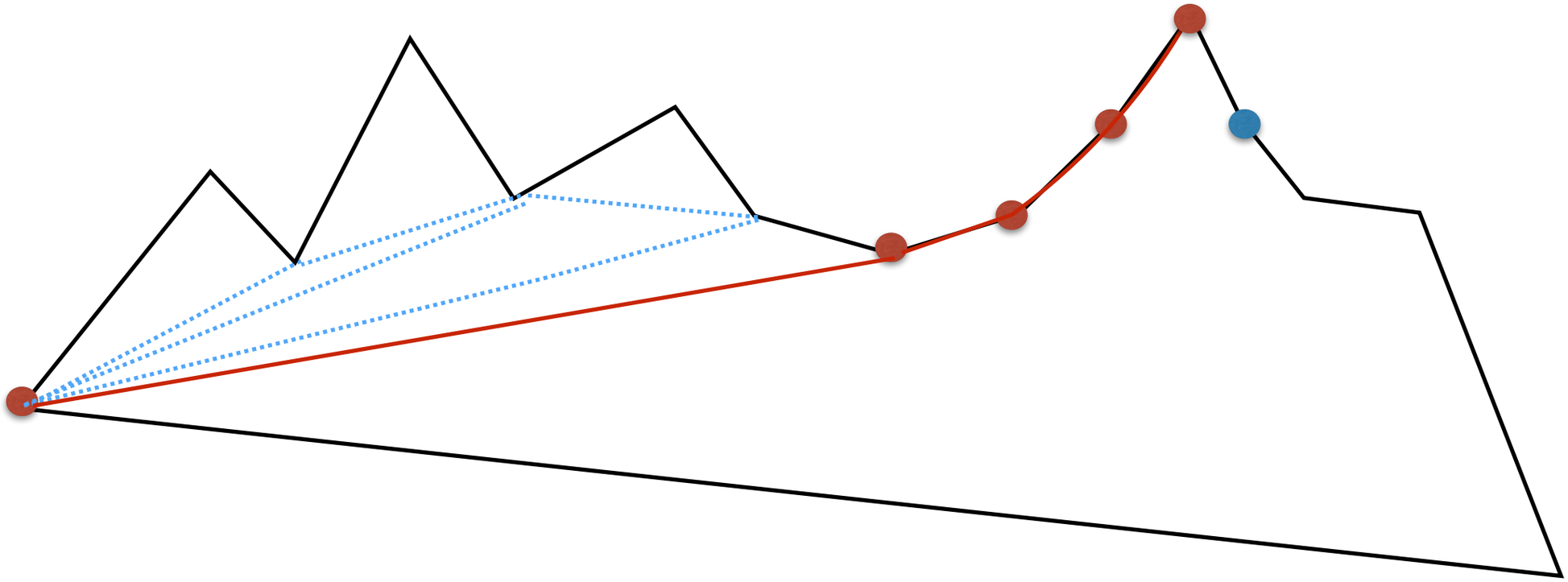

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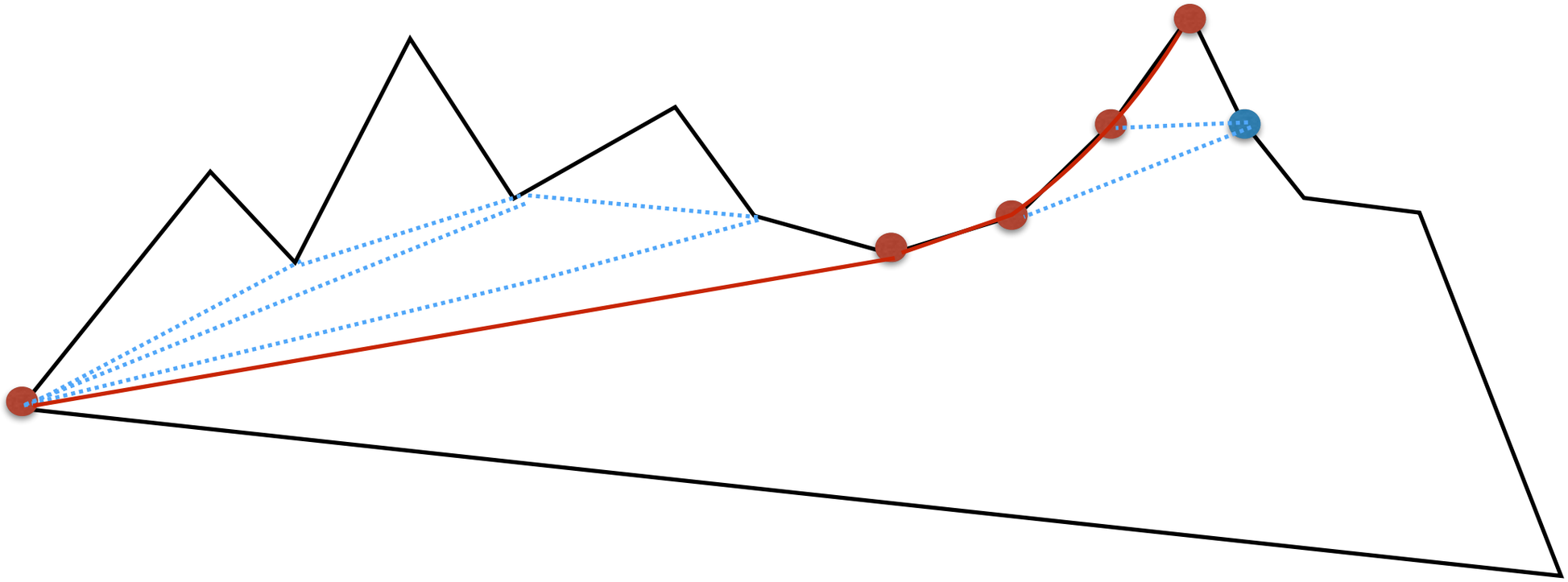

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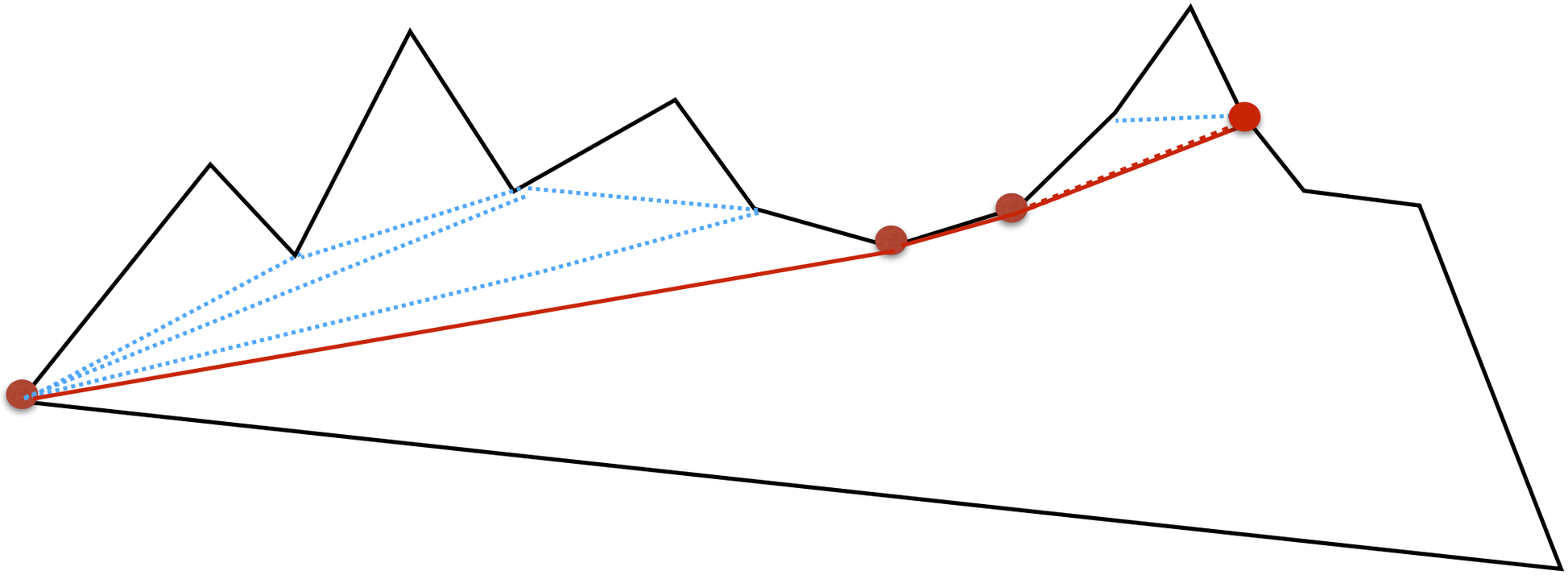

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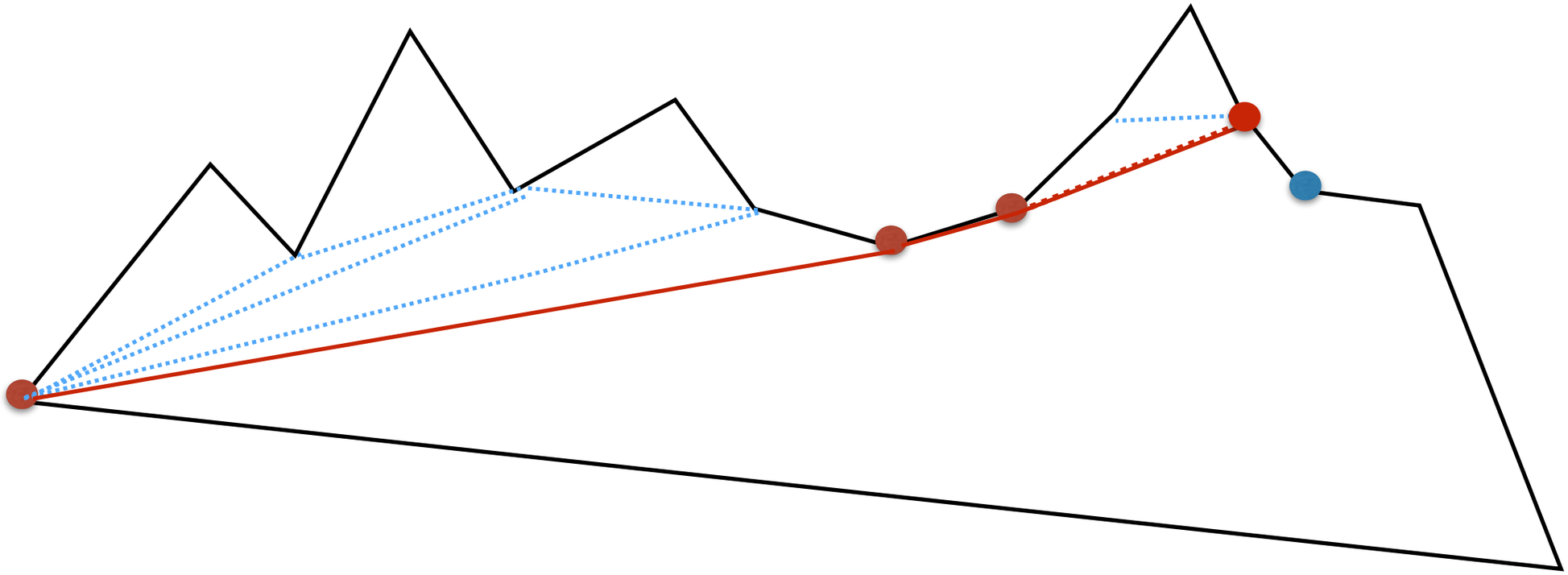

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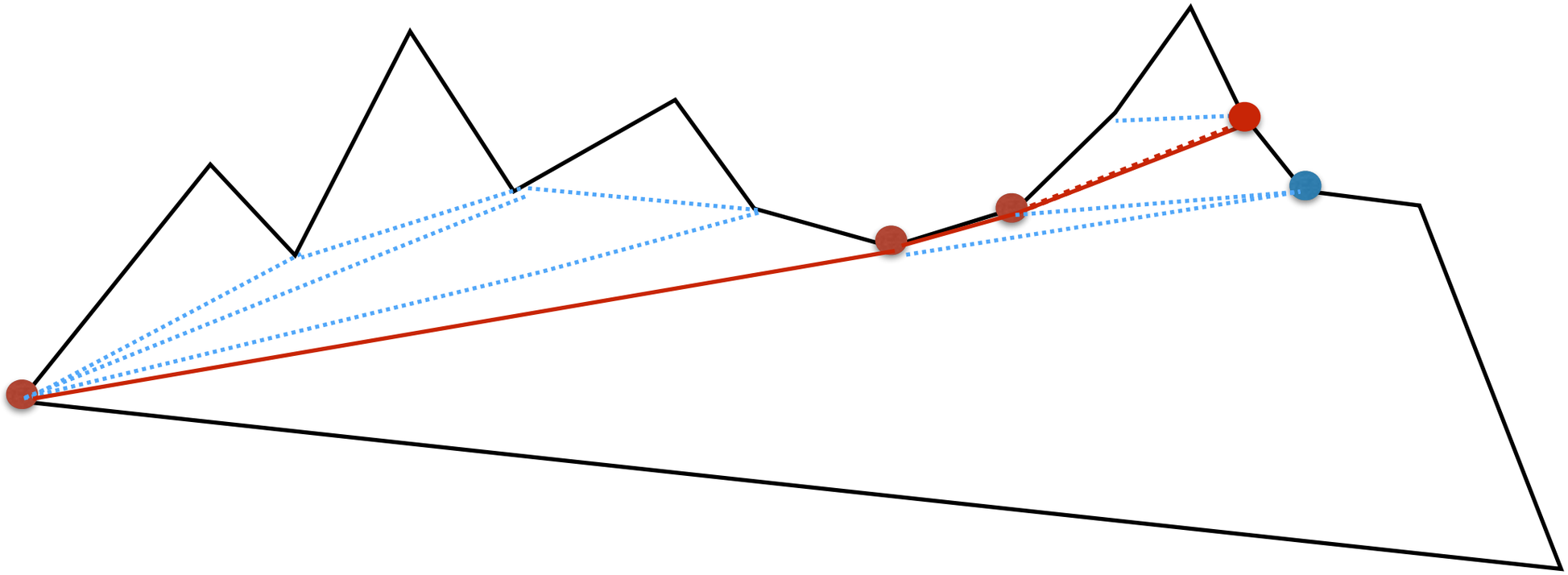

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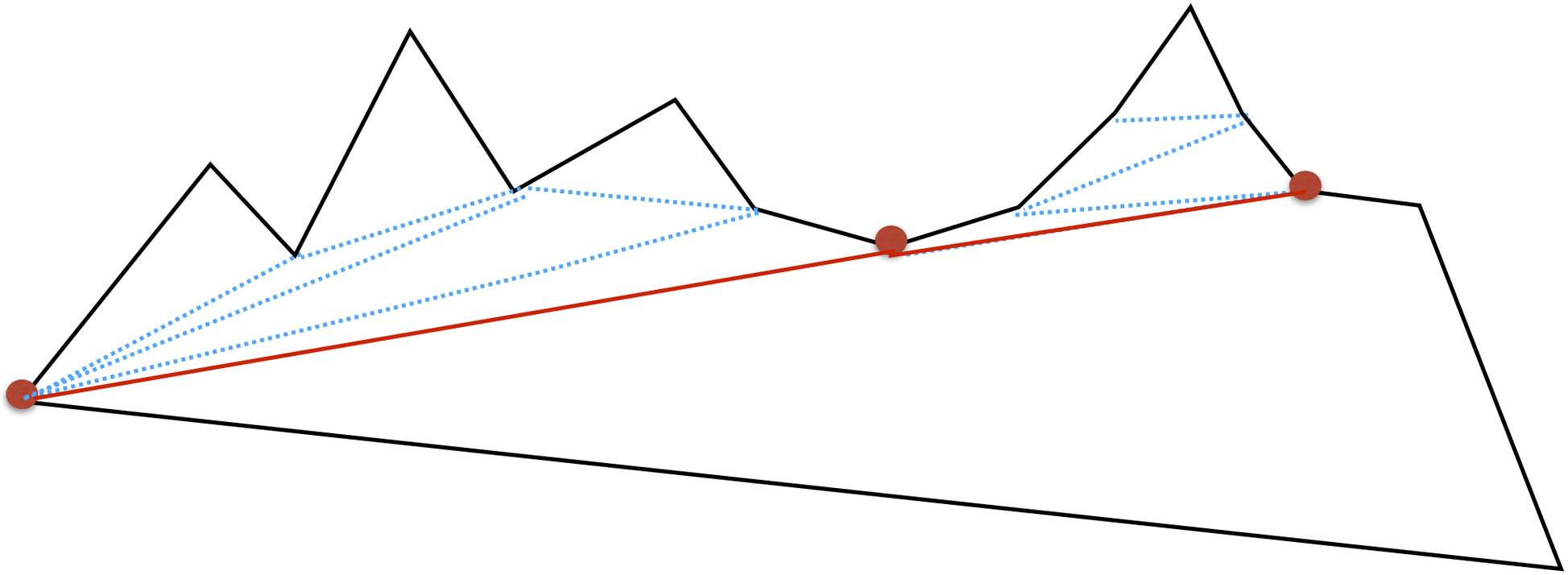

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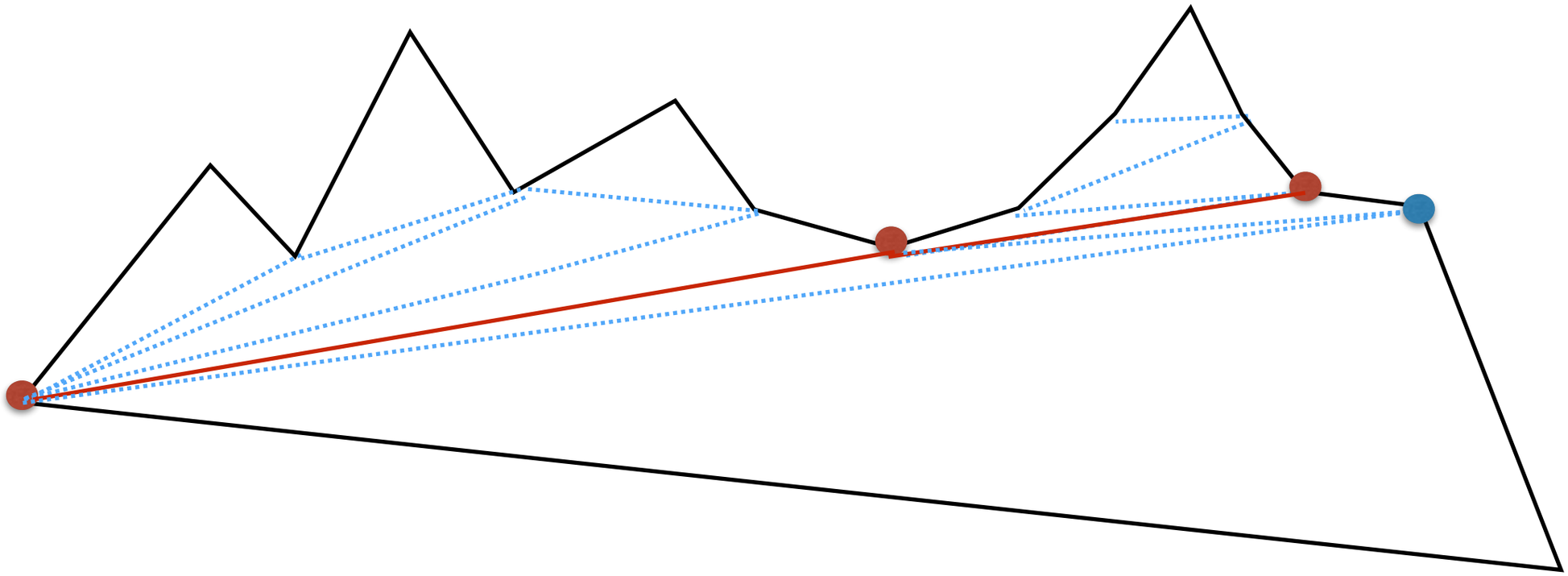

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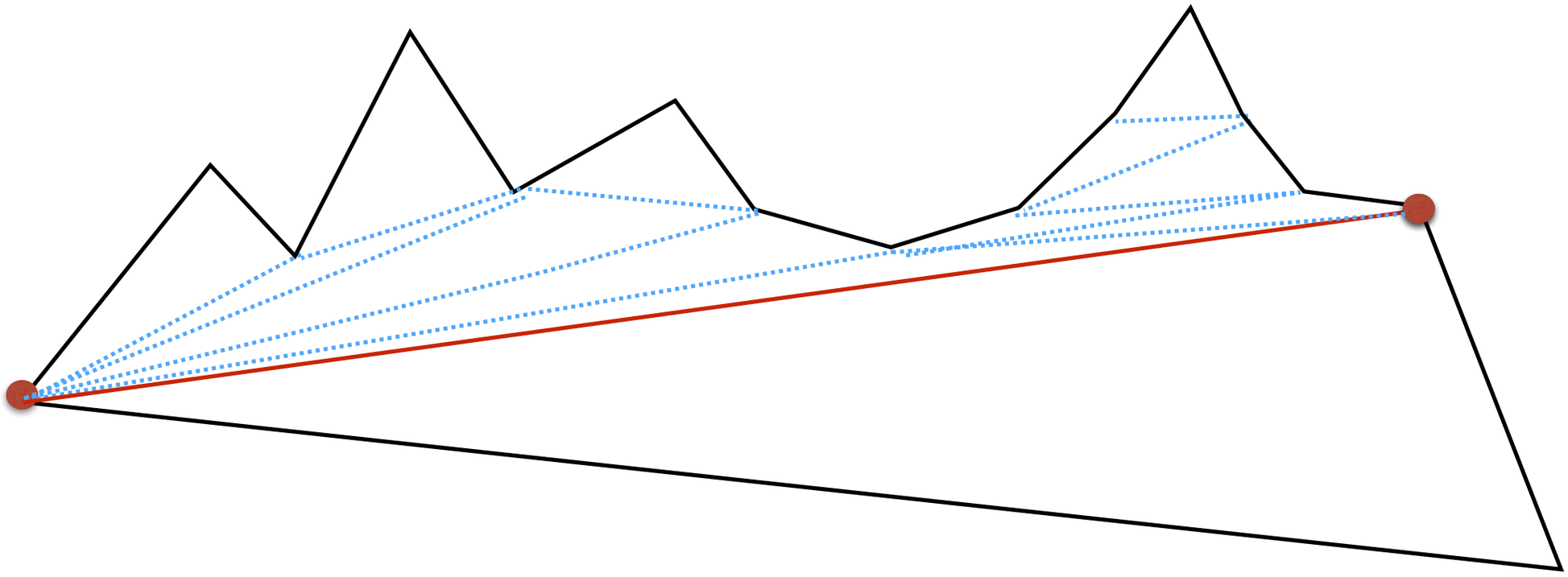
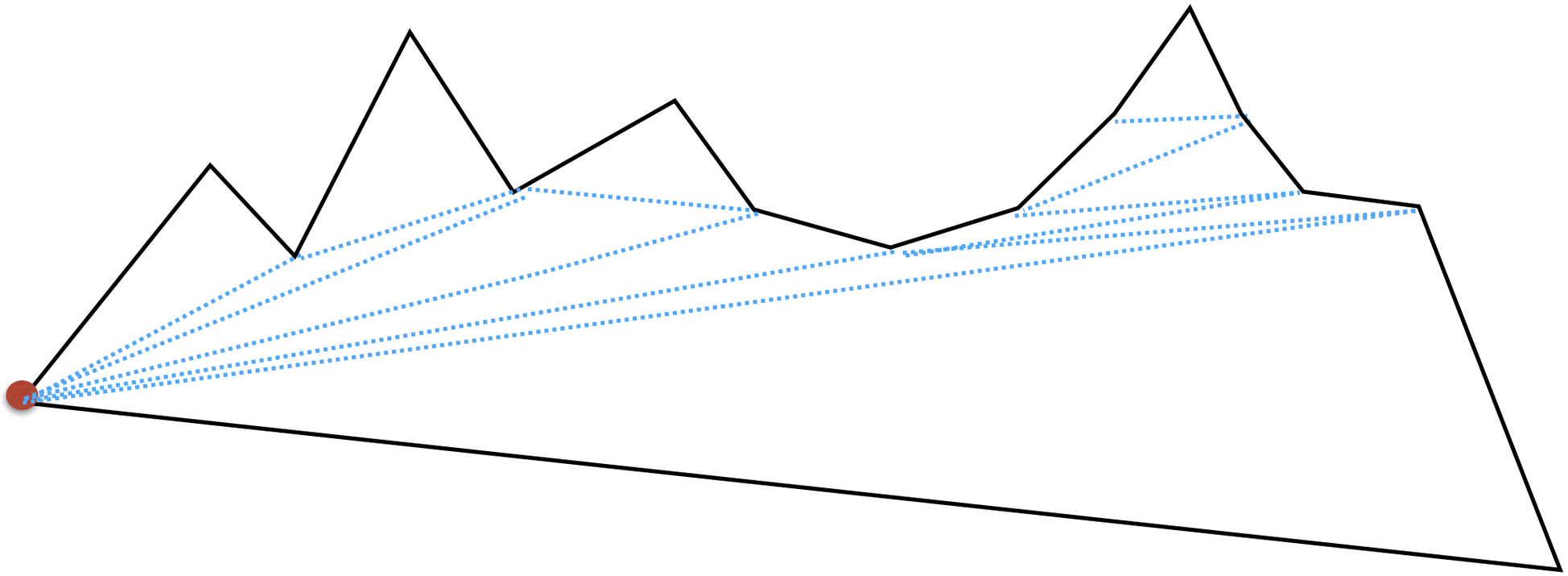


Monotone mountains are easy to triangulate!

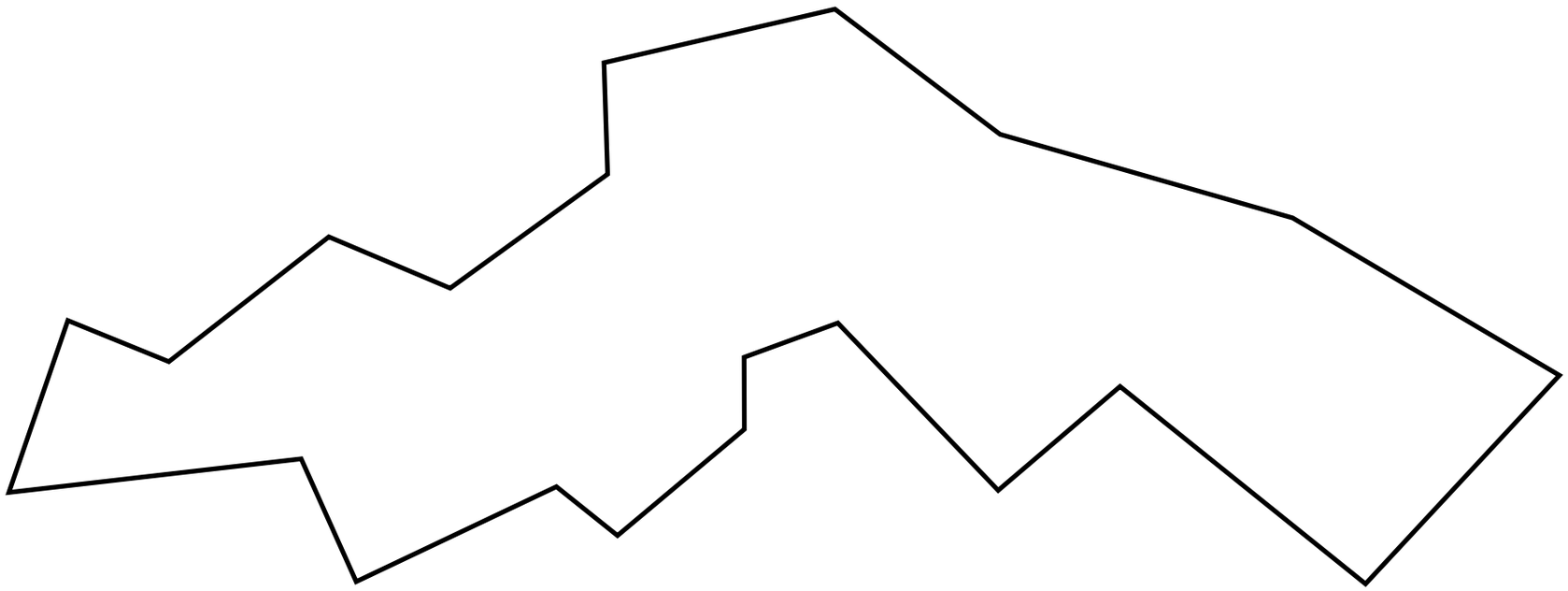


Analysis: O(n) time

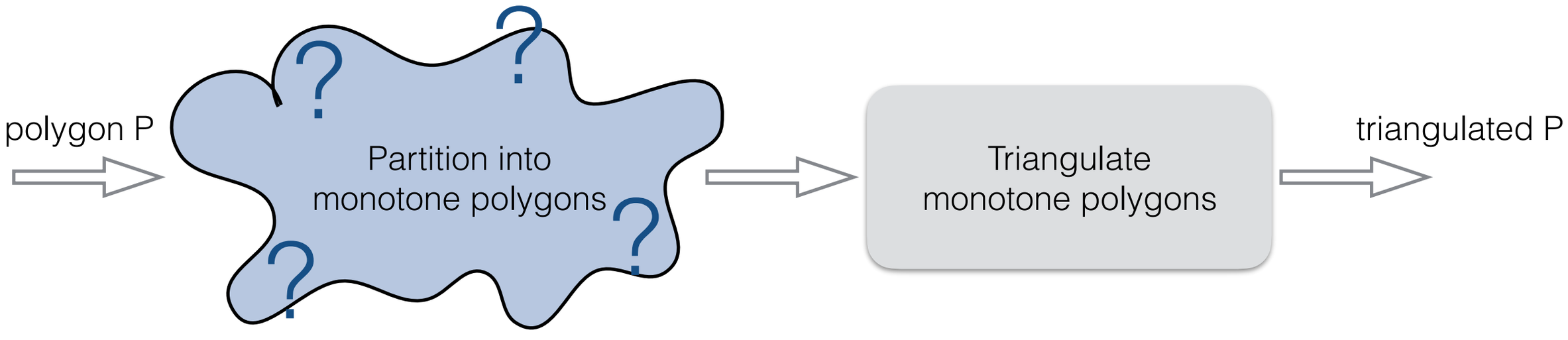
Triangulating Monotone Polygons

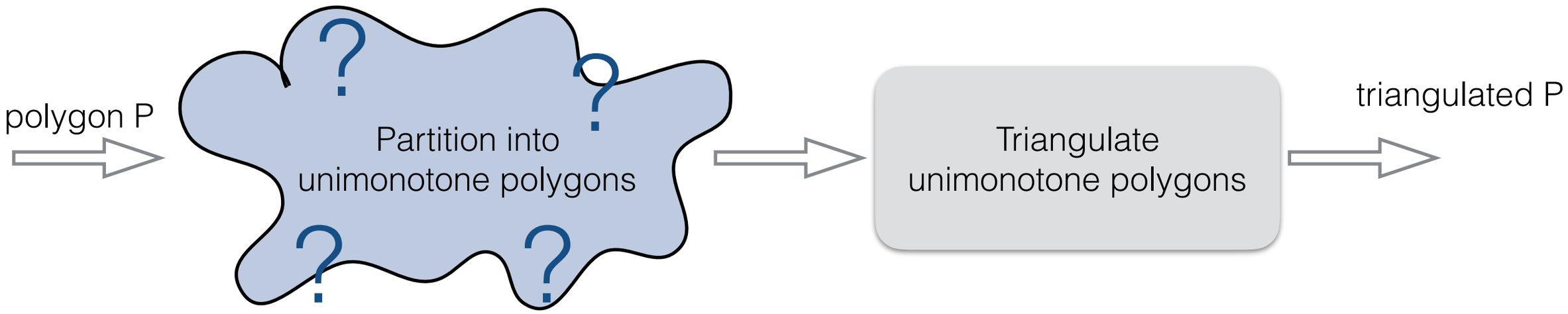
Similar idea, pick the next vertex in x-order. It can be on upper or lower chain.

O(n) time



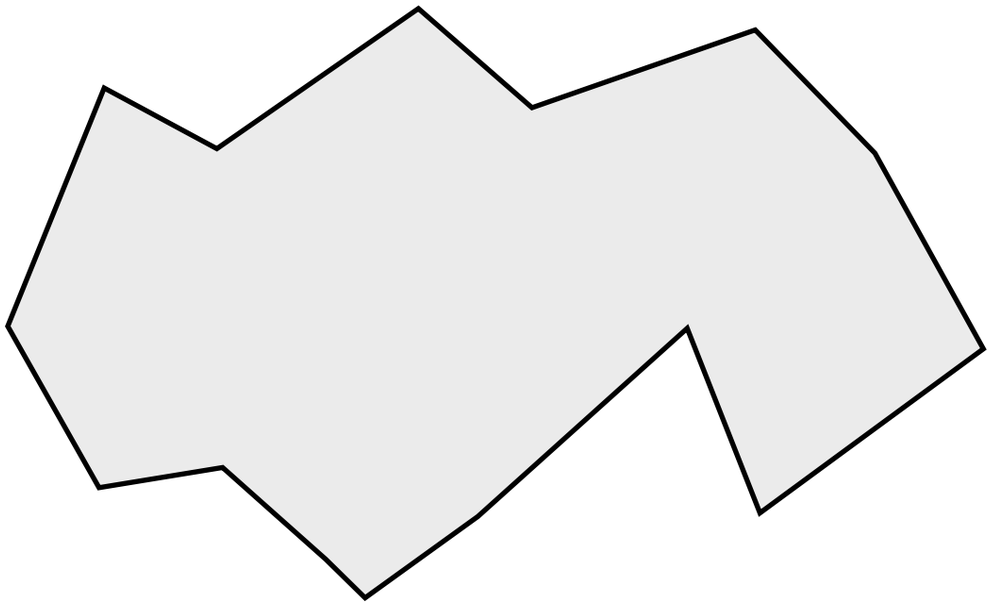
Towards an O(n lg n) Polygon Triangulation Algorithm



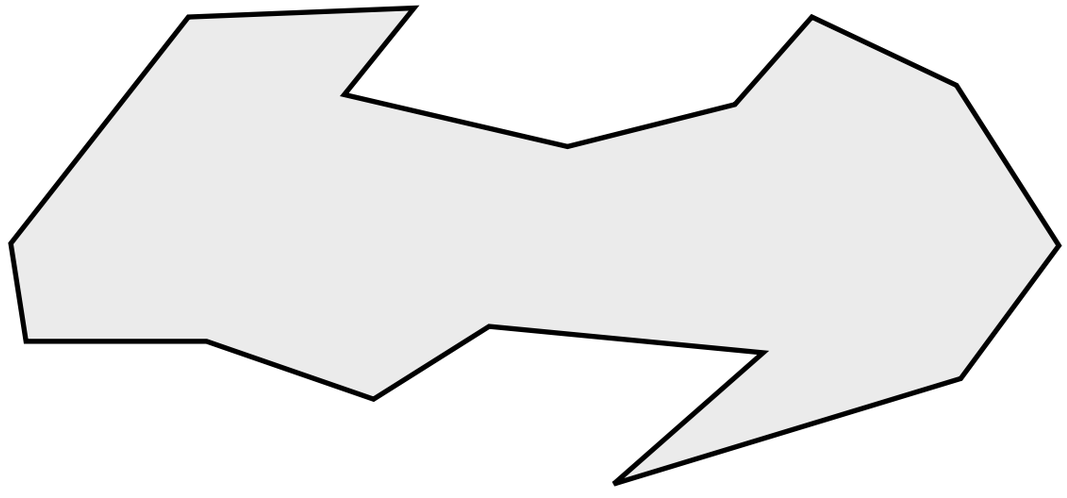


How can we partition a polygon into (uni)monotone pieces?

Intuition



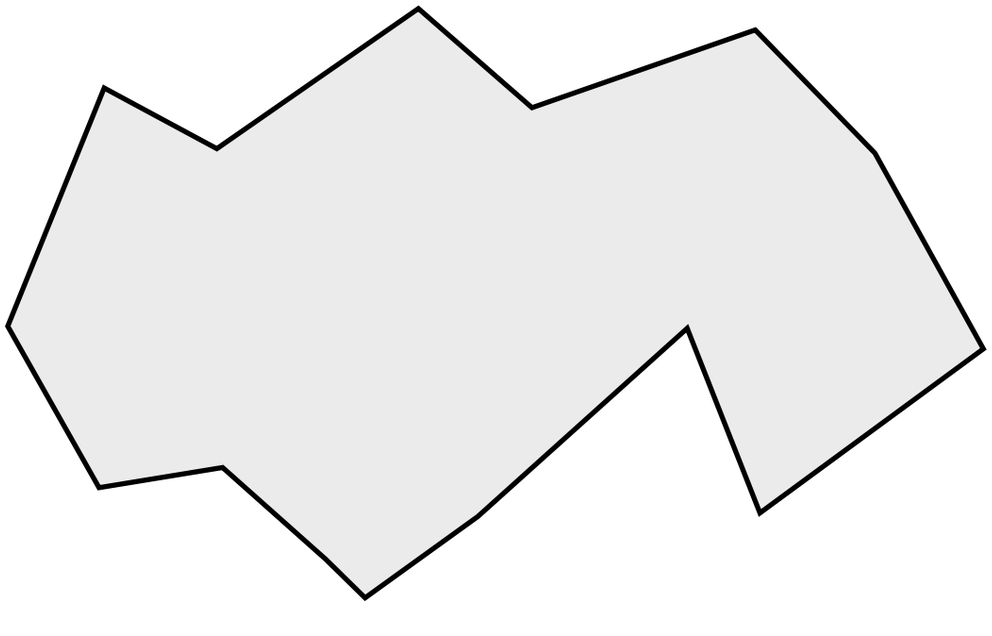
x-monotone



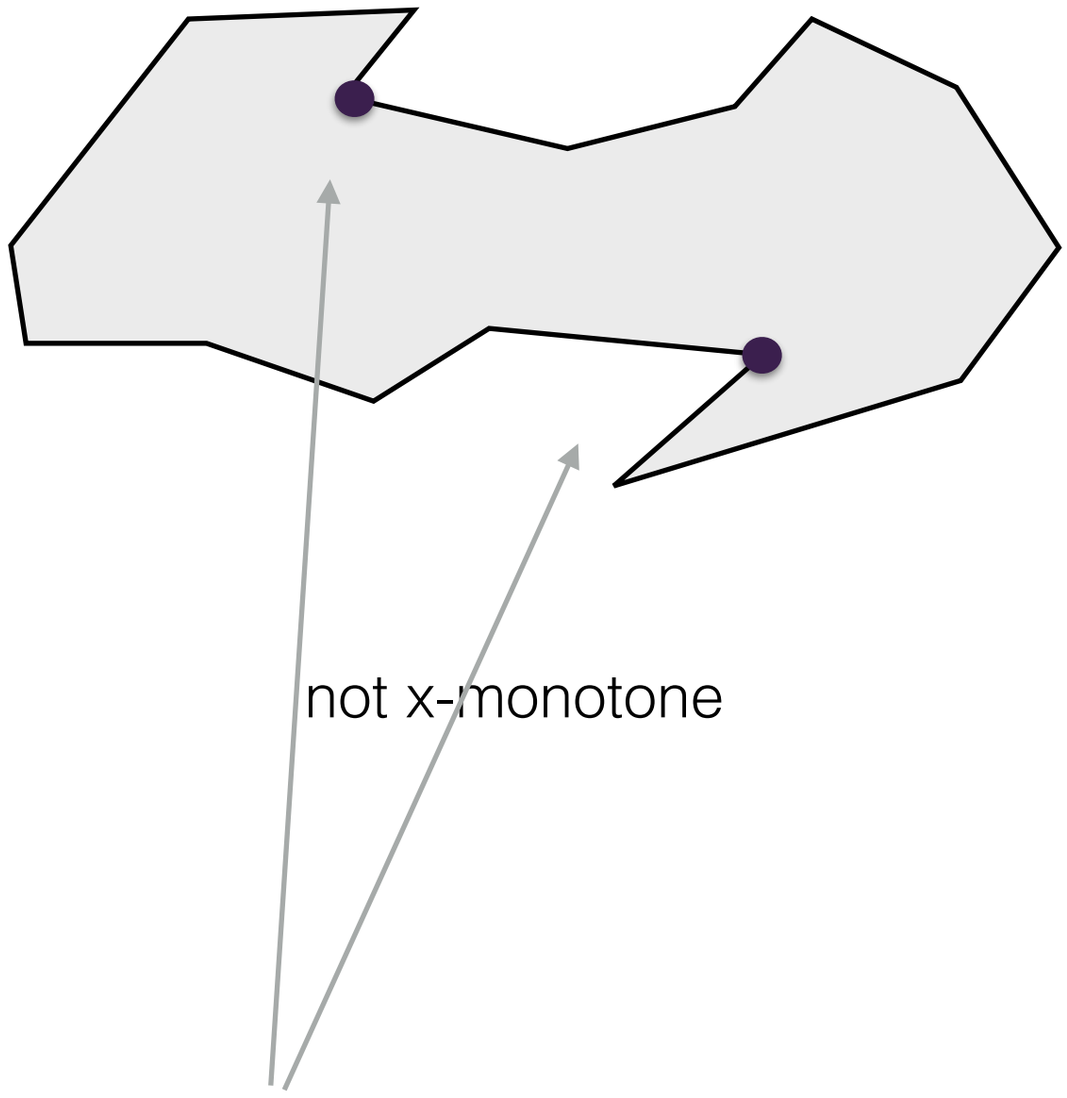
not x-monotone

What makes a polygon **not** monotone?

Intuition



x-monotone

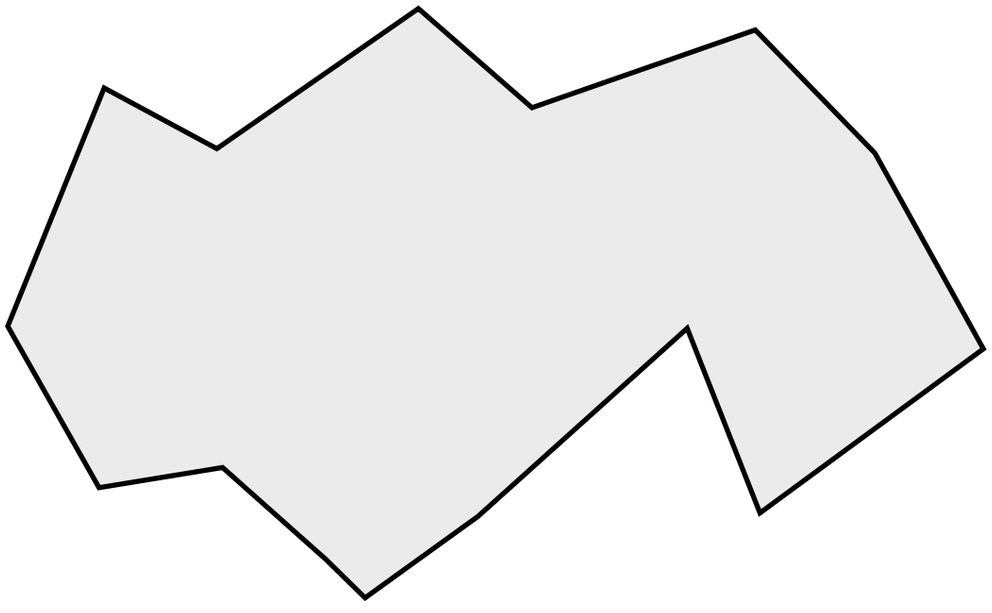


not x-monotone

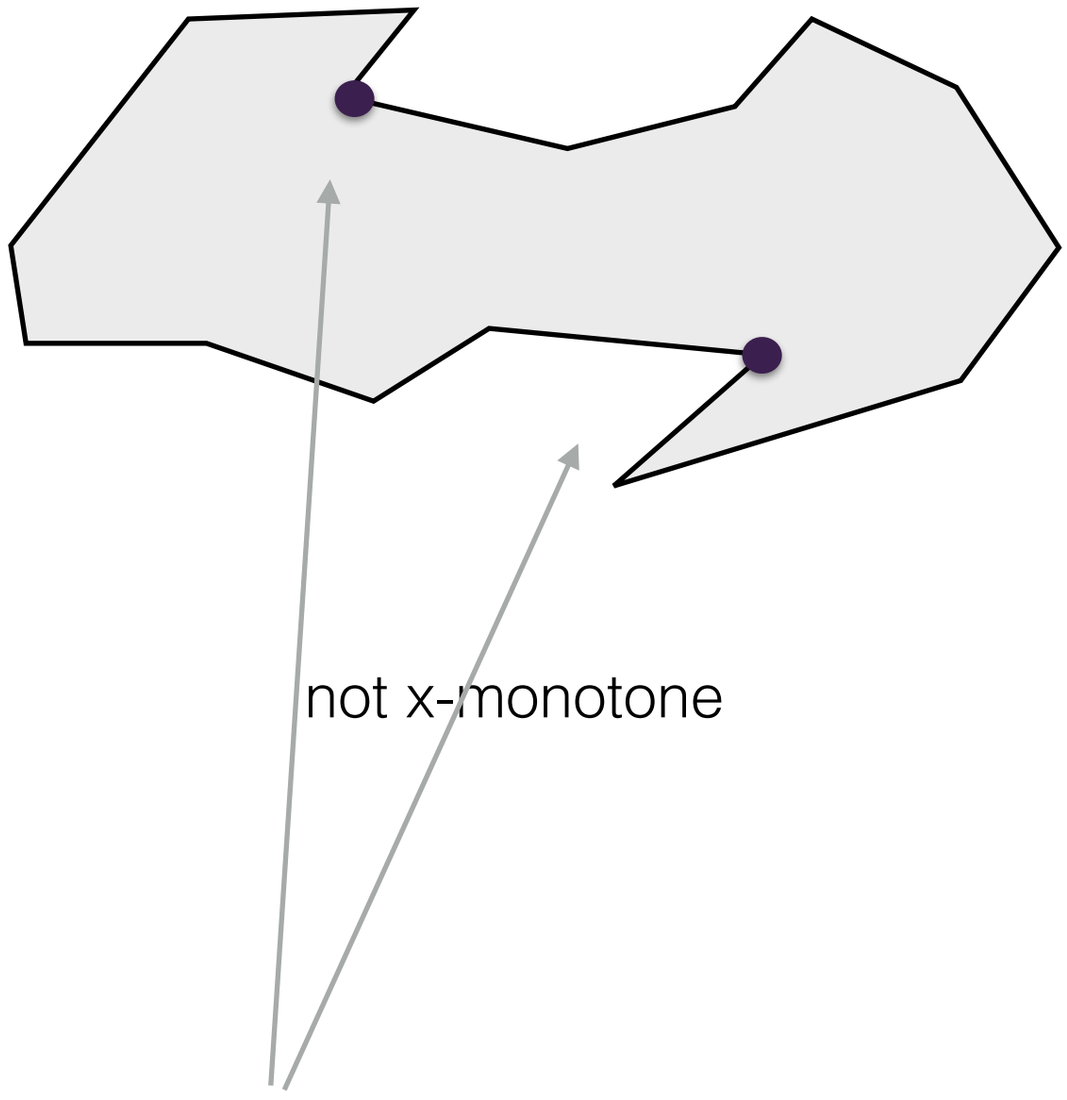
What makes a polygon **not** monotone?

Intuition

Cusp: a reflex vertex v such that the vertices before and after are both smaller or both larger than v (in terms of x-coords).



x-monotone



not x-monotone

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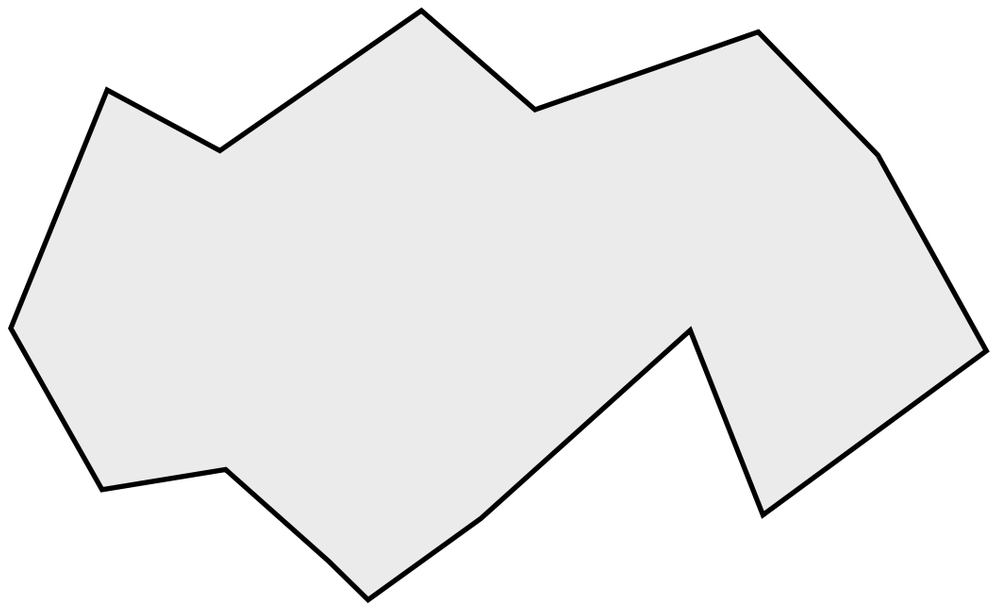
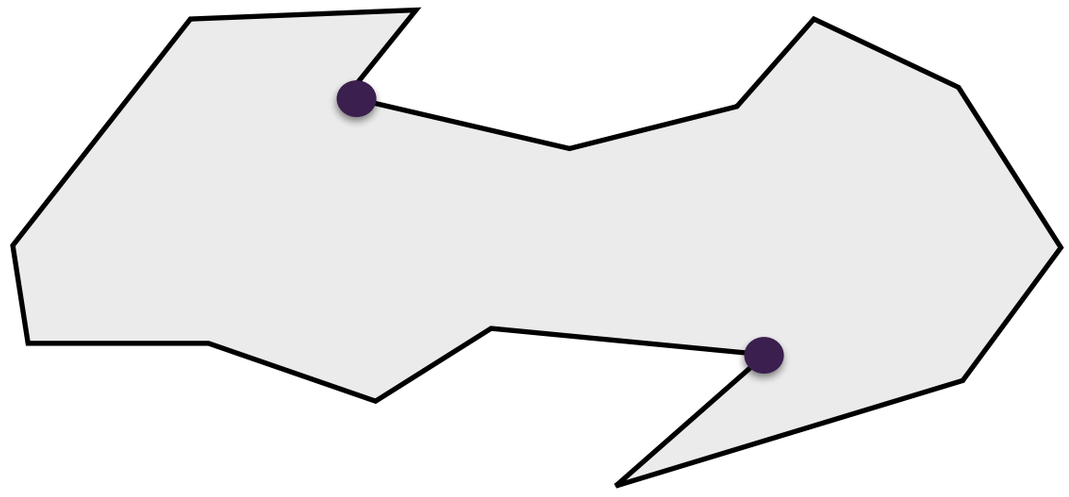


x-monotone



not x-monotone

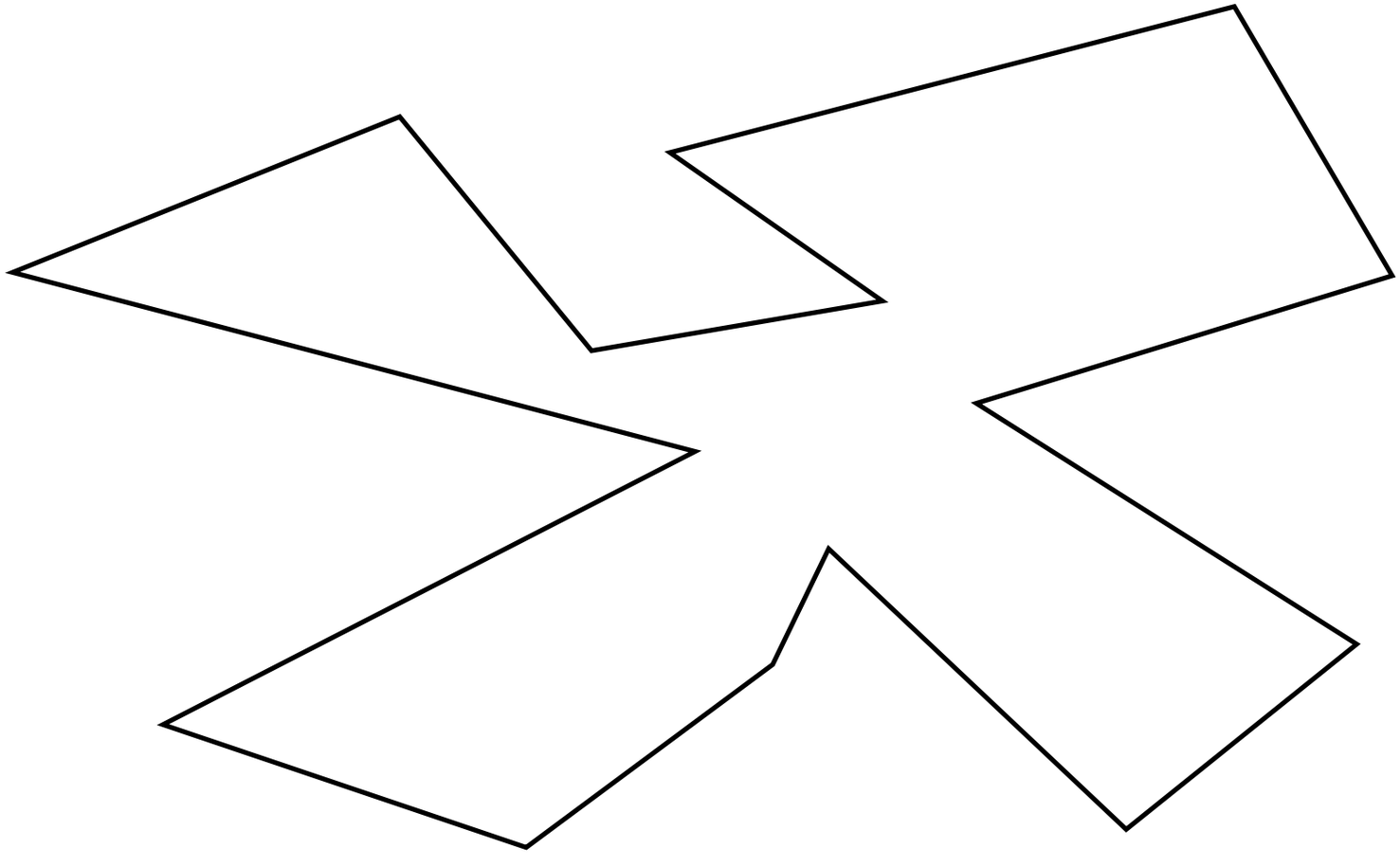
- Theorem: If a polygon has no cusps, then it's monotone.
- Proof: Intuitively clear, but proof a little tedious. Maybe later..

We’ll get rid of cusps using a trapezoidation of P.

Trapezoid partitions

Shoot vertical rays

- If polygon is above vertex, shoot vertical ray up until reaches boundary
- If f polygon is below vertex, shoot down
- If polygon is above and below vertex, shoot both up and down



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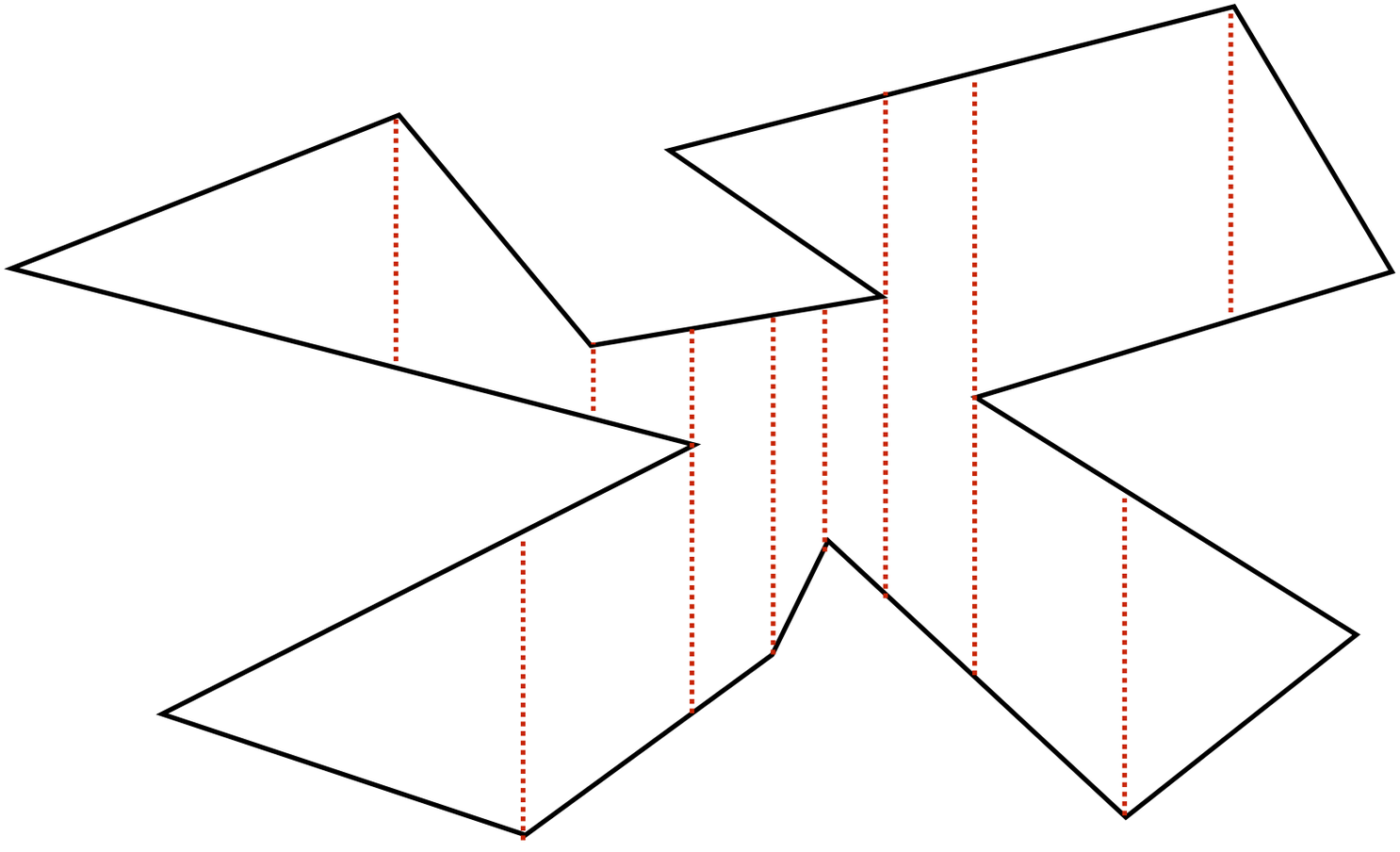
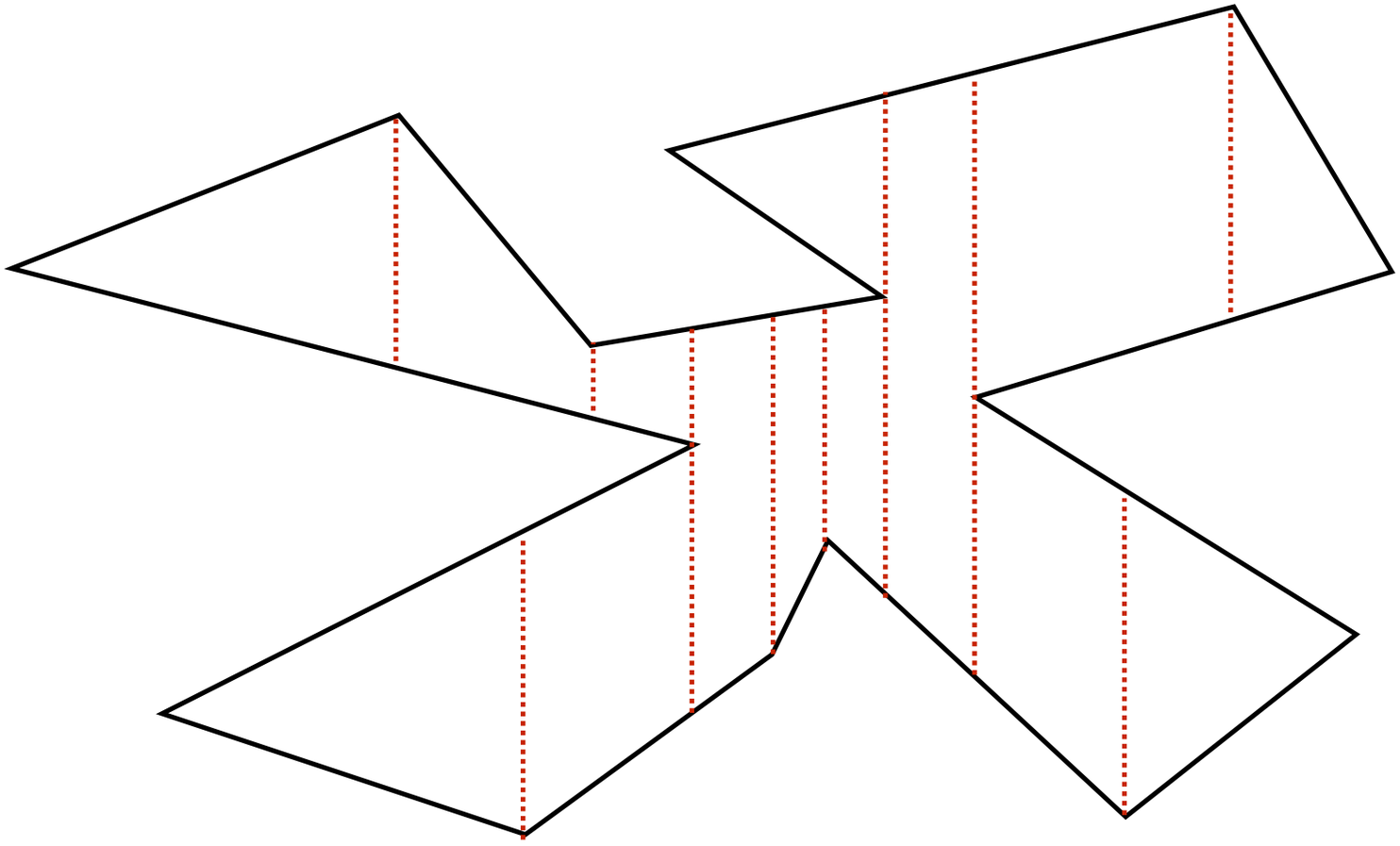


Trapezoid partitions

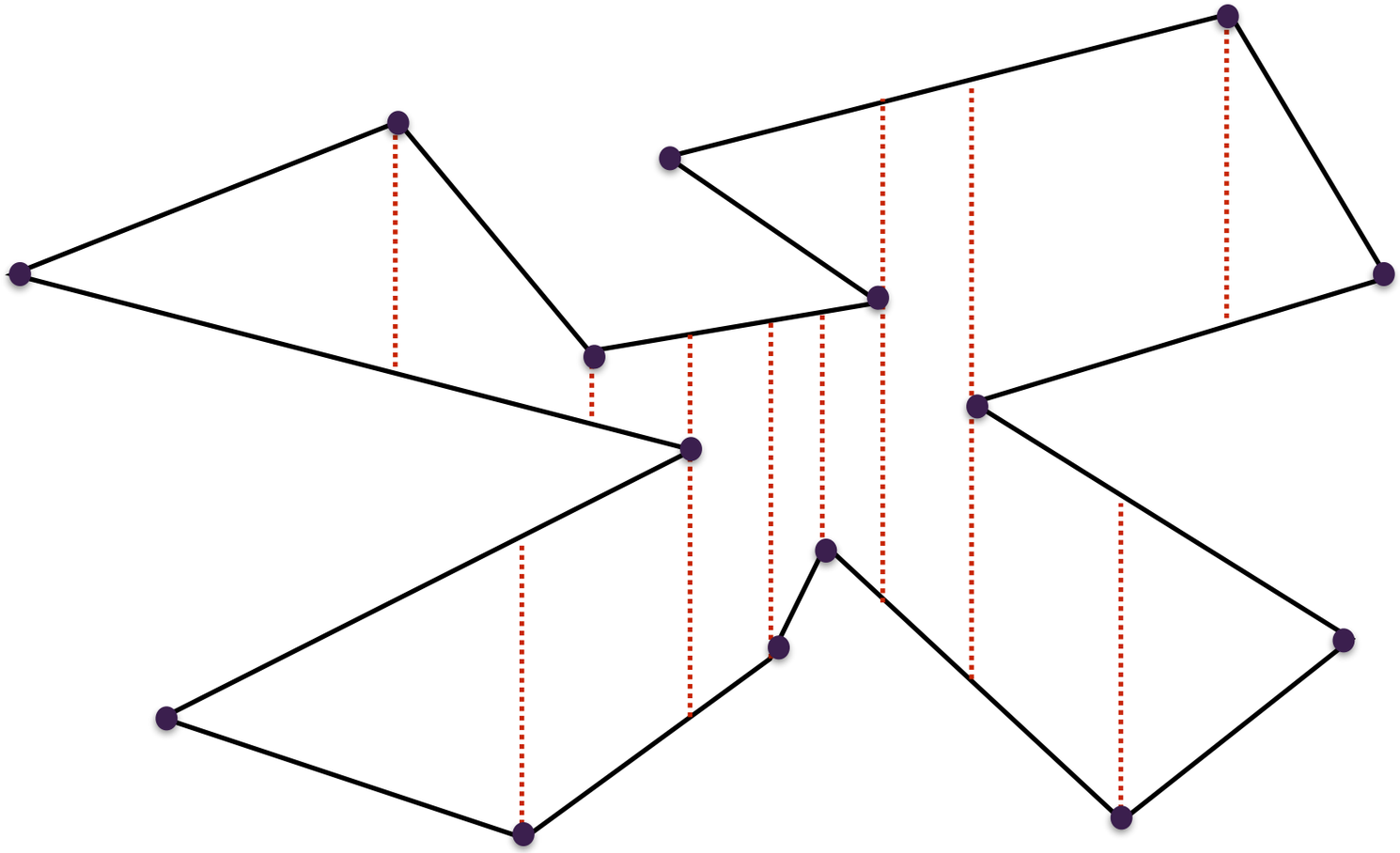
Properties

- Each polygon in the partition is a trapezoid, because:
    - It has one or two threads as sides.
    - If it has two, then they must both hit the same edge above, and the same edge below.
- At most one thread through each vertex => O(n) threads => O(n) trapezoids



Trapezoid partitions

- Each trapezoid has precisely two vertices of the polygon, one on the left and one on the right. They can be on the top, bottom or middle of the trapezoid.



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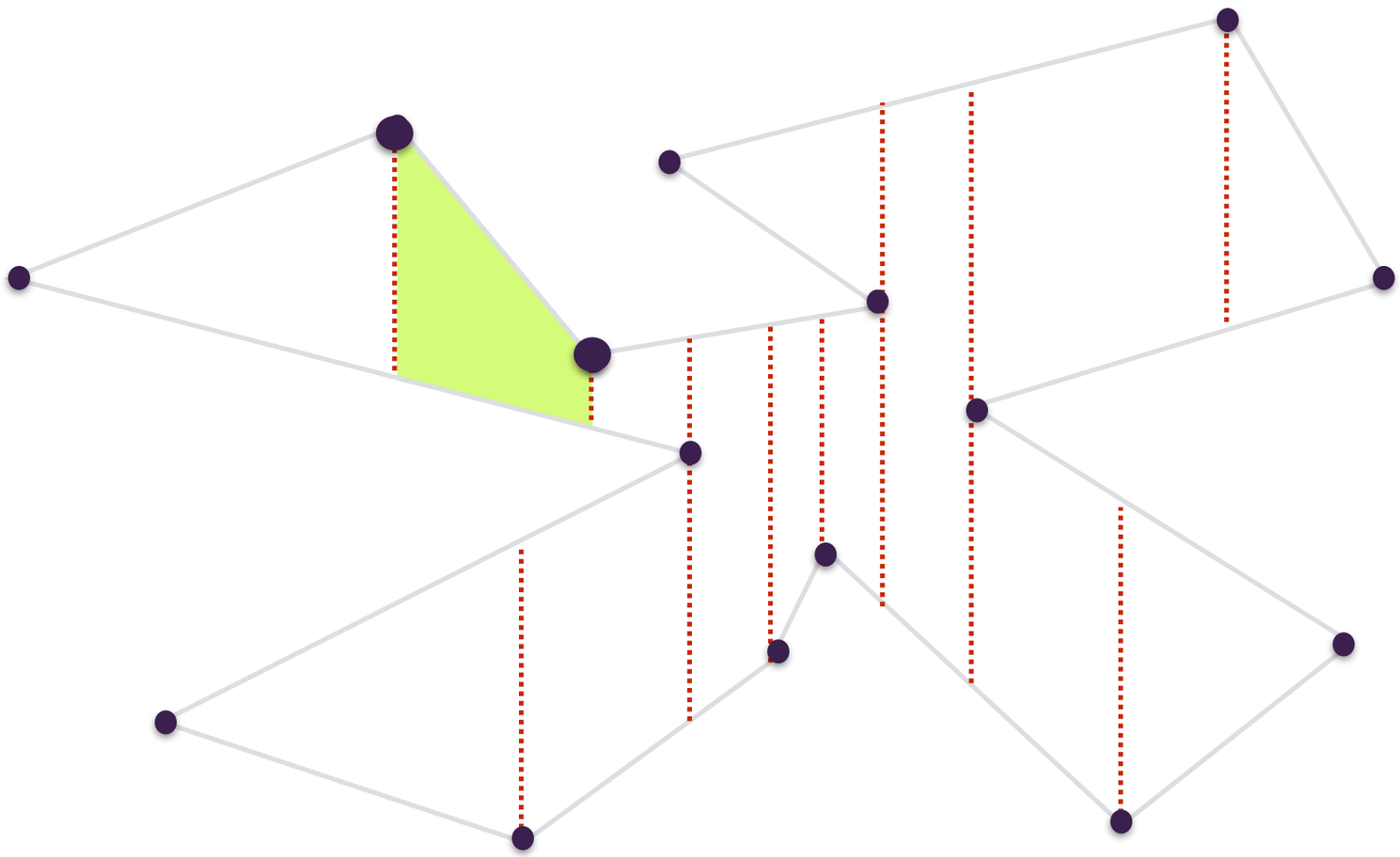

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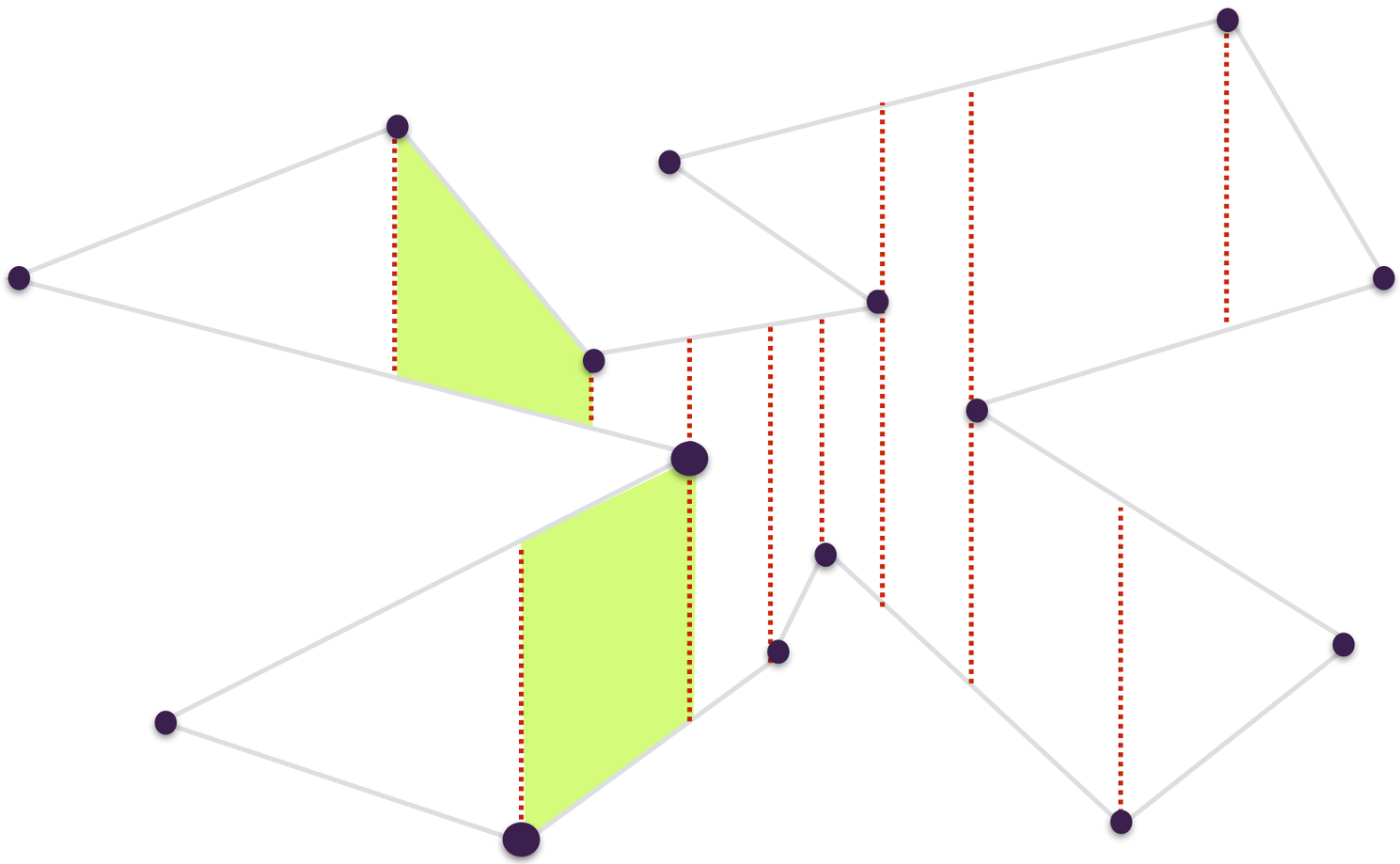

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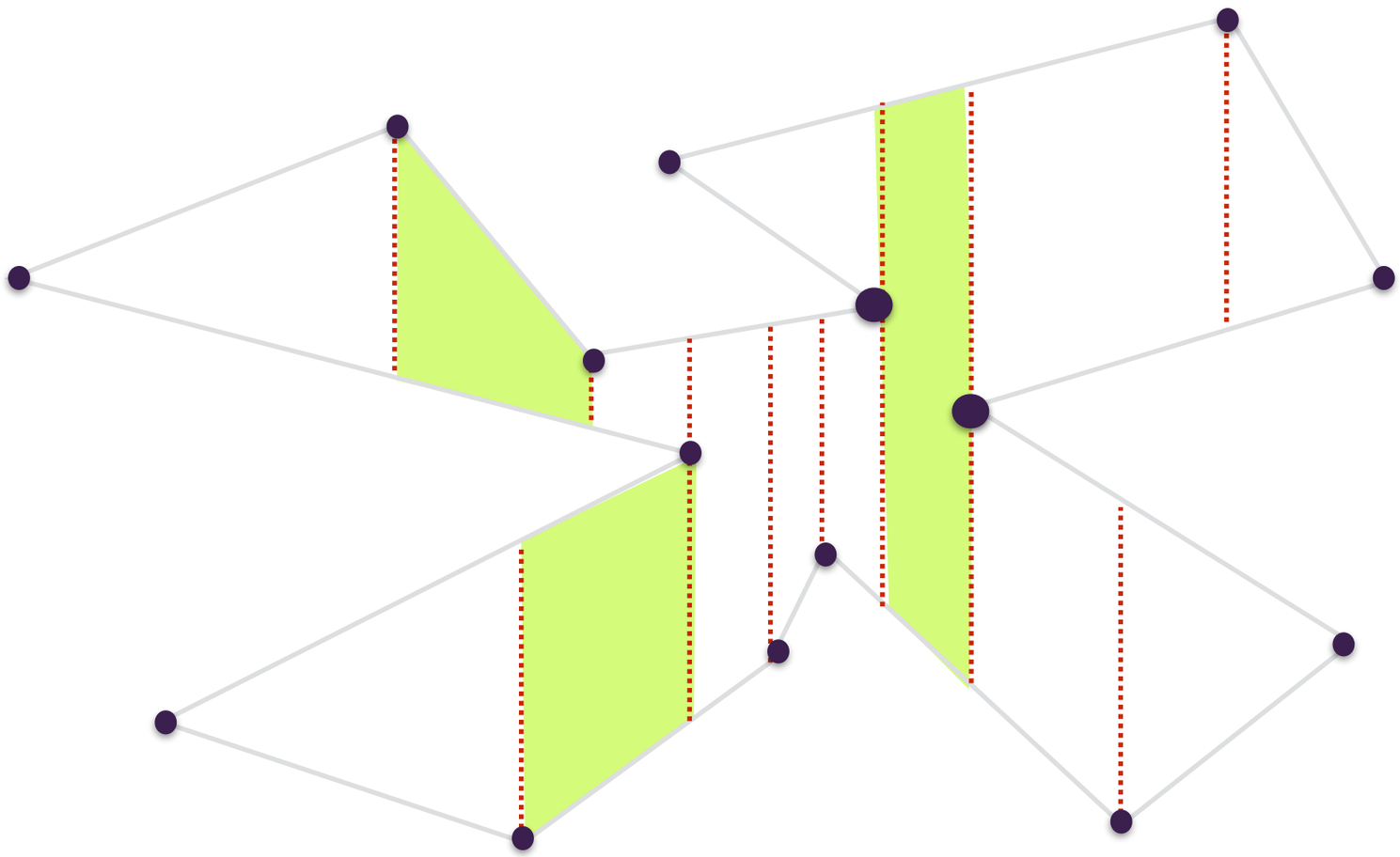
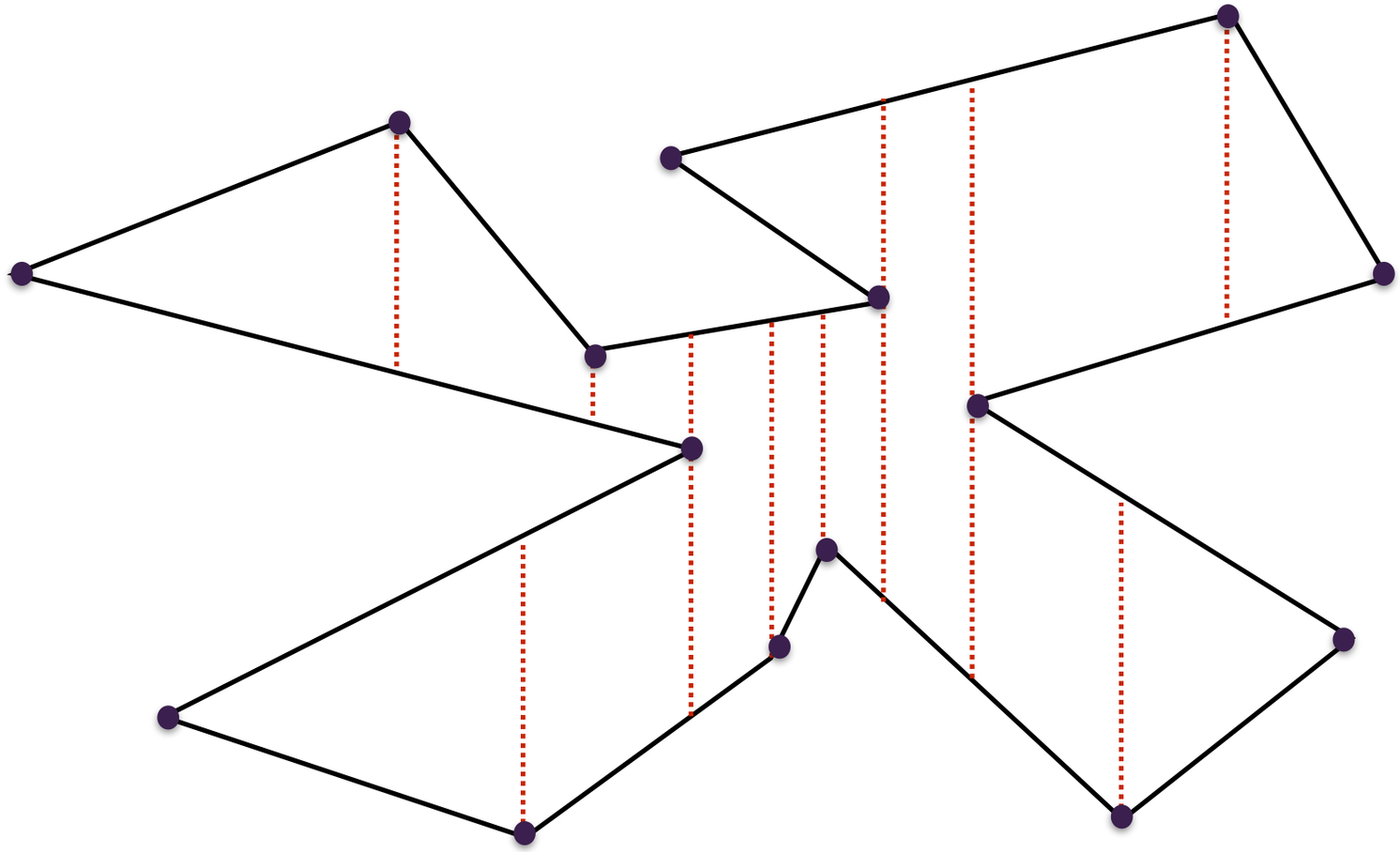


Diagonals

- In each trapezoid: if its two vertices are not on the same edge, they define a **diagonal**.



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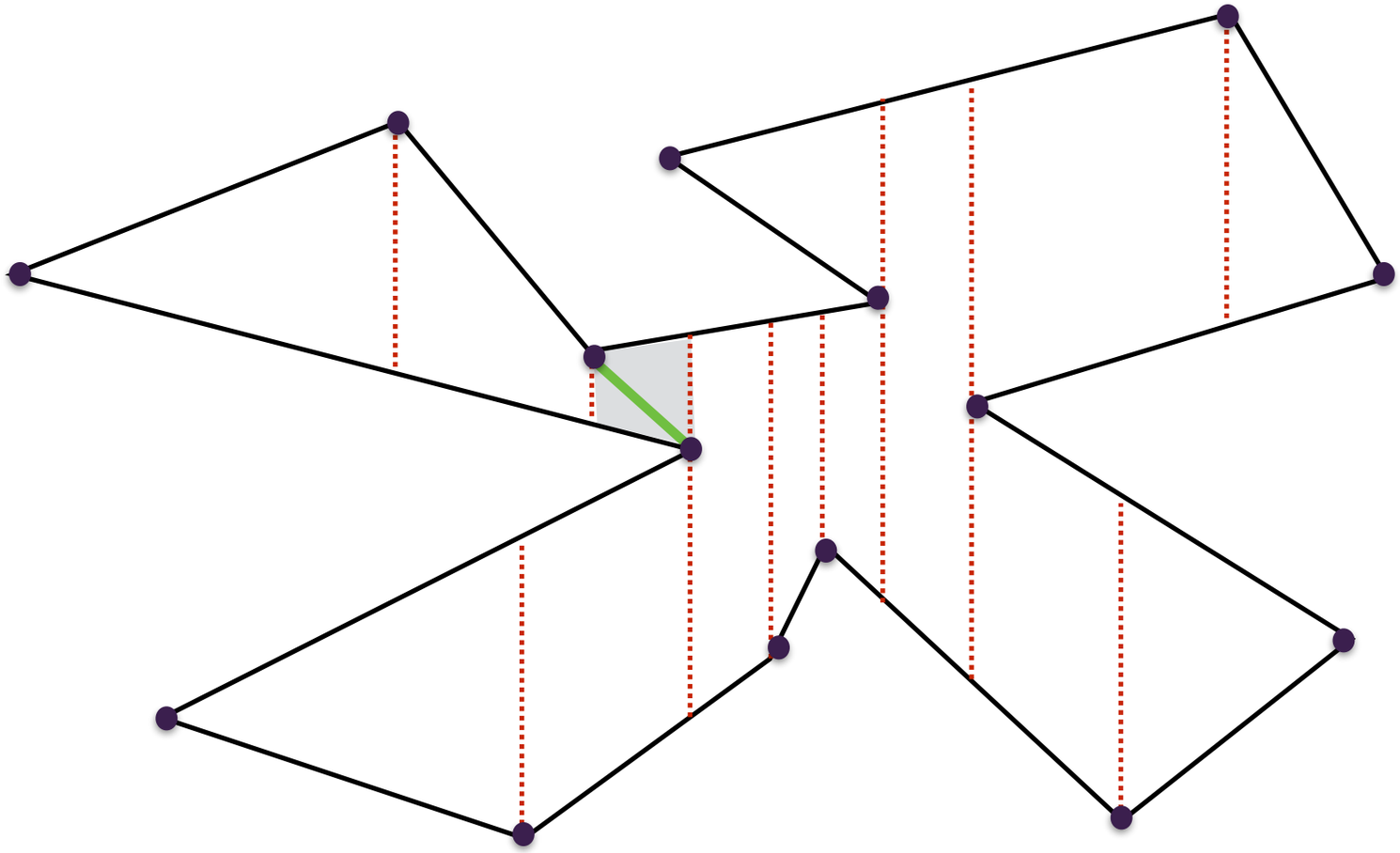

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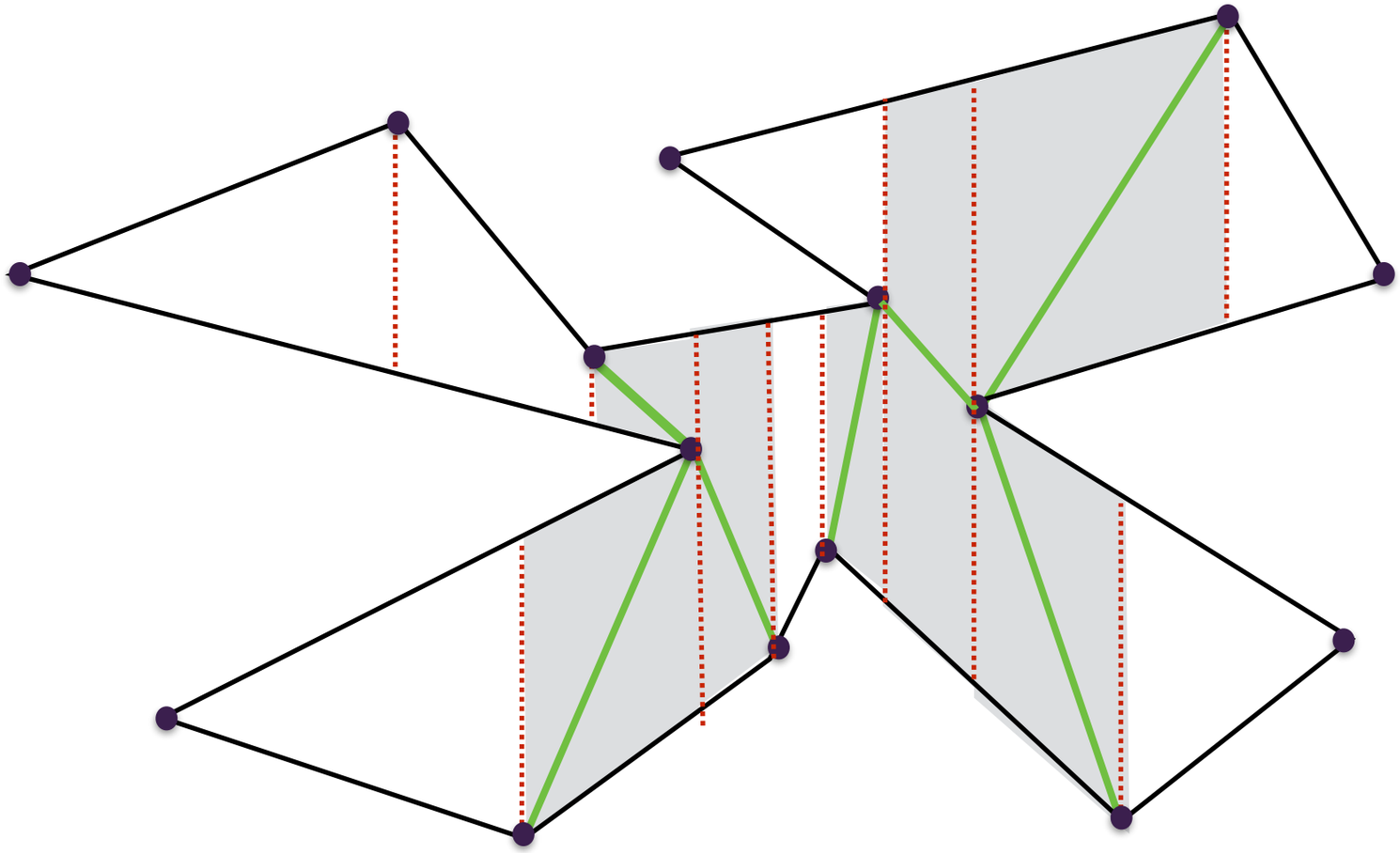
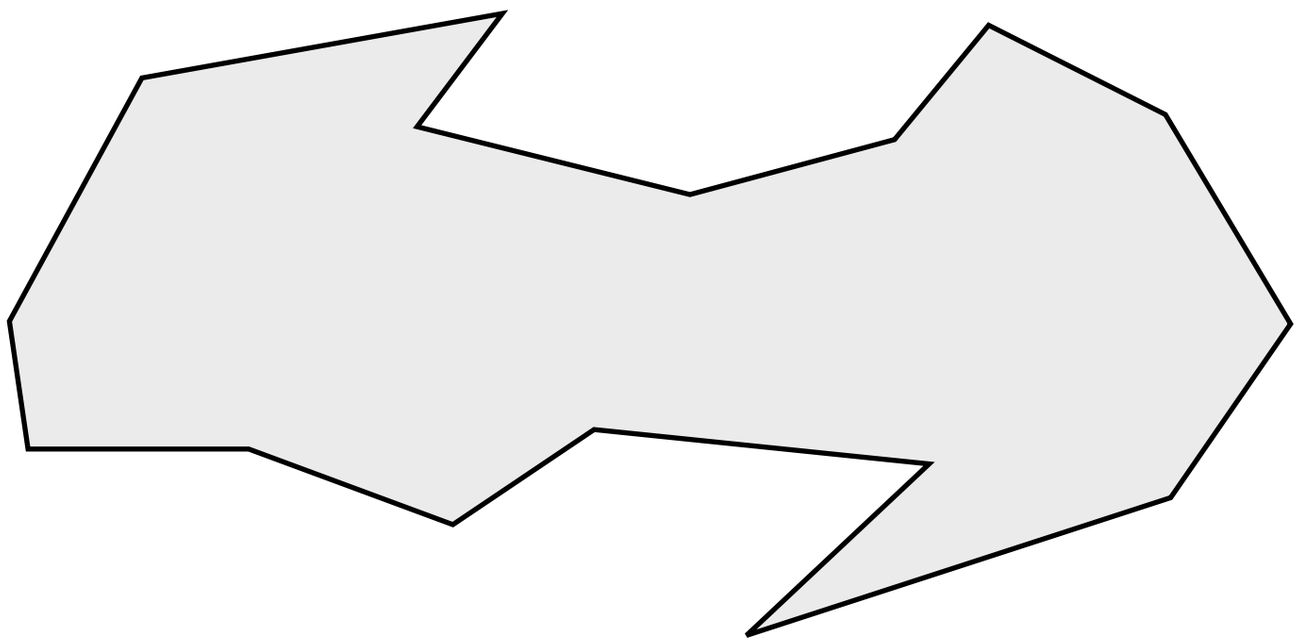
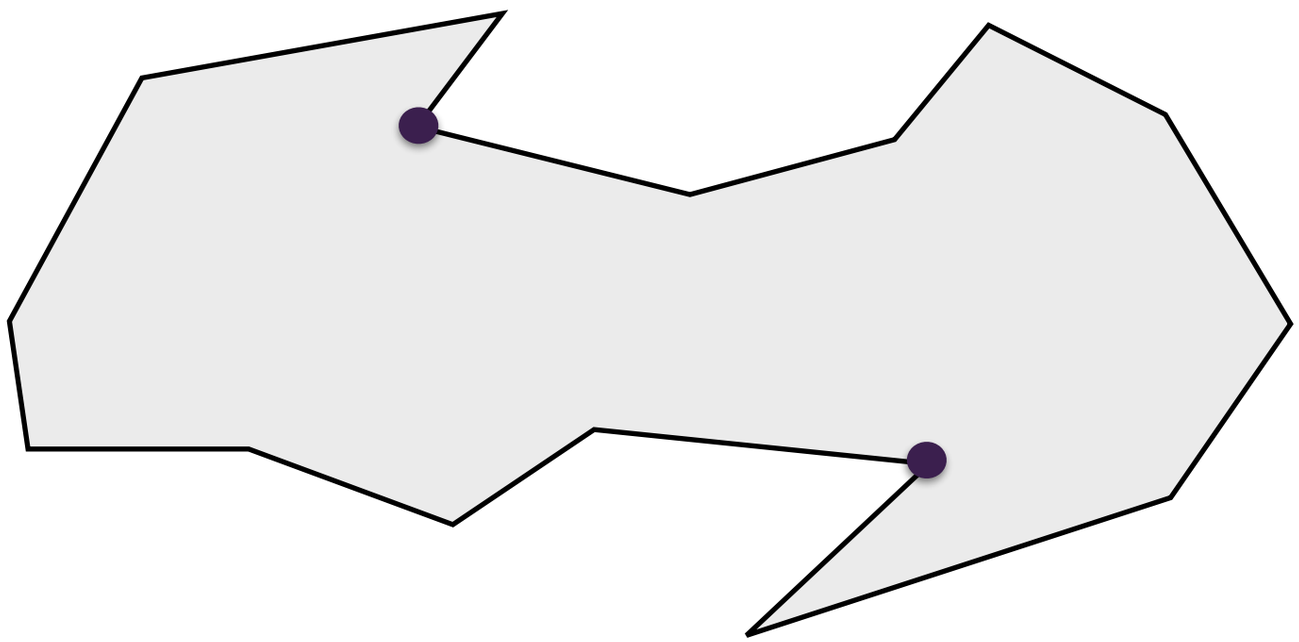


We can use the trapezoid partition of P to “split” the cusps

Removing cusps

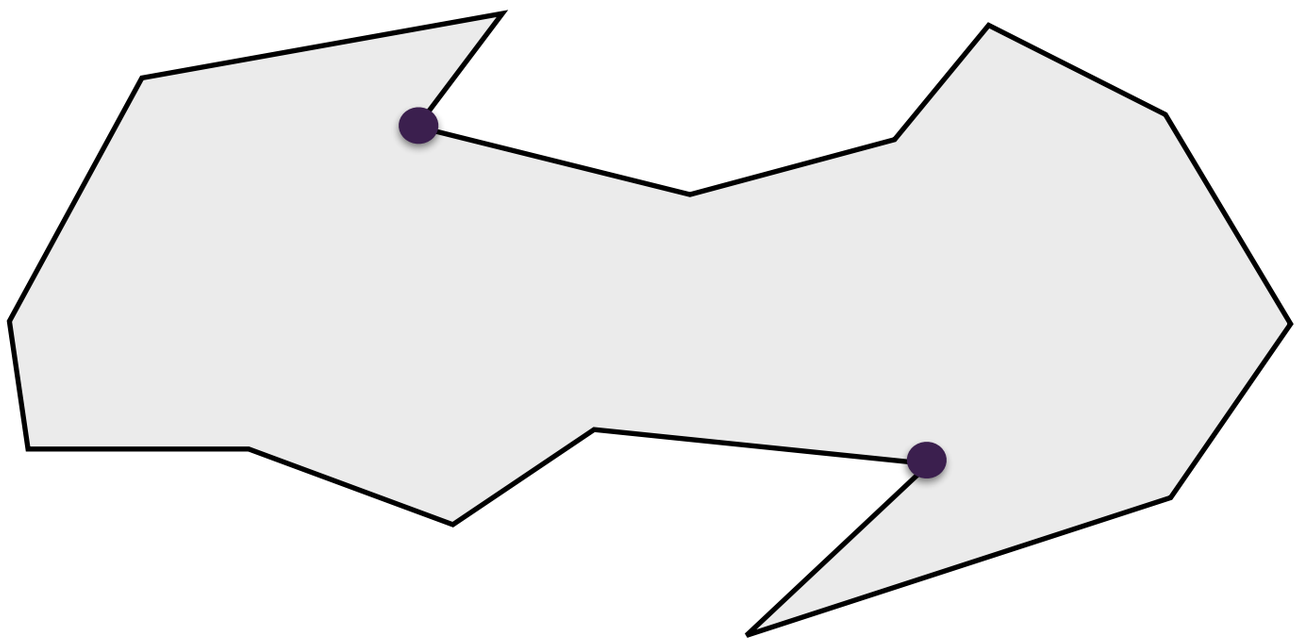


Removing cusps



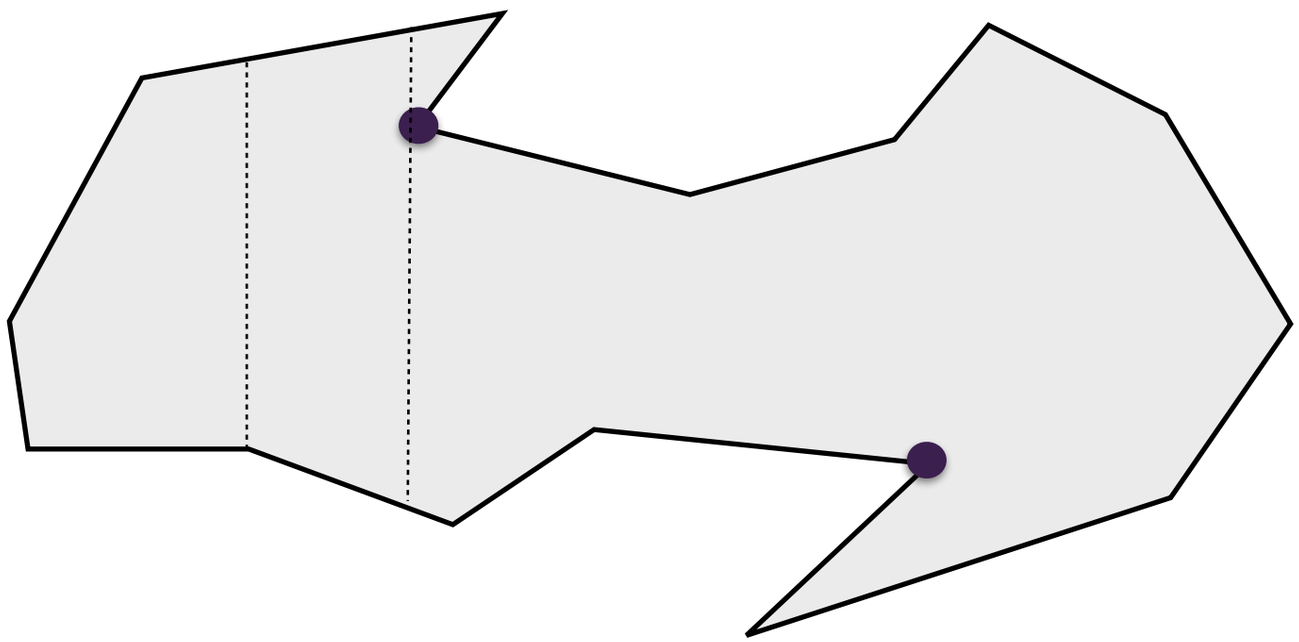
1. Identify cusp vertices

Removing cusps



1. Identify cusp vertices
2. Compute a trapezoid partition of P
3. For each cusp vertex, add diagonal in trapezoid before/after the cusp

Removing cusps



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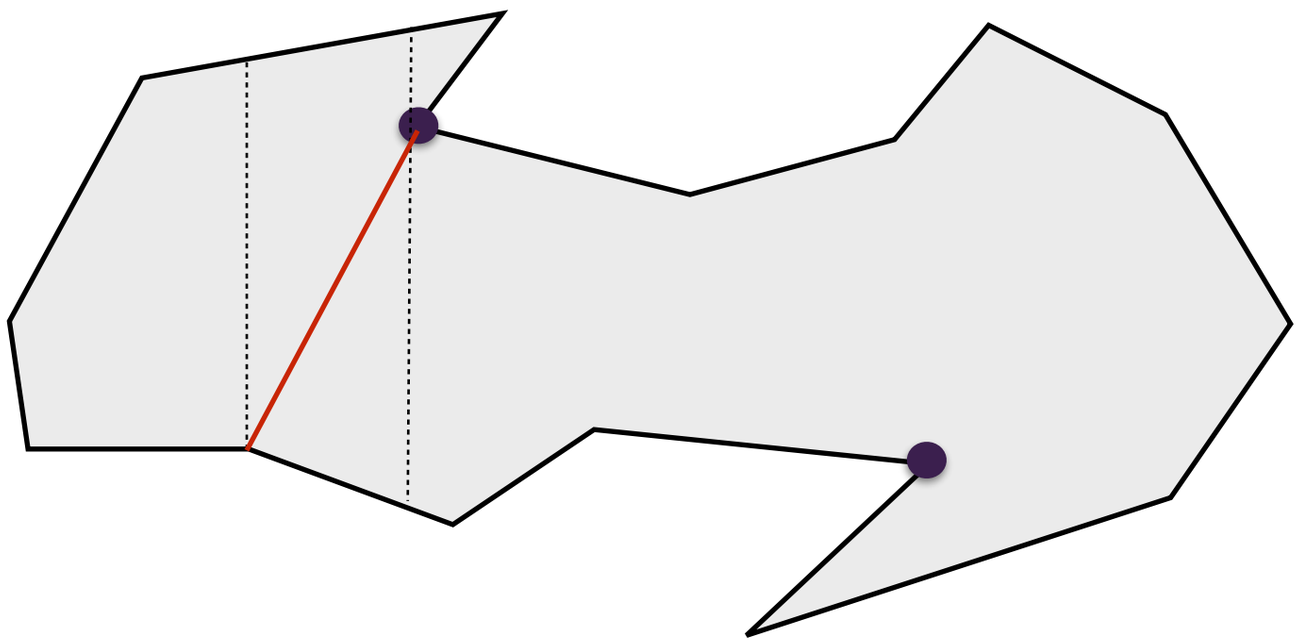

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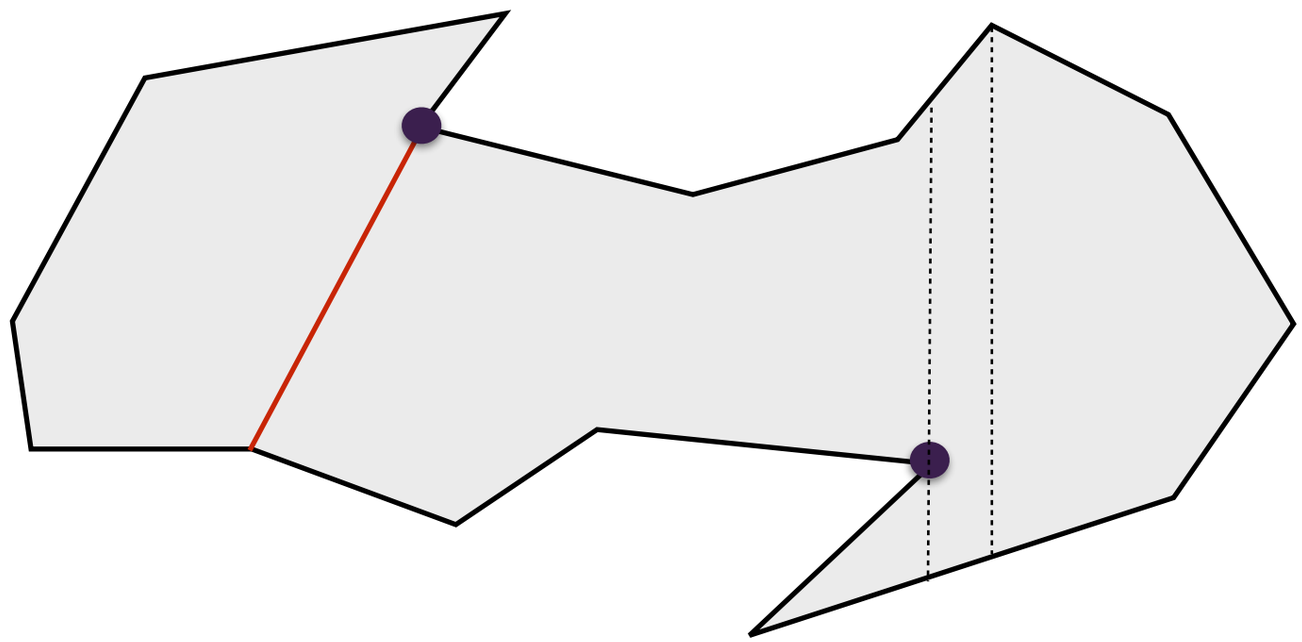

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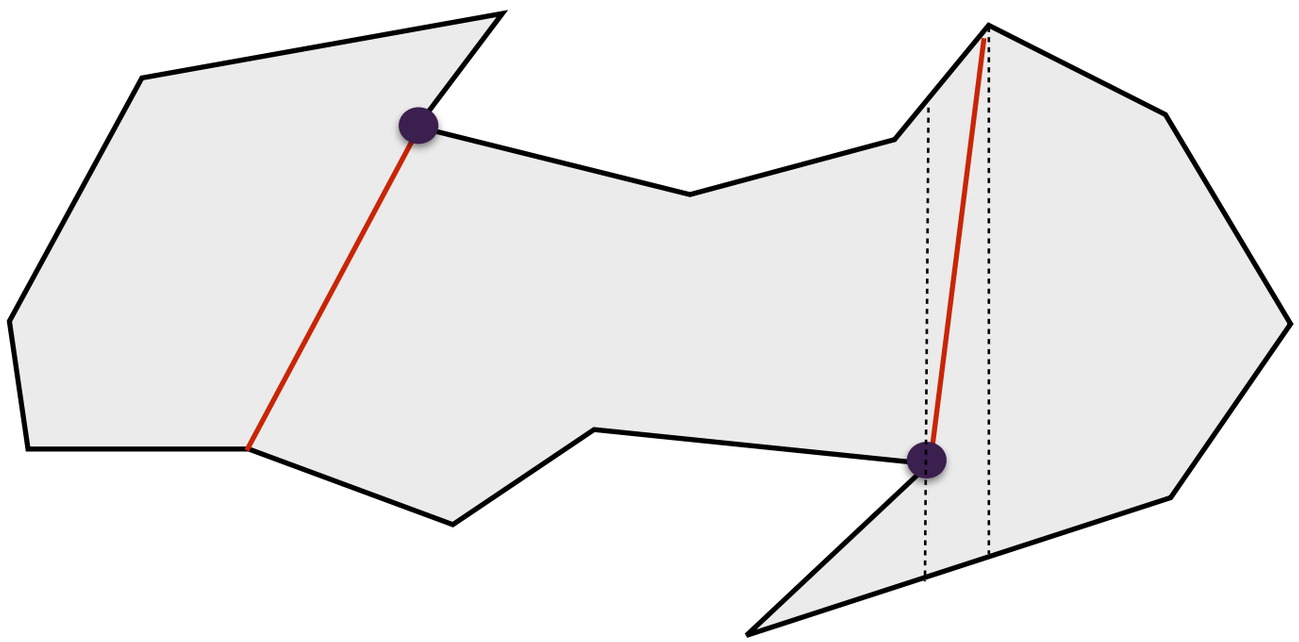

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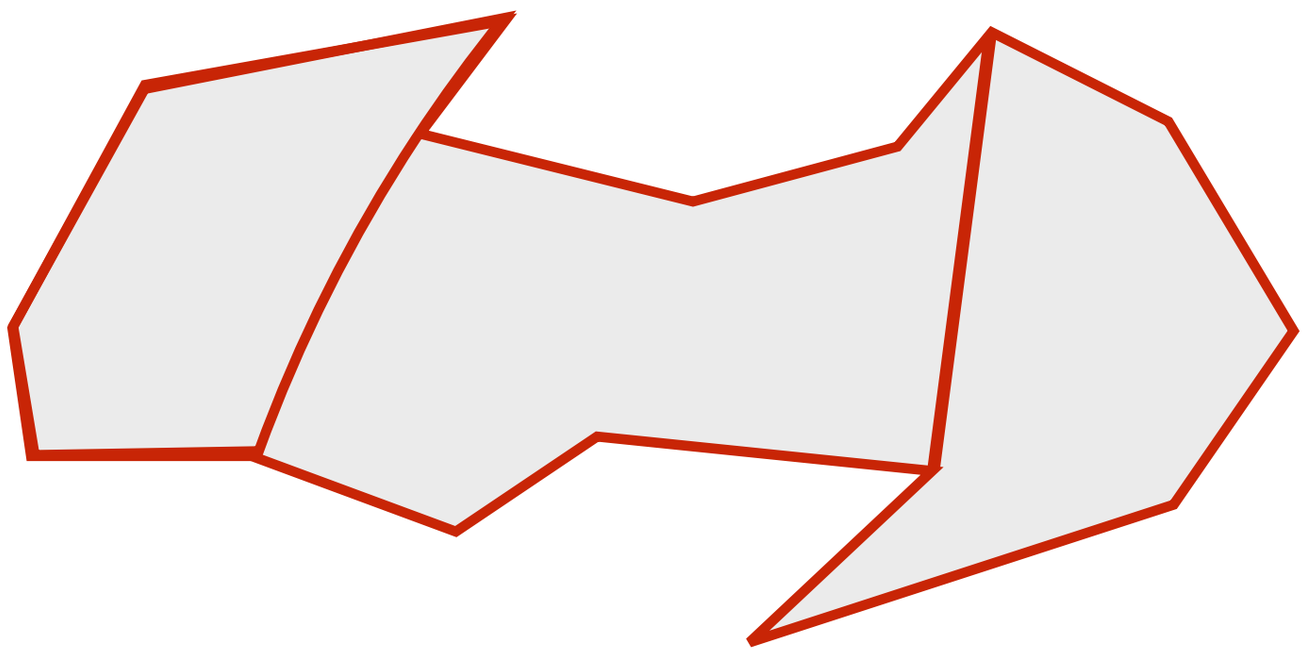
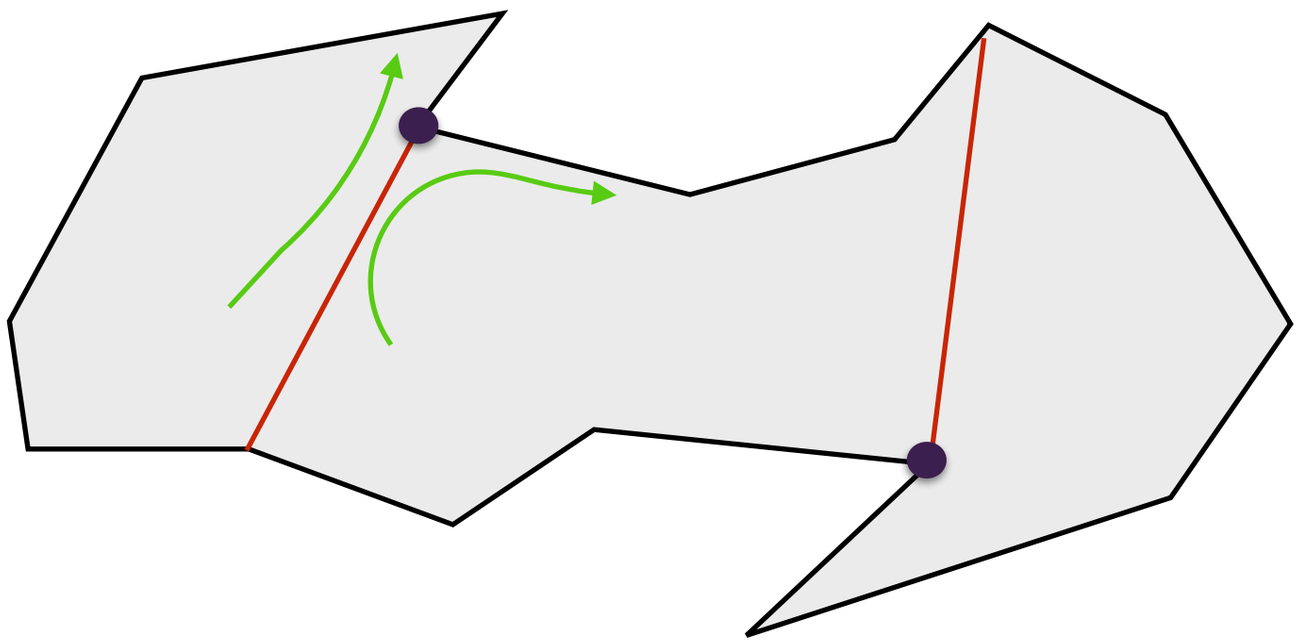
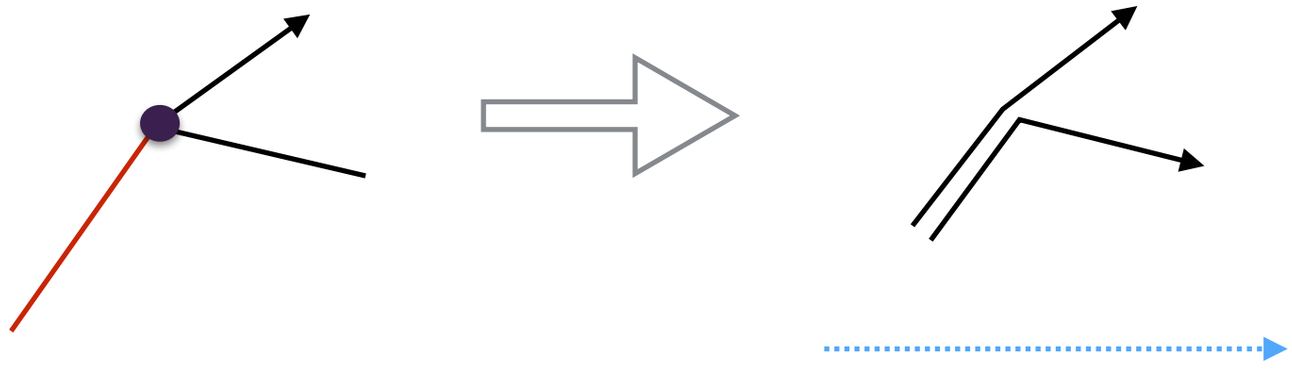


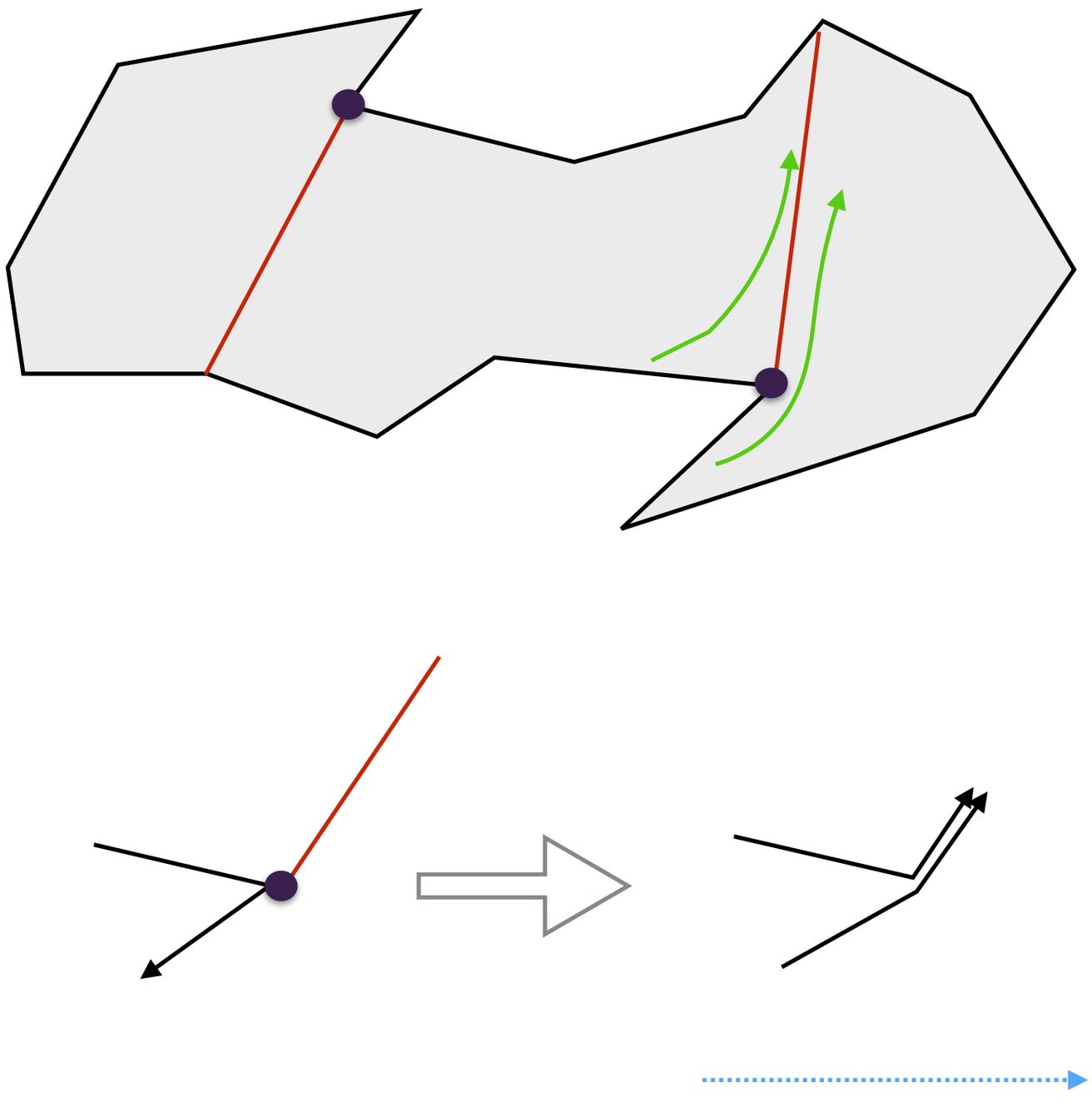
1. Identify cusp vertices
2. Compute a trapezoid partition of P
3. For each cusp vertex, add diagonal in trapezoid before/after the cusp

This creates a partition of P.

Claim: The resulting polygons have no cusps and thus are monotone (by theorem).

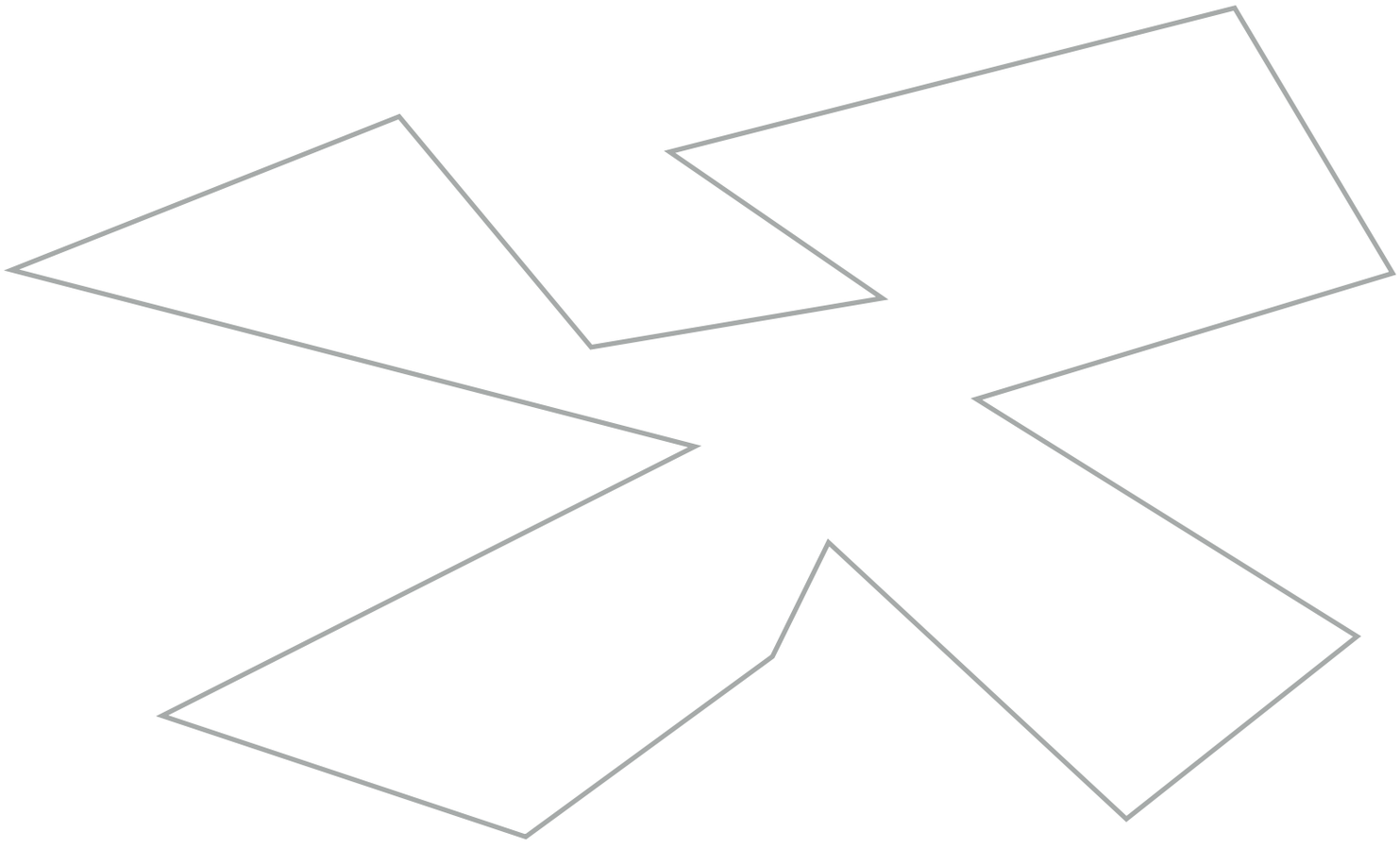




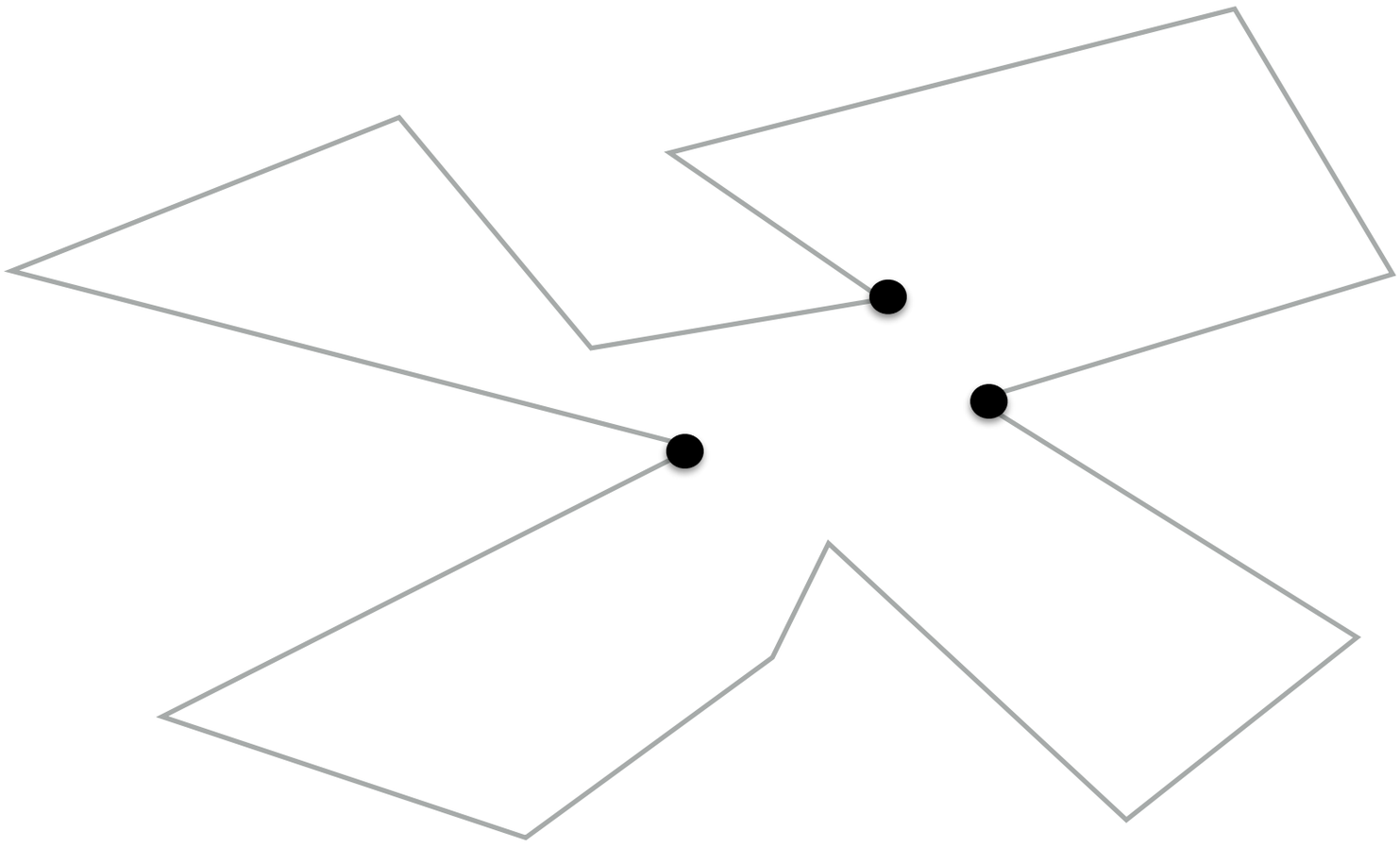


Removing cusps

- Another example

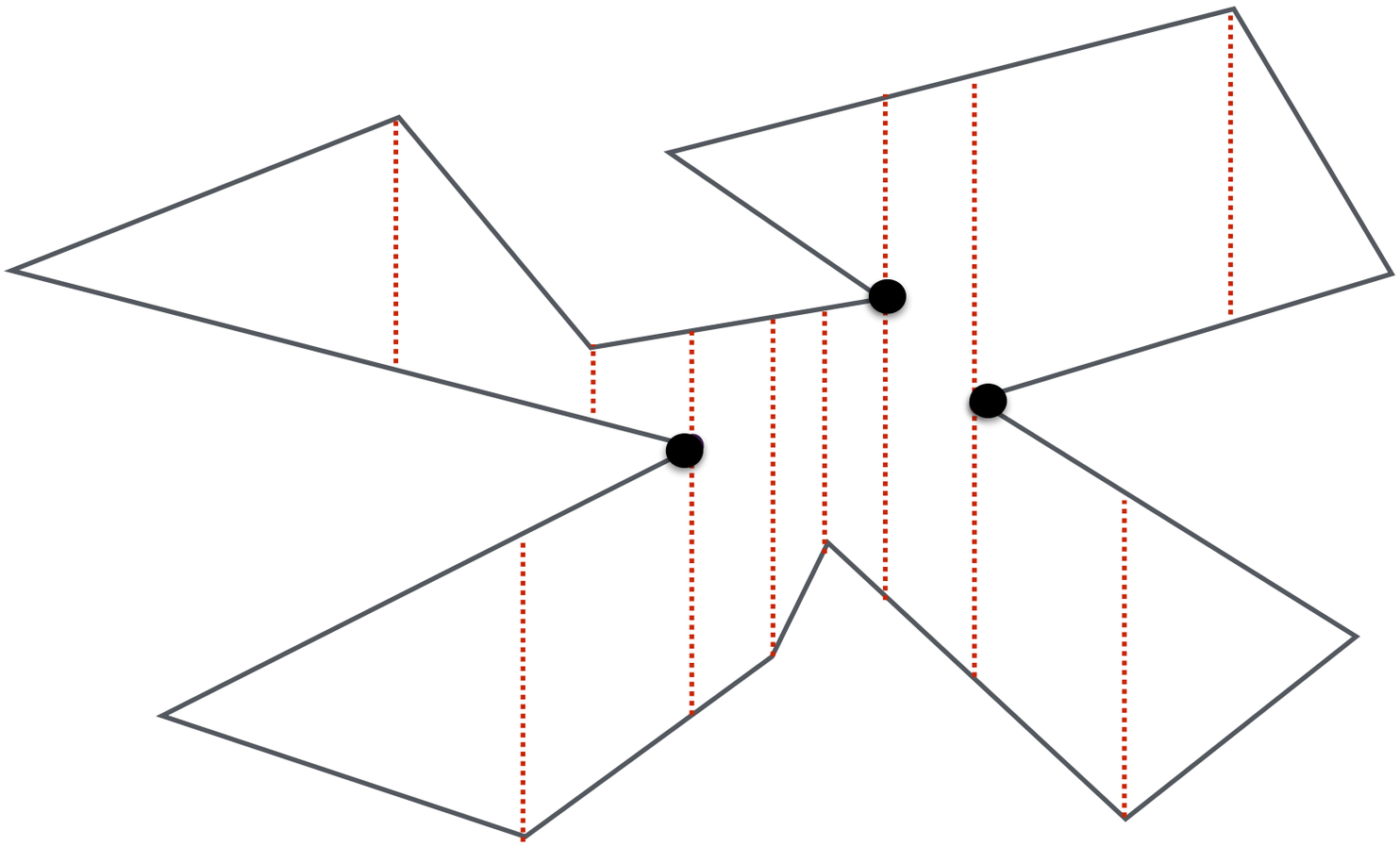


Removing cusps



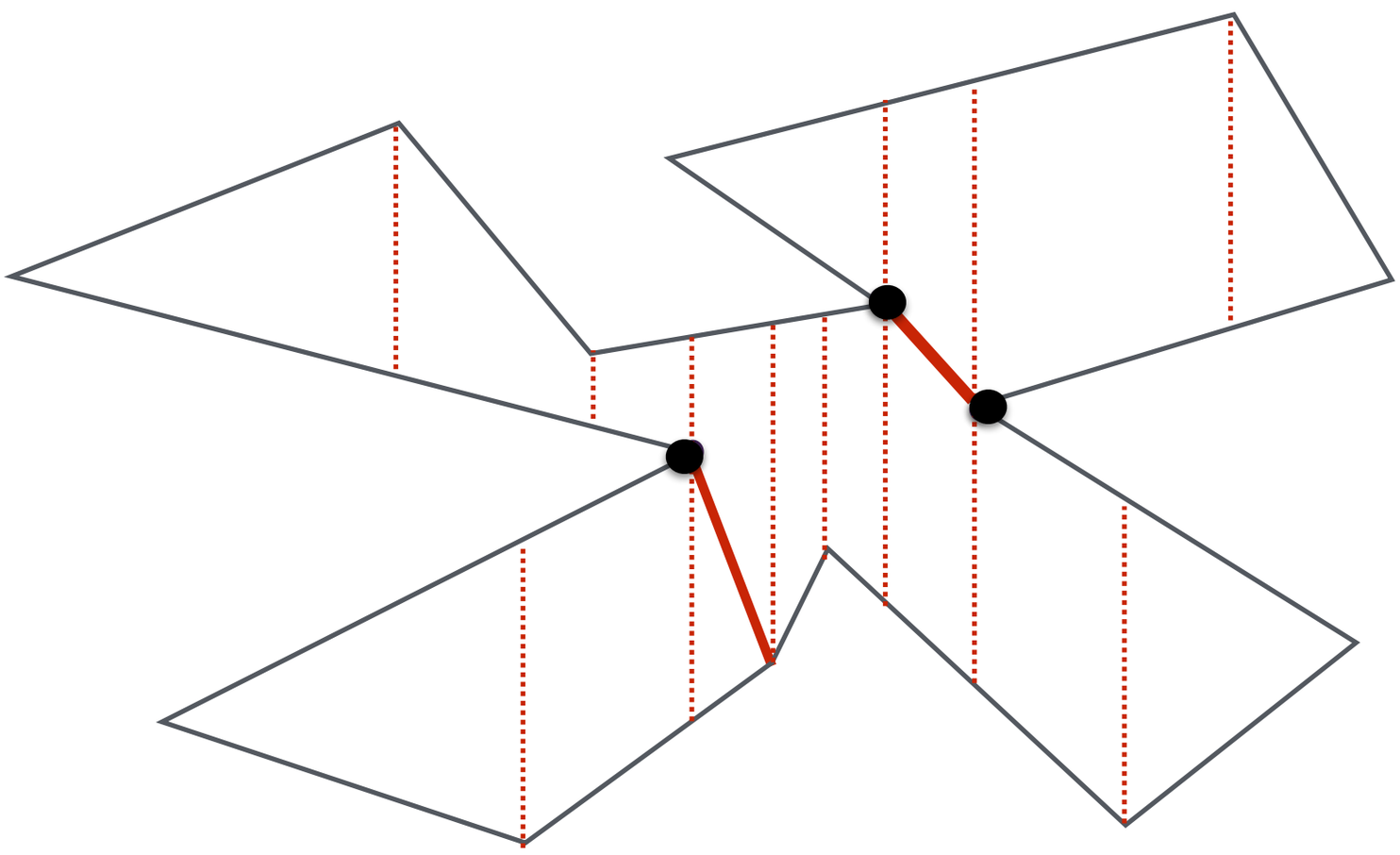
1. Identify cusp vertices

Removing cusps



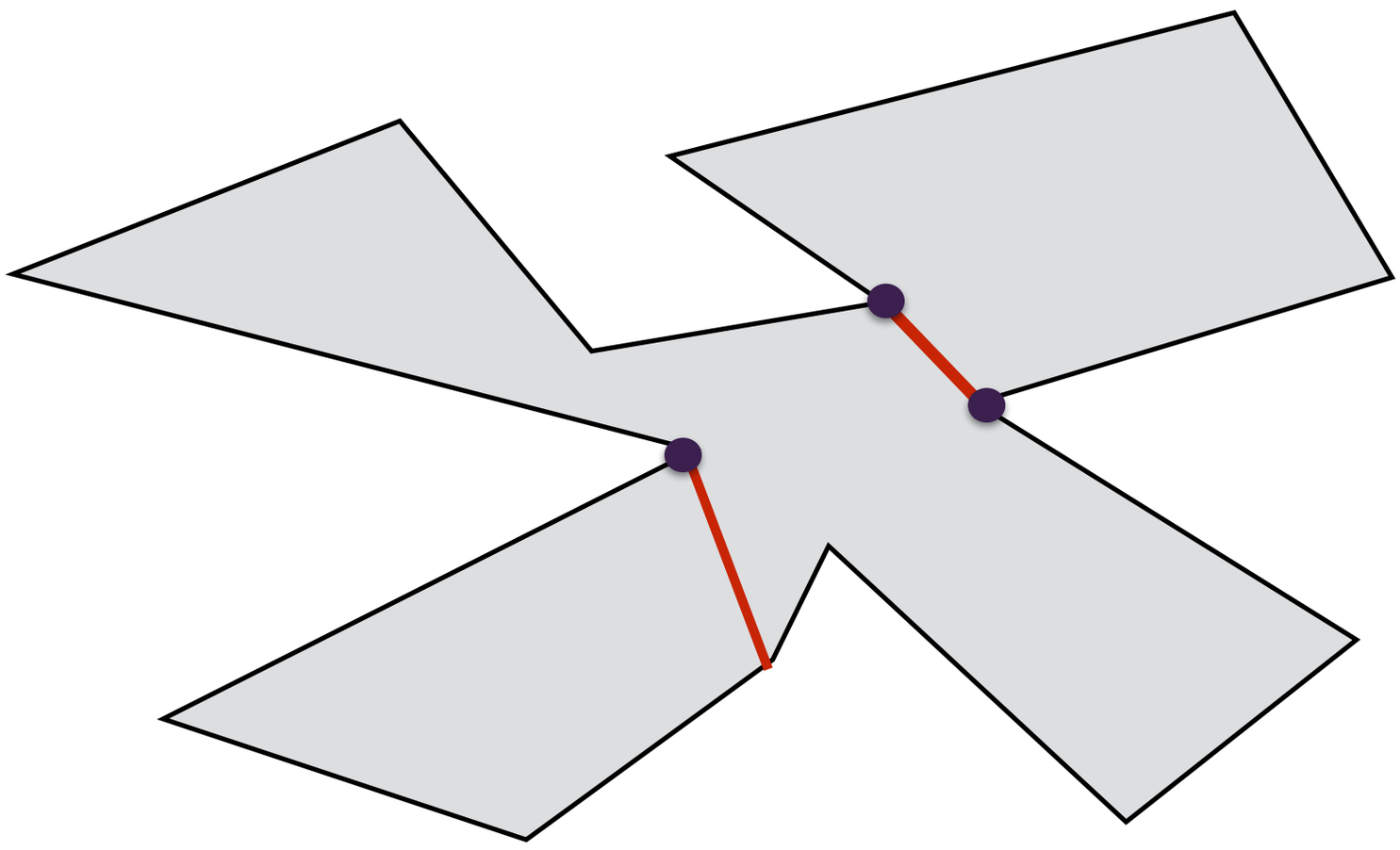
1. Identify cusp vertices
2. Compute a trapezoid partition of P

Removing cusps



1. Identify cusp vertices
2. Compute a trapezoid partition of P
3. Add obvious diagonal before/after each cusp

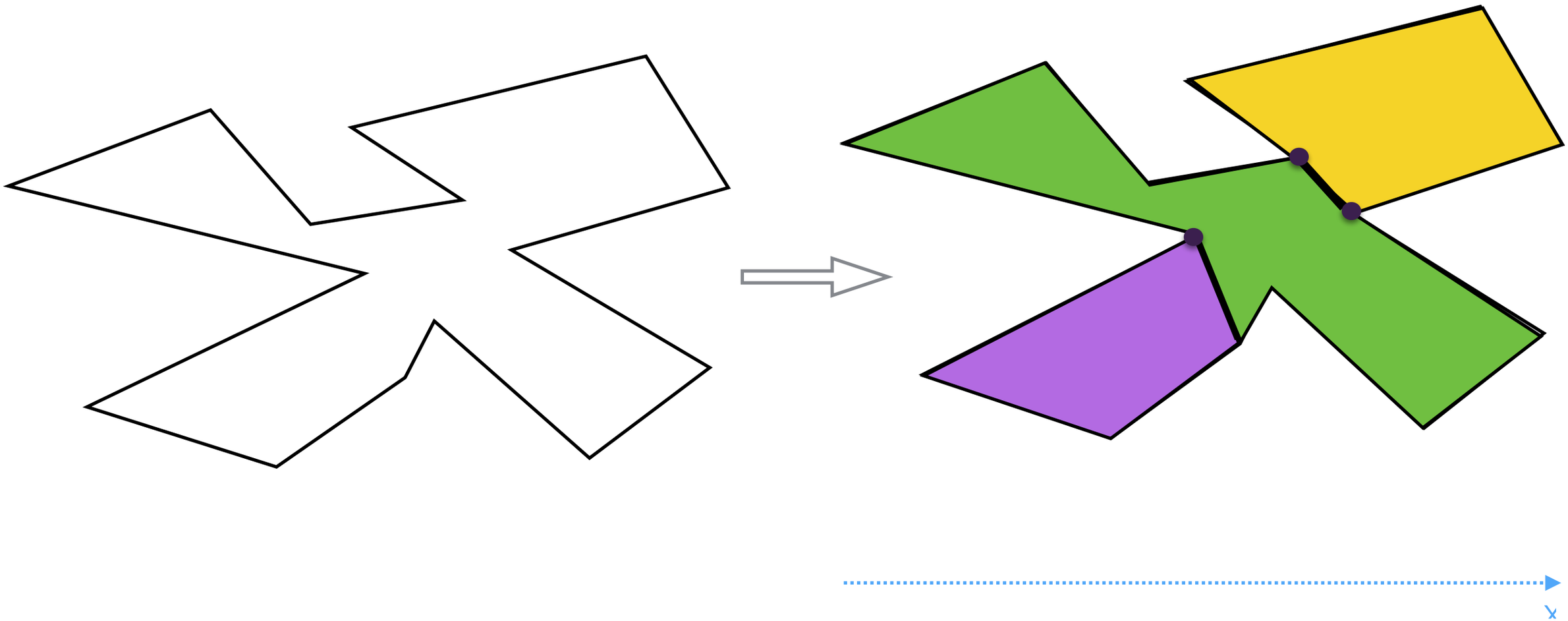
Removing cusps



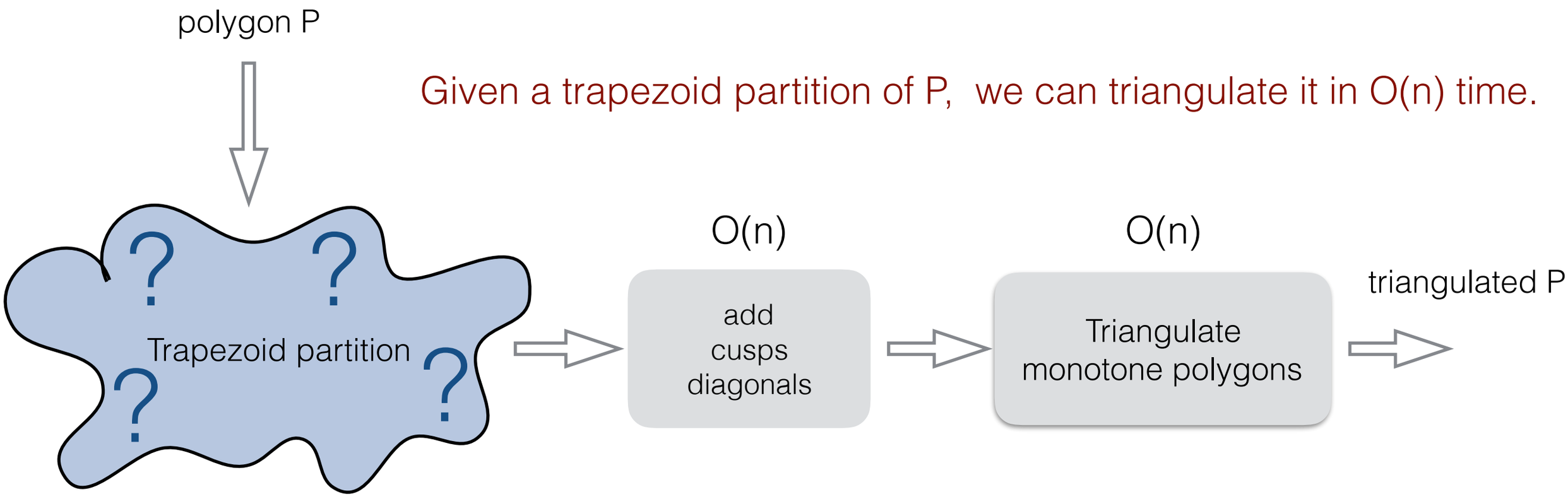
This partitions the polygon into monotone pieces.

Partition P into monotone polygons

1. Identify cusp vertices
2. Compute a trapezoid partition of P
3. Add obvious diagonal before/after each cusp

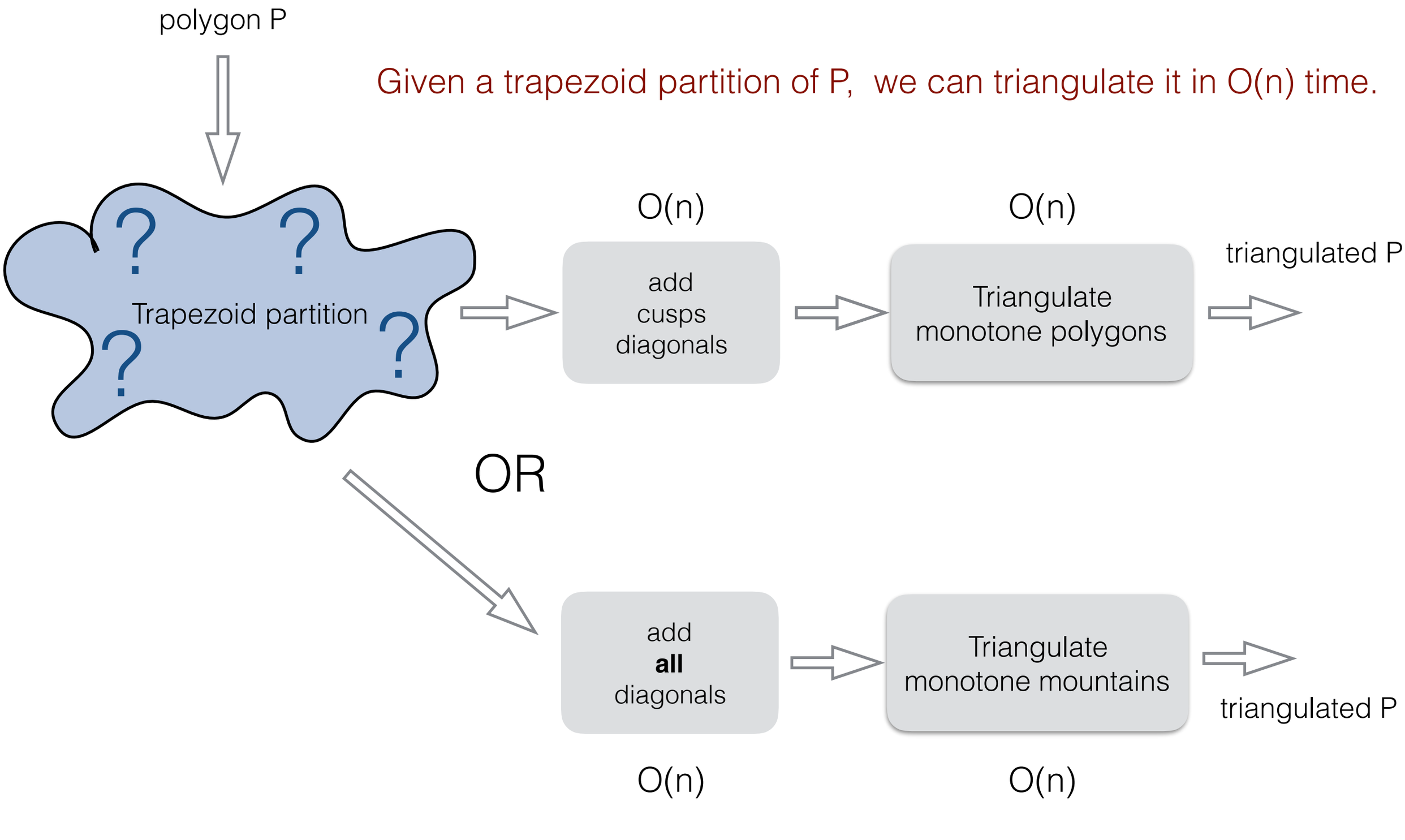


Towards an O(n lg n) Polygon Triangulation Algorithm

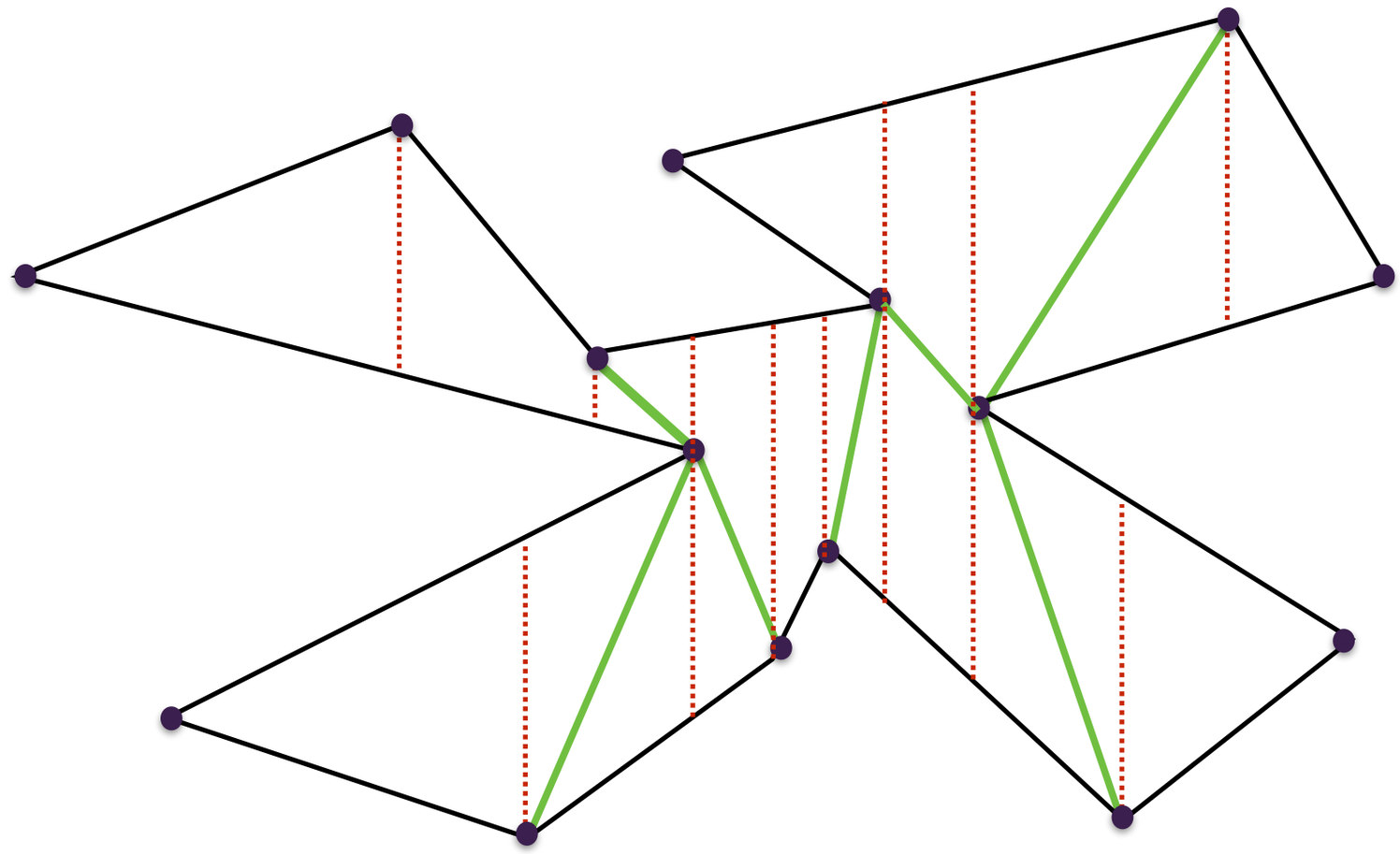


Given a trapezoid partition of P, we can triangulate it in O(n) time.

Towards an O(n lg n) Polygon Triangulation Algorithm

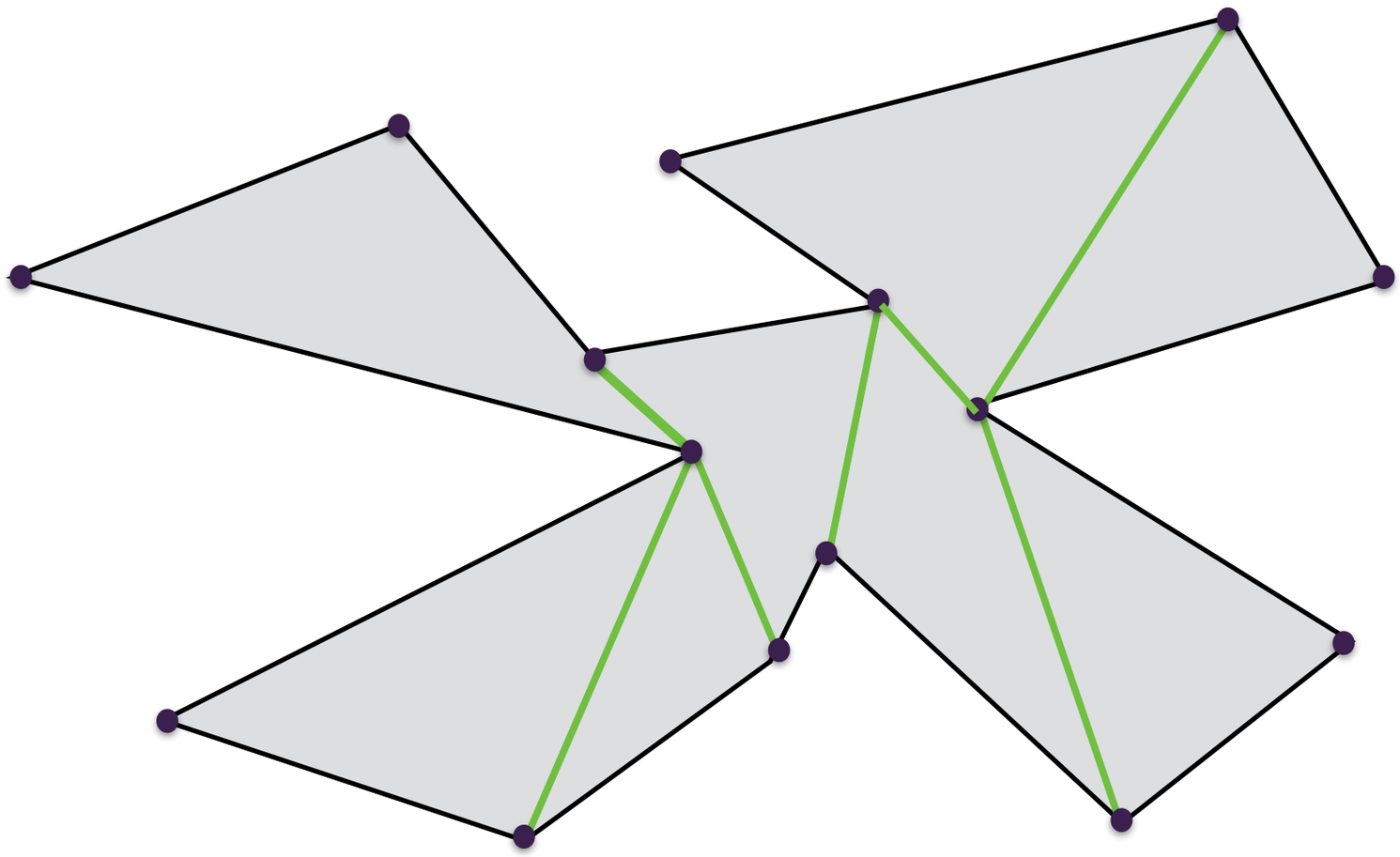


Partitioning into monotone mountains



1. Compute a trapezoid partition of P
2. Output **all** diagonals.

Partitioning into monotone mountains



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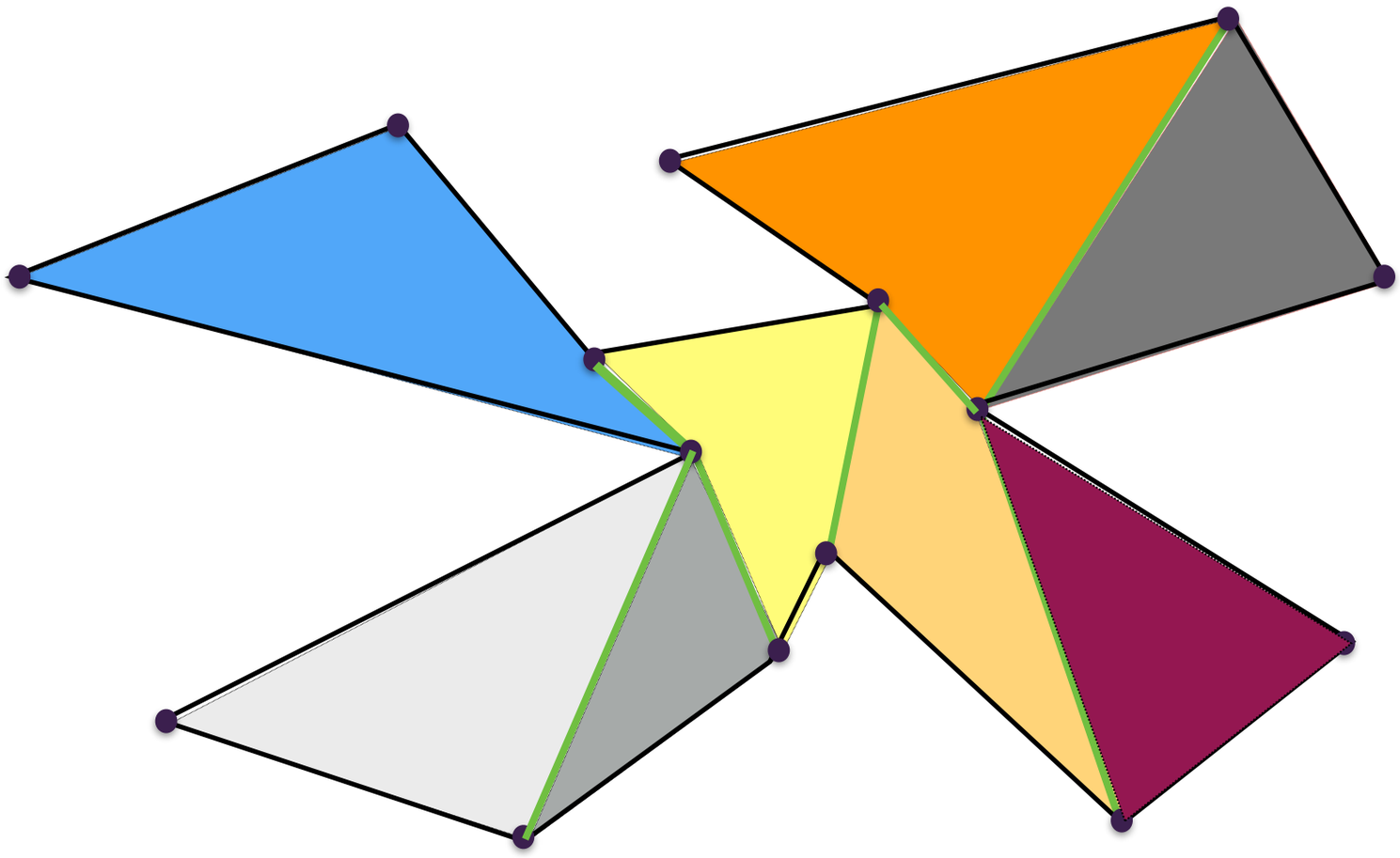
   Claim: All diagonals partition the polygon into monotone mountains.

Partitioning into monotone mountains



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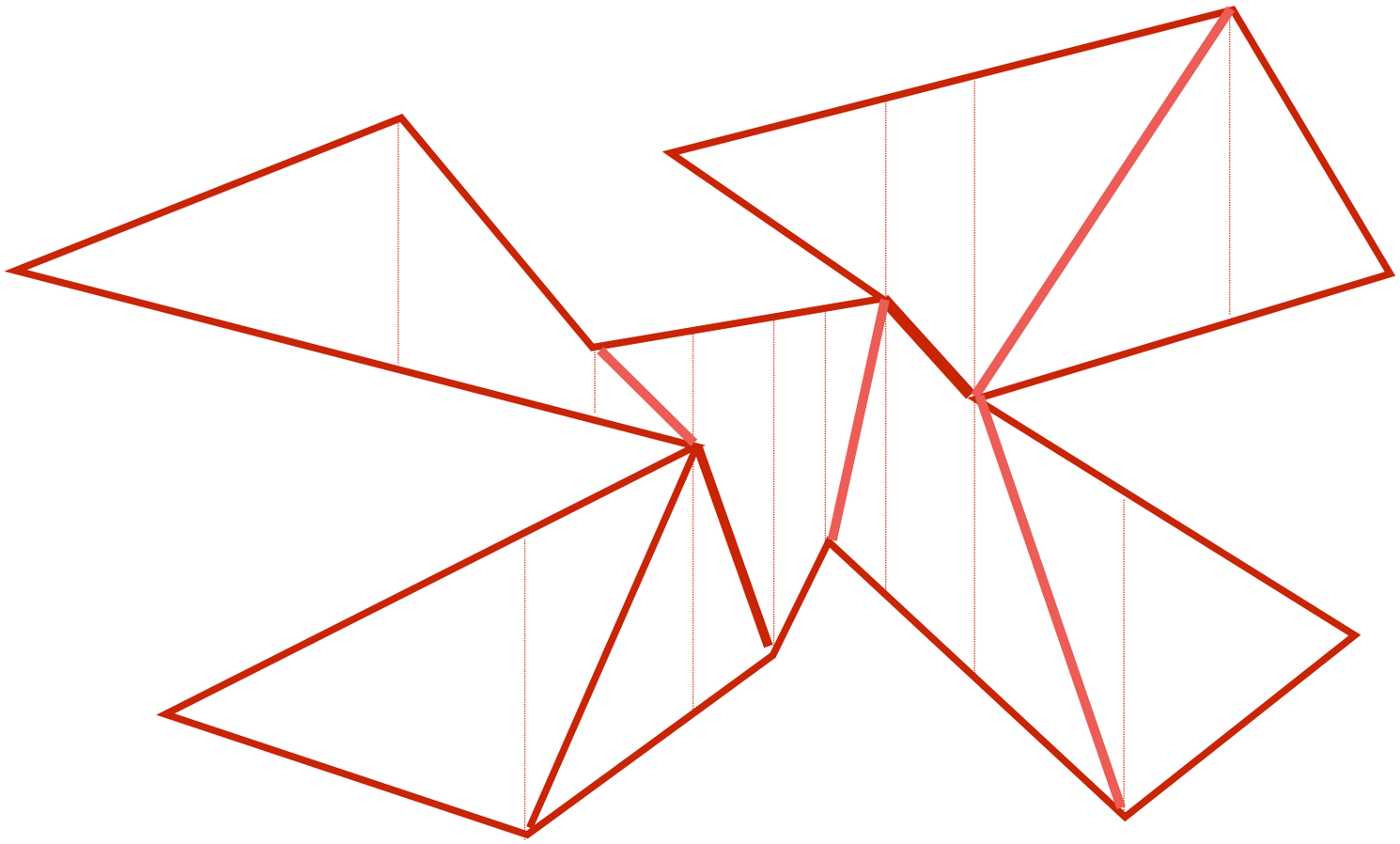
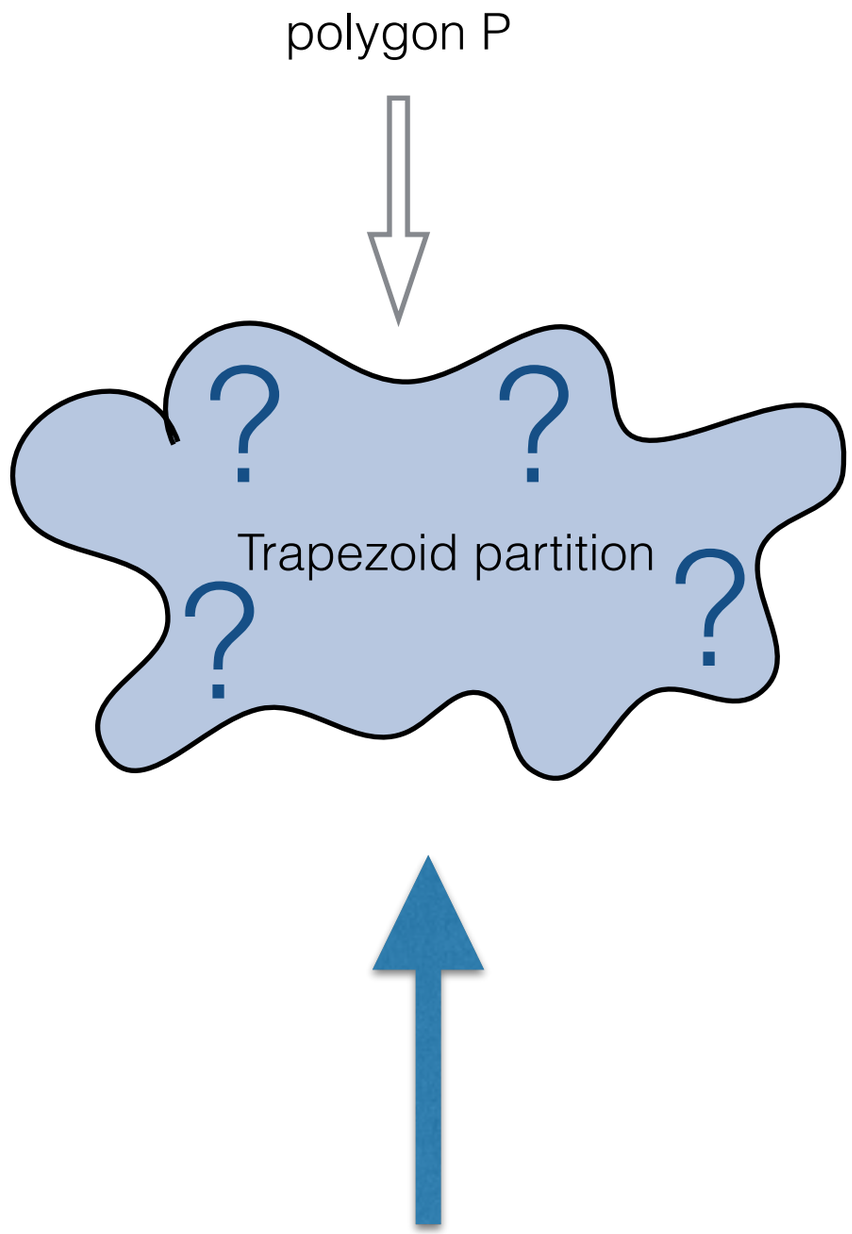
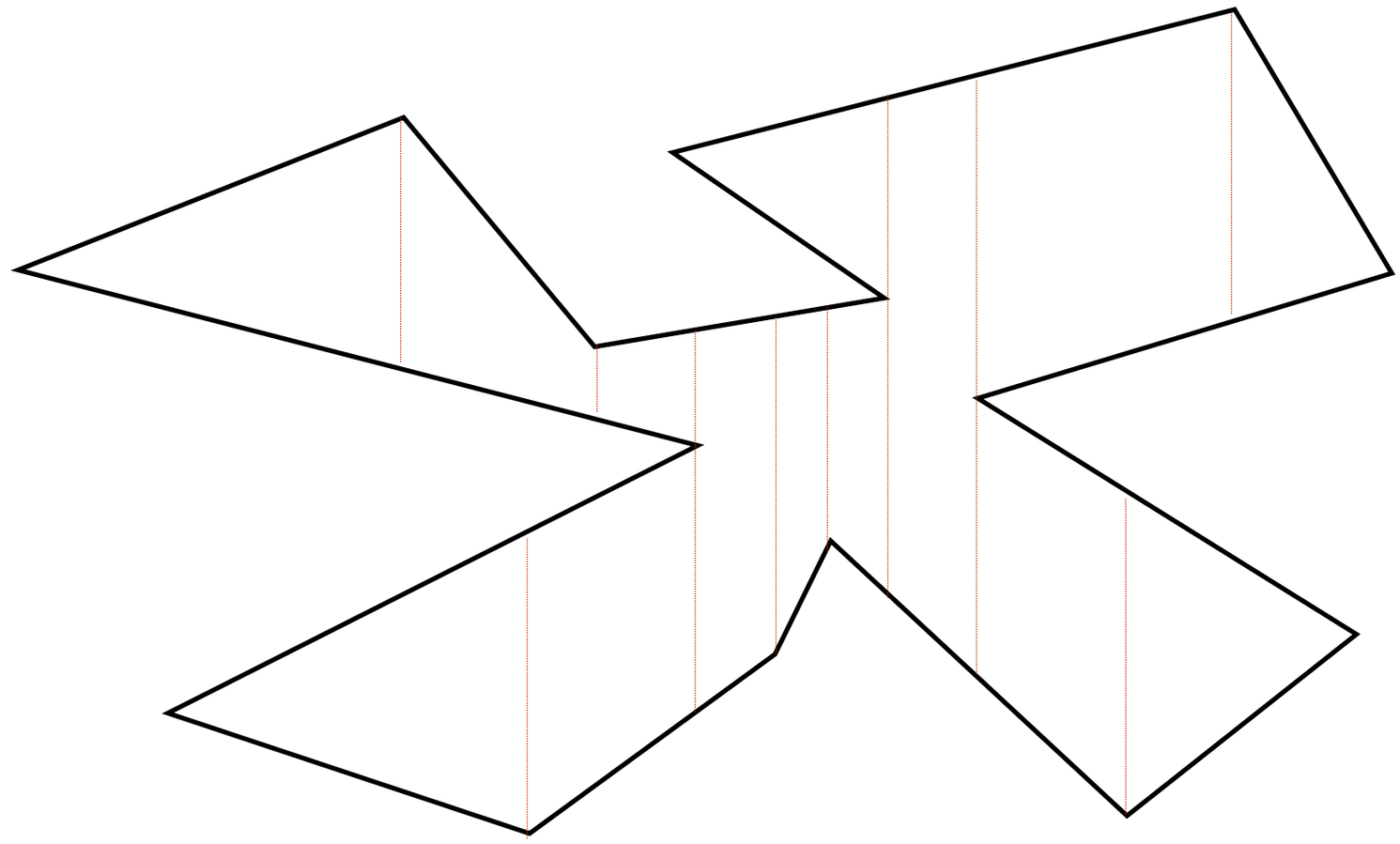


Proof:

- Each polygon is monotone
- One of the chains must be a segment (because if it had another point, that point would generate a diagonal)

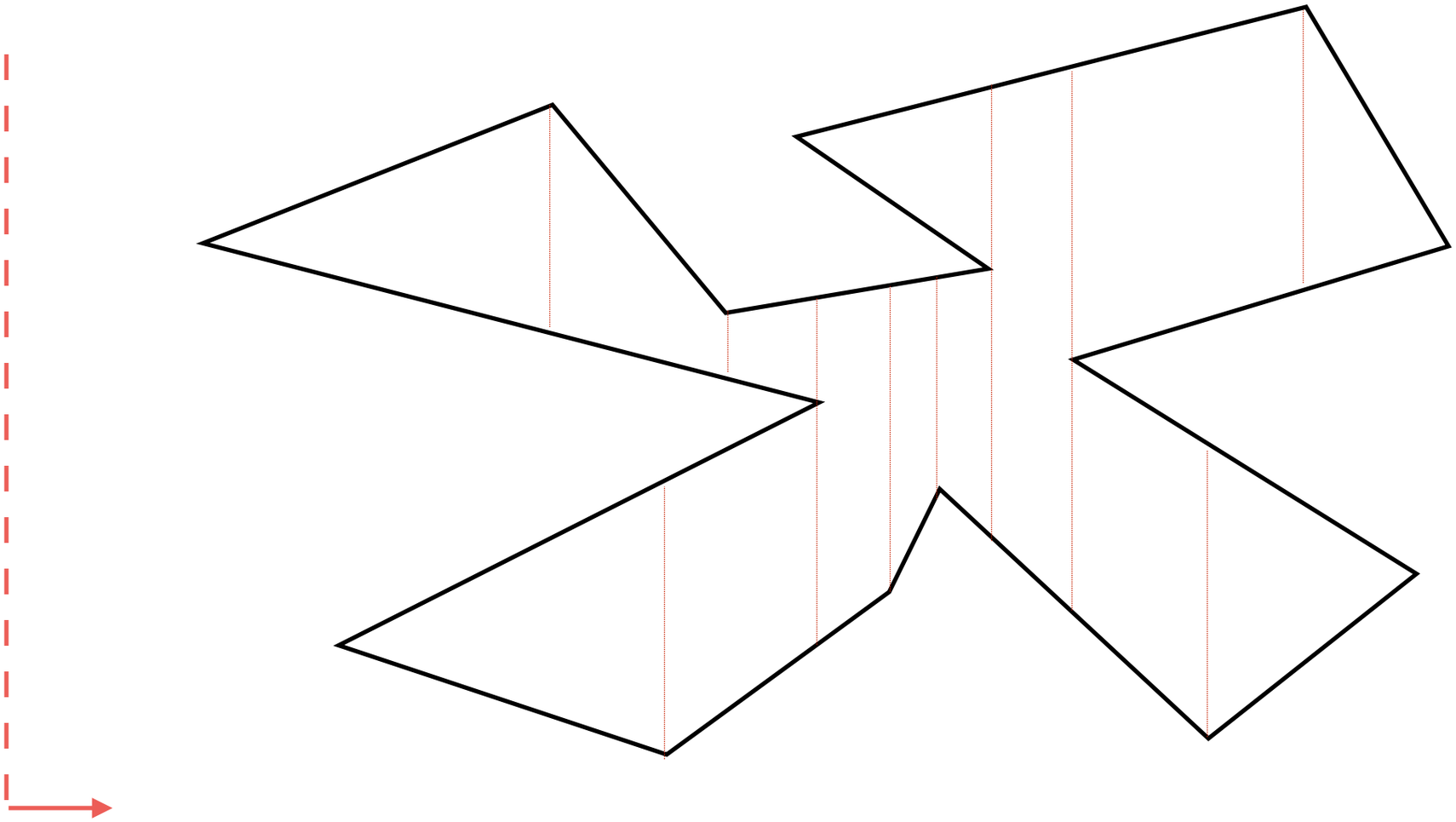

polygon P
?
?
Trapezoid partition
?
?
How do we get
the trapezoid partition?

Computing the trapezoid partition



Computing the trapezoid partition in O(n lg n)

- Plane sweep



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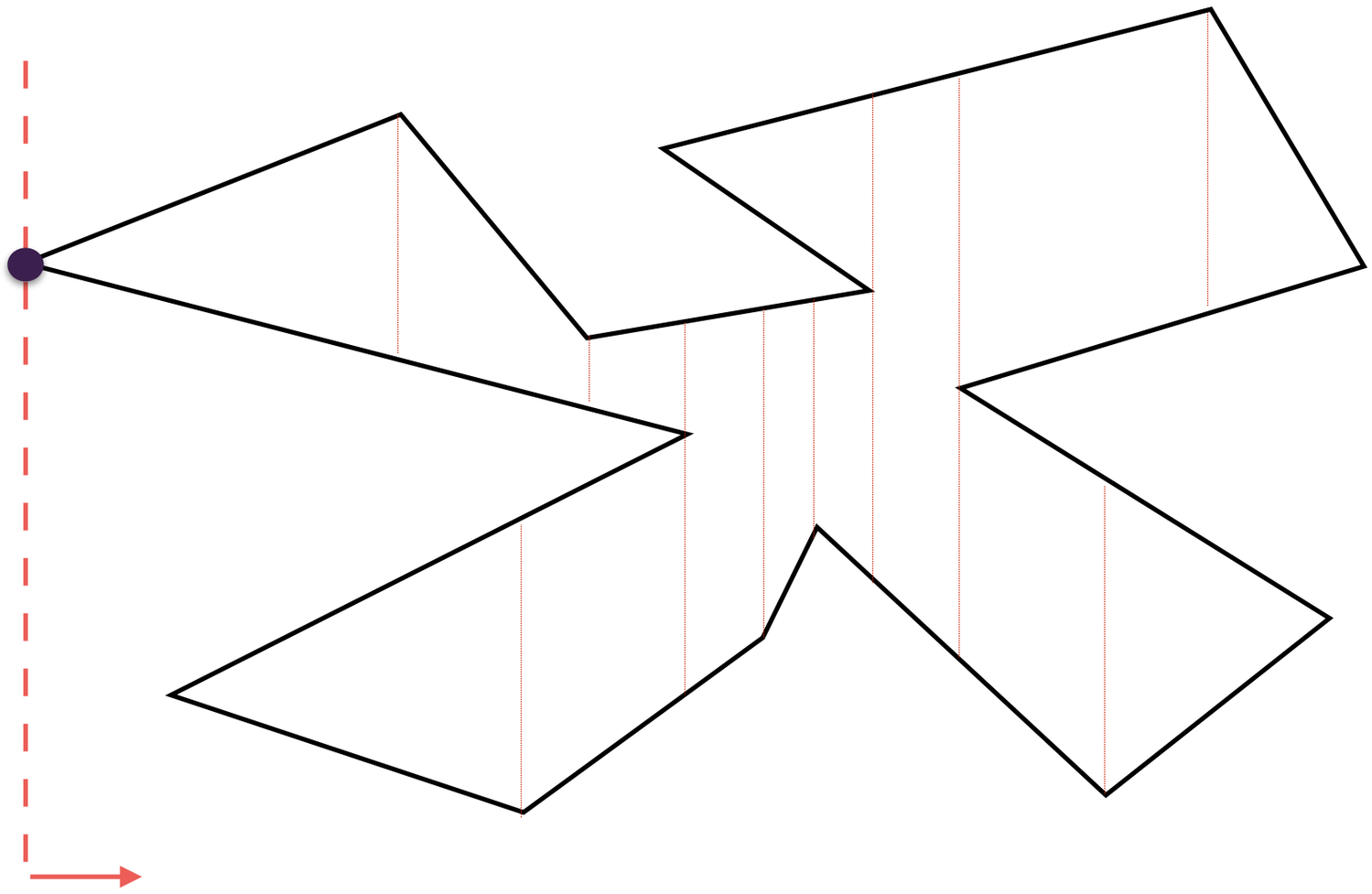
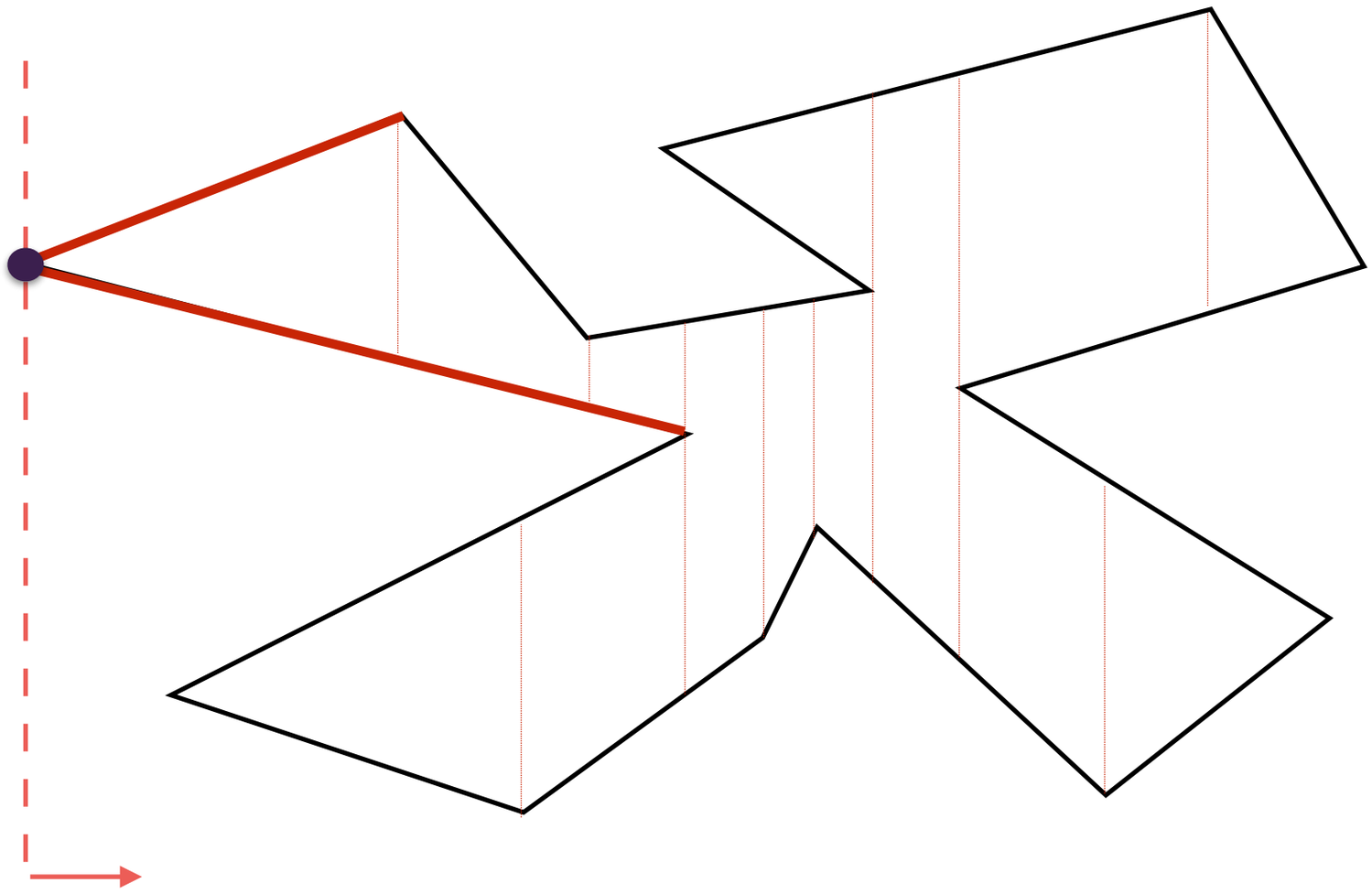
- Plane sweep



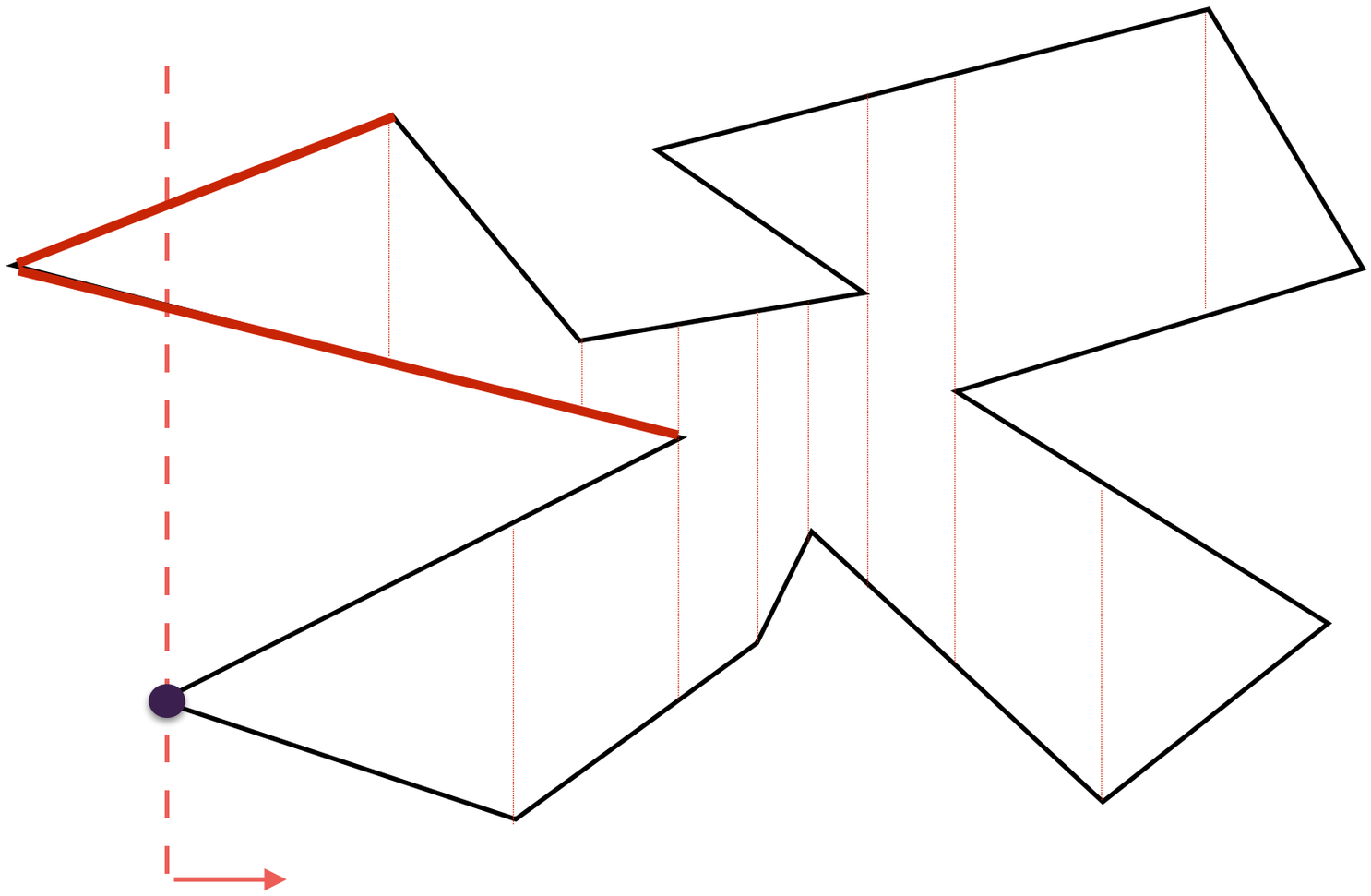
Computing the trapezoid partition in O(n lg n)

- Plane sweep



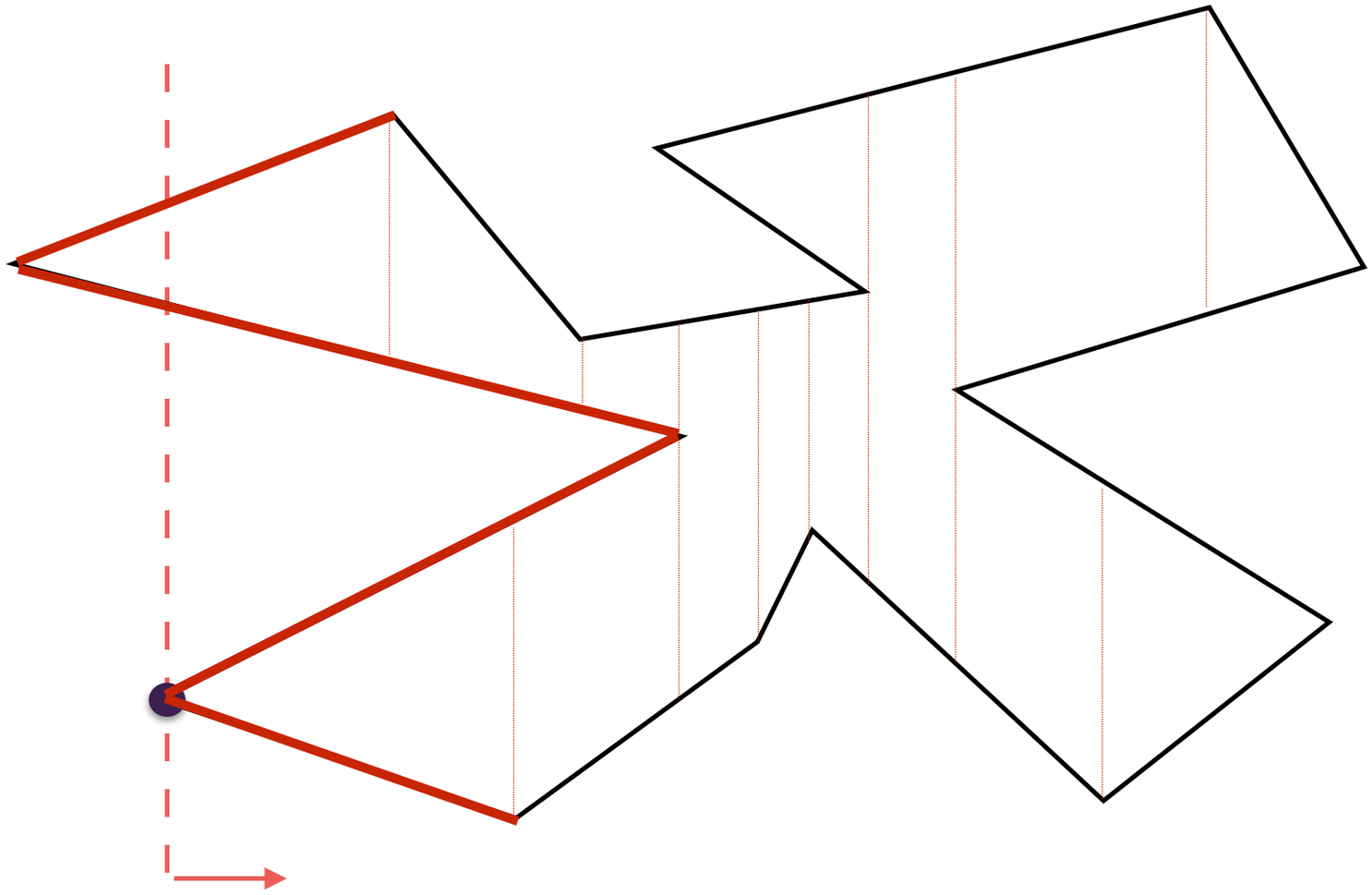
Computing the trapezoid partition in O(n lg n)

- Plane sweep



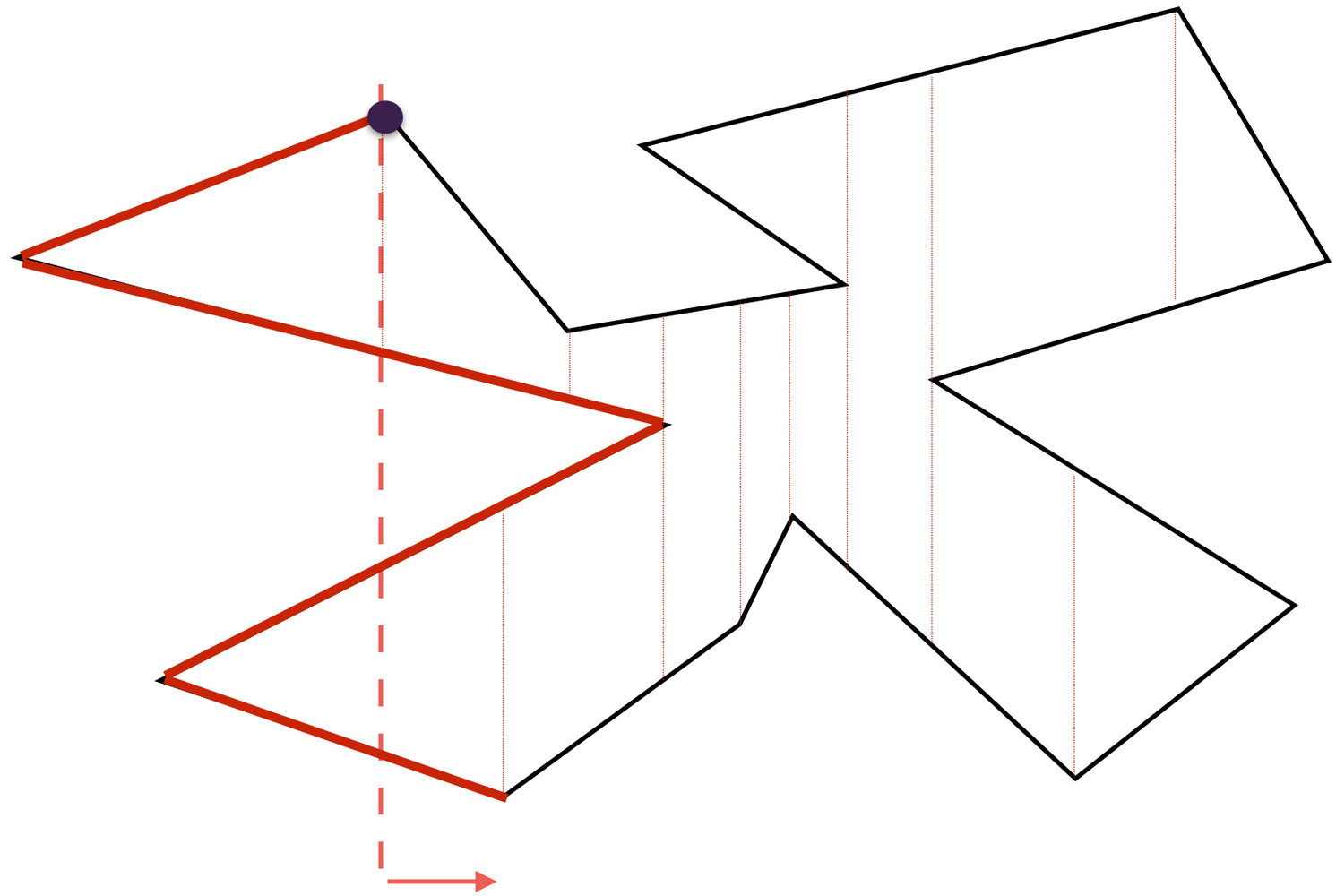
Computing the trapezoid partition in O(n lg n)

- Plane sweep



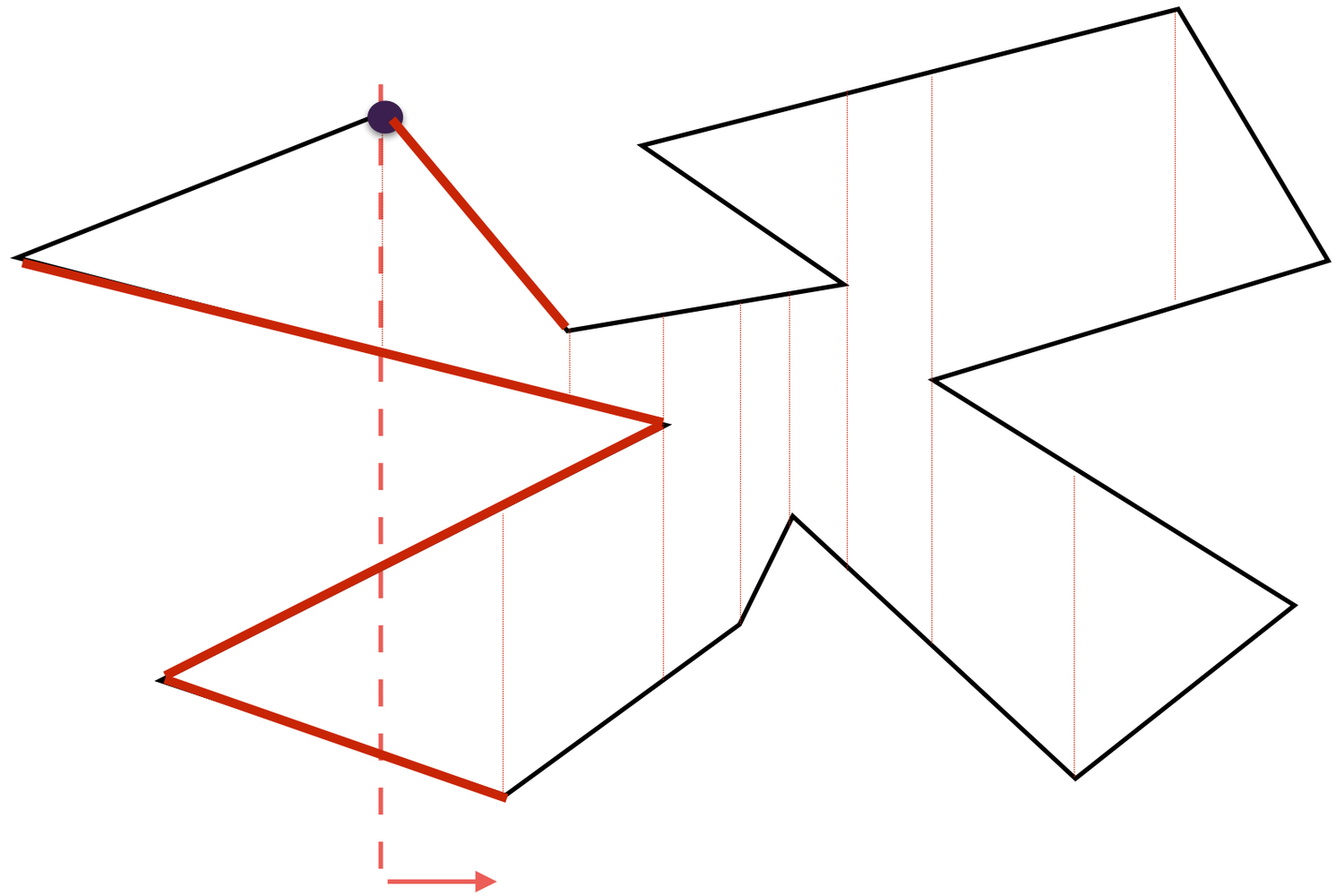
Computing the trapezoid partition in O(n lg n)

- Plane sweep



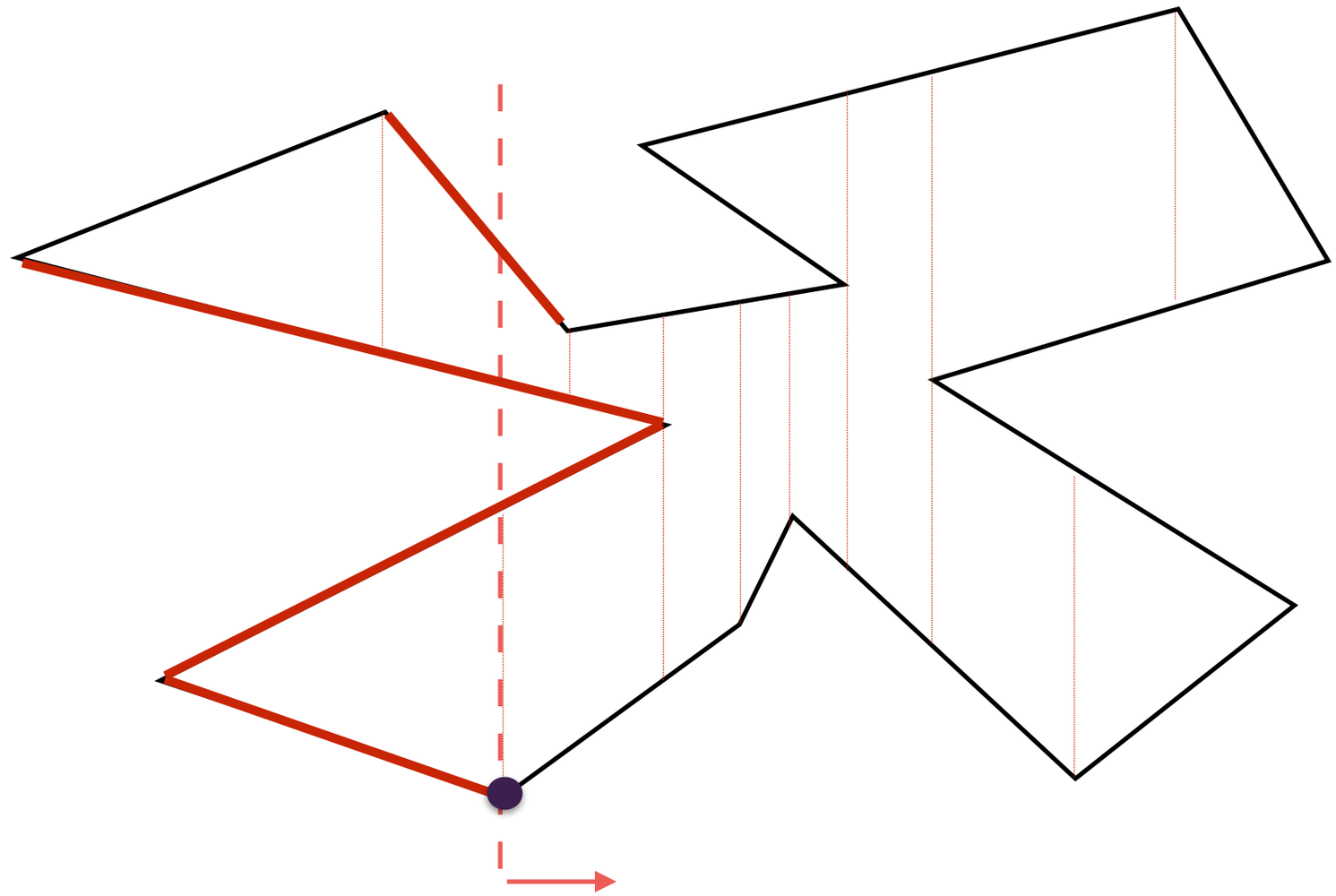
Computing the trapezoid partition in O(n lg n)

- Plane sweep



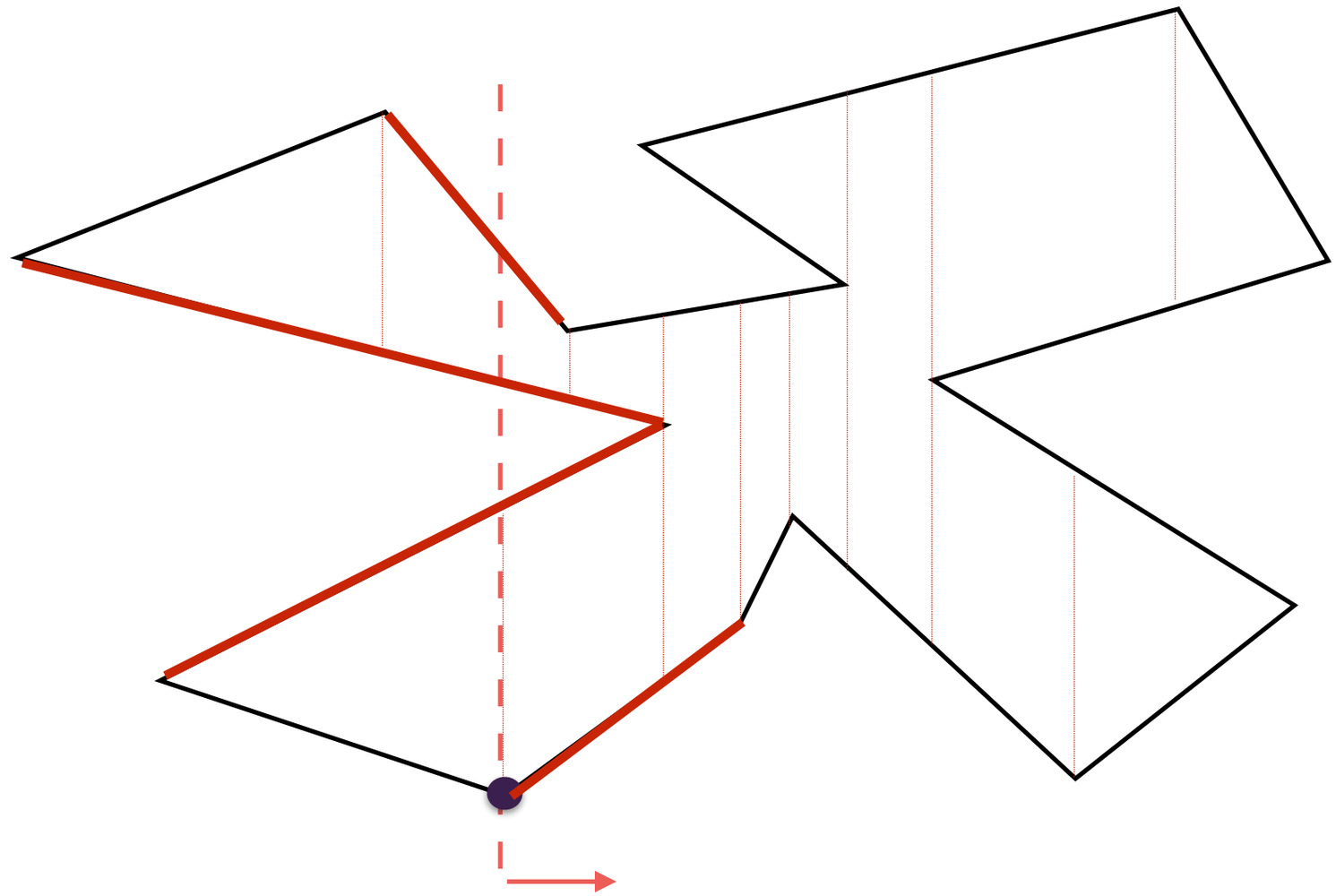
Computing the trapezoid partition in O(n lg n)

- Plane sweep



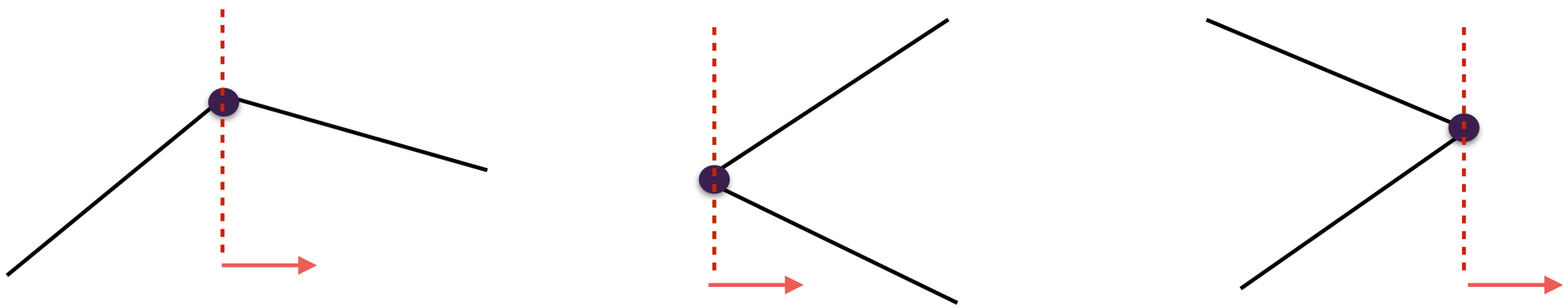
Computing the trapezoid partition in O(n lg n)

- Plane sweep



Computing the trapezoid partition in O(n lg n)

- Plane sweep
- Events: polygon vertices
- Status structure: edges that intersect current sweep line, in y-order
- Events:



How do we determine the trapezoids?

Computing the trapezoid partition in O(n lg n)

- Algorithm