Computational Geometry
[csci 3250]

Laura Toma

Bowdoin College

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The Art Gallery Problem

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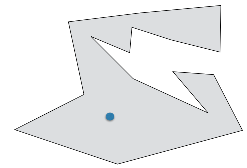
## The Art Gallery Problem

Imagine an art gallery whose floor plan is a simple polygon, and a guard (a point) inside the gallery.



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## The Art Gallery Problem

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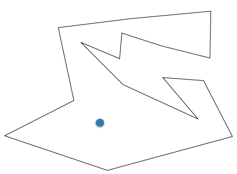
What does the guard see?



We say that two points a, b are visible if segment ab stays inside P (touching boundary is ok).

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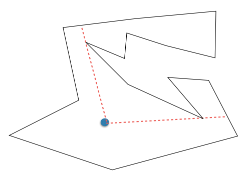

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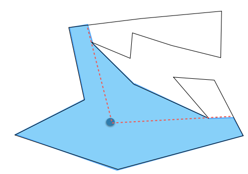

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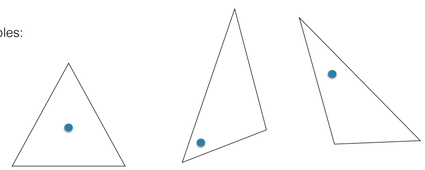
## The Art Gallery Problem(s)

We say that a set of guards **covers** polygon P if every point in P is visible to at least one guard.

Examples:



Does the point guard the triangle?

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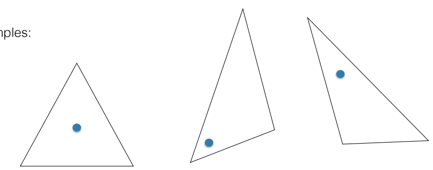

Can all triangles be guarded with one point?

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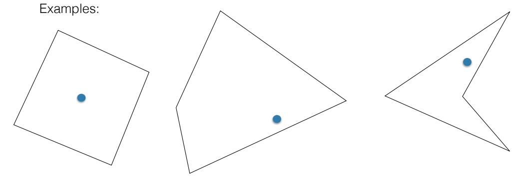

Does the point guard the quadrilateral?

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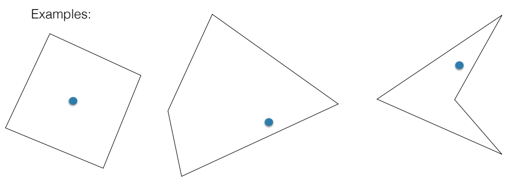

Can all quadrilaterals be guarded with one point?

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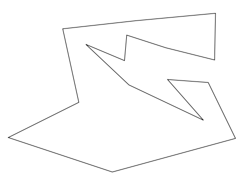
## The Art Gallery Problem(s)



Questions:

1. Given a polygon P of size n, what is the smallest number of guards (and their locations) to cover P?

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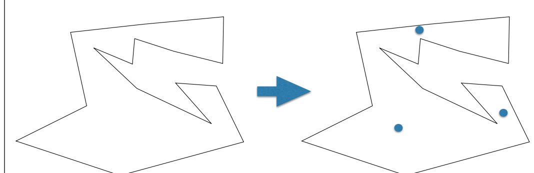

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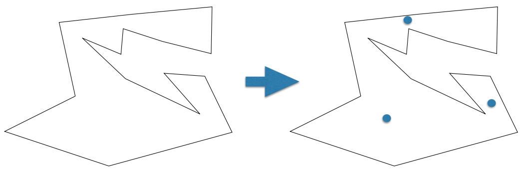

Questions:

1. Given a polygon P of size n, what is the smallest number of guards (and their locations) to cover P?  NP-Complete

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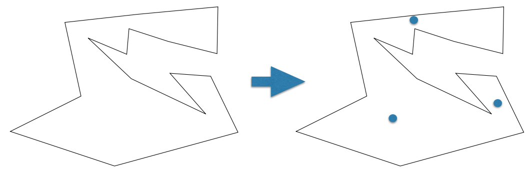
## The Art Gallery Problem(s)



Questions:

1. Given a polygon P of size n, what is the smallest number of guards (and their locations) to cover P?  NP-Complete
2. **Klee's problem:** Consider all polygons of n vertices, and the smallest number of guards to cover each of them. What is the worst-case?

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## Klee's problem

Notation

- Let $P_n$: polygon of n vertices
- Let g(P) = the smallest number of guards to cover P
- Let G(n) = max { g($P_n$) | all $P_n$}.

- G(n) is the smallest number that always works for any n-gon. It is sometimes necessary and always sufficient to guard a polygon of n vertices.
  - G(n) is necessary: there exists a $P_n$ that requires G(n) guards
  - G(n) is sufficient: any $P_n$ can be guarded with G(n) guards

- Klee's problem: find G(n)

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## Klee's problem: find G(n)

Our goal (i.e. Klee's goal) is to find G(n).

Trivial bounds

- G(n) >= 1 : obviously, you need at least one guard.
- G(n) <= n : place one guard in each vertex

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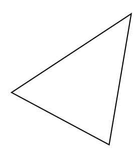
## Klee's Problem

n=3



Any triangle needs at least one guard.
One guard is always sufficient.

G(3) = 1

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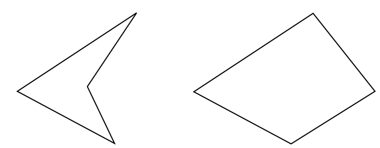
## Klee's Problem

n=4



Any quadrilateral needs at least one guard.
One guard is always sufficient.

G(4) = 1

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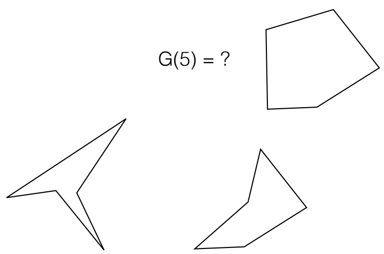
## Klee's Problem

n=5

G(5) = ?



Can all 5-gons be guarded by one point?

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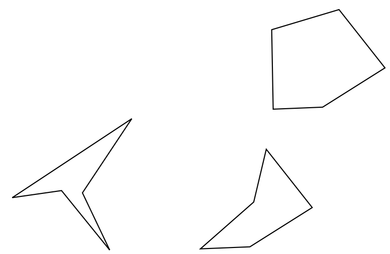
## Klee's Problem

n=5



G(5) = 1

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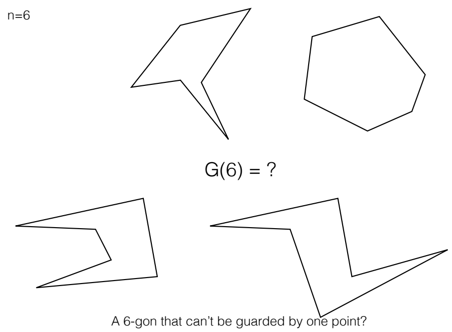
## Klee's Problem

n=6



G(6) = ?

A 6-gon that can't be guarded by one point?

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## Klee's Problem

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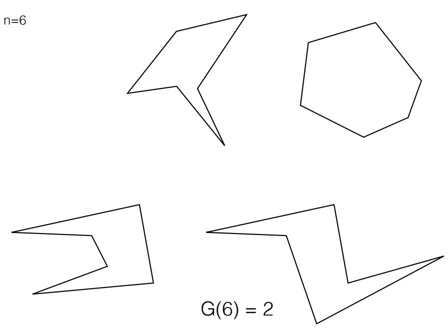

G(6) = 2

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## Klee's Problem

G(n) = ?

Come up with a $P_n$ that requires as many guards as possible.

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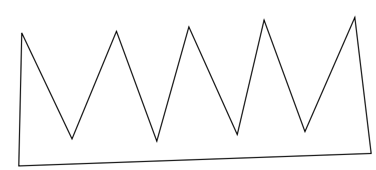
## Klee's Problem

G(n) = ?

Come up with a $P_n$ that requires as many guards as possible.



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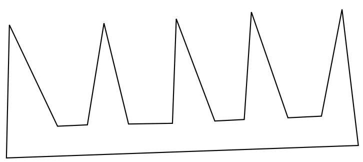
## Klee’s Problem

G(n) = ?

Come up with a $P_n$ that requires as many guards as possible.



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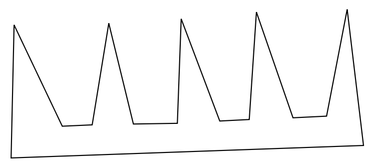
## Klee’s Problem

$\lfloor n/3 \rfloor$ necessary



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## Klee’s Problem

It was shown that $\lfloor n/3 \rfloor$ is also sufficient. That is,

| Any $P_n$ can be guarded with at most $\lfloor n/3 \rfloor$ guards. |
|---|

- (Complex) proof by induction
- Subsequently, simple and beautiful proof due to Steve Fisk, who was Bowdoin Math faculty.
- Proof in The Book.

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### *Proofs from THE BOOK*

From Wikipedia, the free encyclopedia

***Proofs from THE BOOK*** is a book of mathematical proofs by Martin Aigner and Günter M. Ziegler. The book is dedicated to the mathematician Paul Erdős, who often referred to "The Book" in which God keeps the most elegant proof of each mathematical theorem. During a lecture in 1985, Erdős said, "You don't have to believe in God, but you should believe in The Book."

#### Content [ edit ]

*Proofs from THE BOOK* contains 32 sections (44 in the fifth edition), each devoted to one theorem but often containing multiple proofs and related results. It spans a broad range of mathematical fields: number theory, geometry, analysis, combinatorics and graph theory. Erdős himself made many suggestions for the book, but died before its publication. The book is illustrated by Karl Heinrich Hofmann. It has gone through five editions in English, and has been translated into Persian, French, German, Hungarian, Italian, Japanese, Chinese, Polish, Portuguese, Korean, Turkish, Russian and Spanish.

The proofs include:

- Proof of Bertrand's postulate
- Proof that e is irrational (also showing the irrationality of certain related numbers)
- Six proofs of the infinitude of the primes, including Euclid's and Furstenberg's
- Monsky's theorem (4th edition)
- Wetzel's problem on families of analytic functions with few distinct values
- Steve Fisk's proof of the The art gallery theorem

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## Fisk’s proof of sufficiency

1. Any simple polygon can be triangulated.
2. A triangulated simple polygon can be 3-colored.
3. Observe that placing the guards at all the vertices assigned to one color guarantees the polygon is covered.
4. There must exist a color that’s used at most n/3 times. Pick that color and place guards at the vertices of that color.

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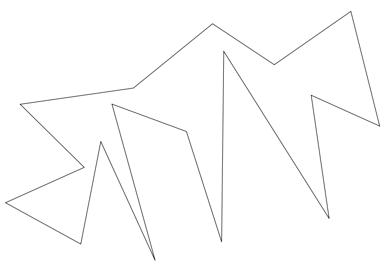
## Fisk’s proof of sufficiency

Claim: Any simple polygon can be triangulated.



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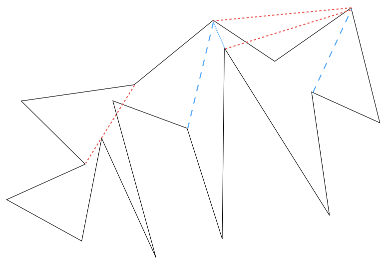
## Polygon triangulation

Given a simple polygon P, a **diagonal** is a segment between 2 non-adjacent vertices that lies entirely within the interior of the polygon.



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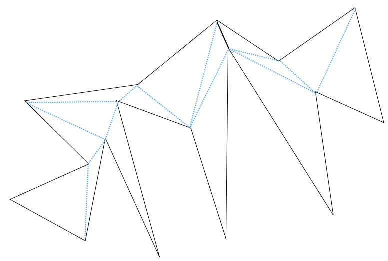
## Polygon triangulation

Claim: Any simple polygon can be triangulated.

Proof idea: induction using the existence of a diagonal.  Later.



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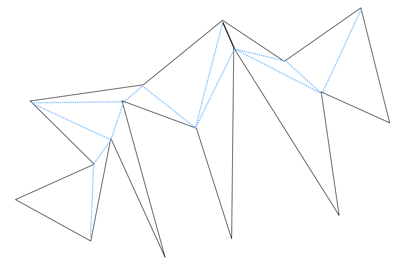

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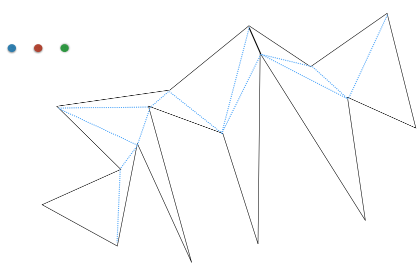

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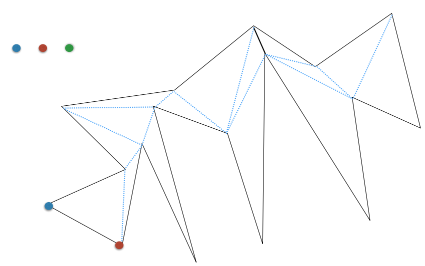

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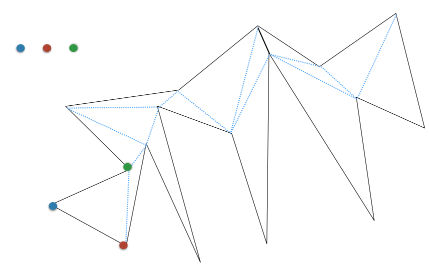

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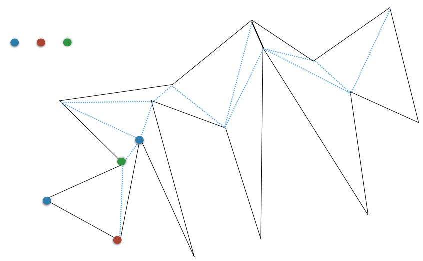

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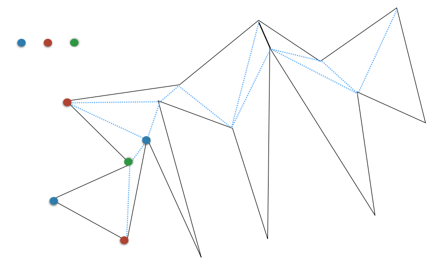

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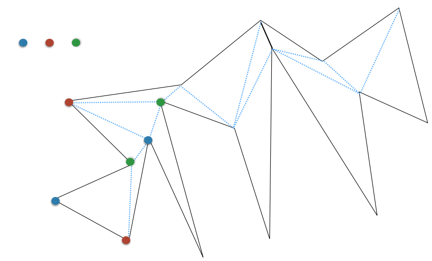

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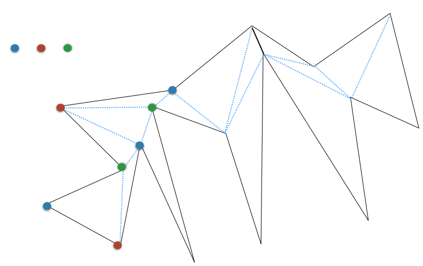

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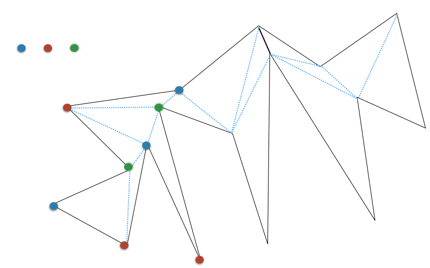

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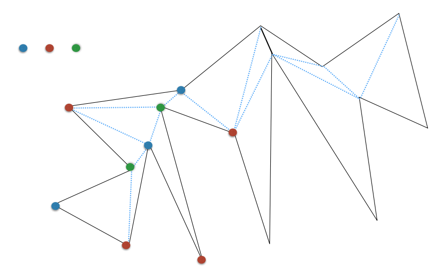

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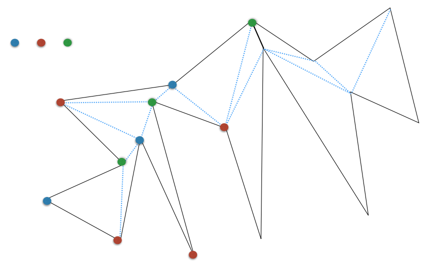

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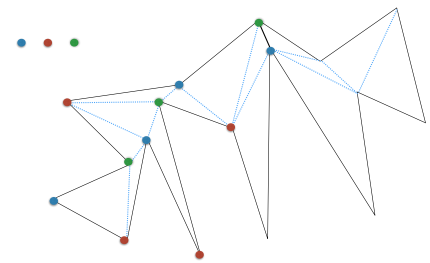

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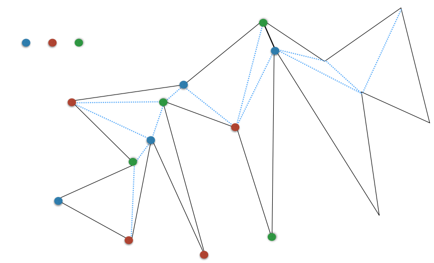

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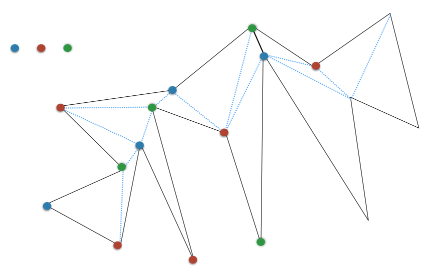

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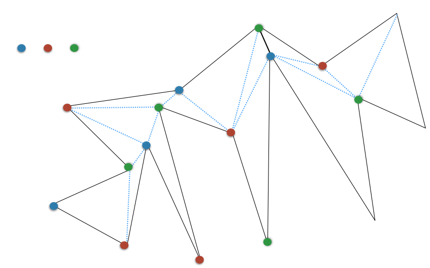

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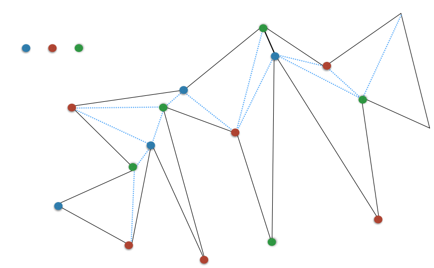

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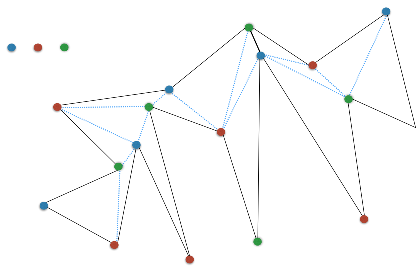

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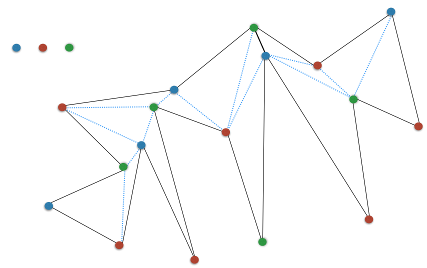

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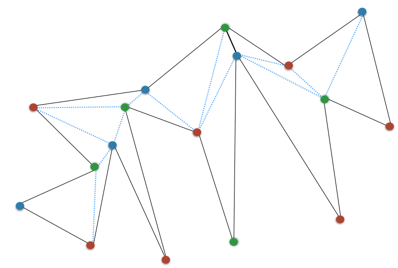
## Fisk's proof of sufficiency

- Placing guards at vertices of one color covers P.



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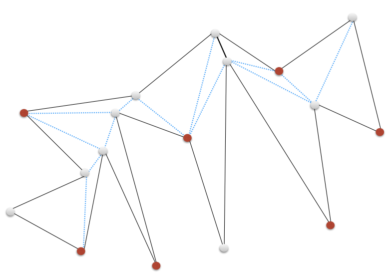

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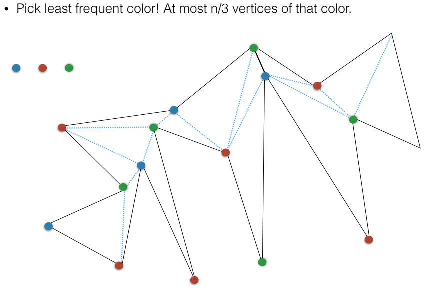
## Fisk's proof of sufficiency

- Placing guards at vertices of one color covers P.
- Pick least frequent color! At most n/3 vertices of that color.



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The proofs

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## Fisk's proof of sufficiency

1. Any polygon can be triangulated
2. Any triangulation can be 3-colored
3. Observe that placing the guards at all the vertices assigned to one color guarantees the polygon is covered.
4. There must exist a color that's used at most n/3 times. Pick that color and place guards at the vertices of that color.

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Claim: The set of red vertices covers the polygon. The set of blue vertices covers the polygon. The set of green vertices covers the polygon.

Proof:

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There are n vertices colored with one of 3 colors.

Claim: There must exist a color that's used at most n/3 times.

Proof:

57

Theorem: Any triangulation can be 3-colored.

Proof:

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## Polygon triangulation

Theorem: Any simple polygon has at least one convex vertex.

Proof:

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## Polygon triangulation

Theorem: Any simple polygon with n>3 vertices contains (at least) a diagonal.

Proof:

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Theorem: Any polygon can be triangulated

Proof: