Computational Geometry
[csci 3250]

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Bowdoin College

Line segment intersection

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Given a set of line segments in 2D, find (report) all their pairwise intersections.

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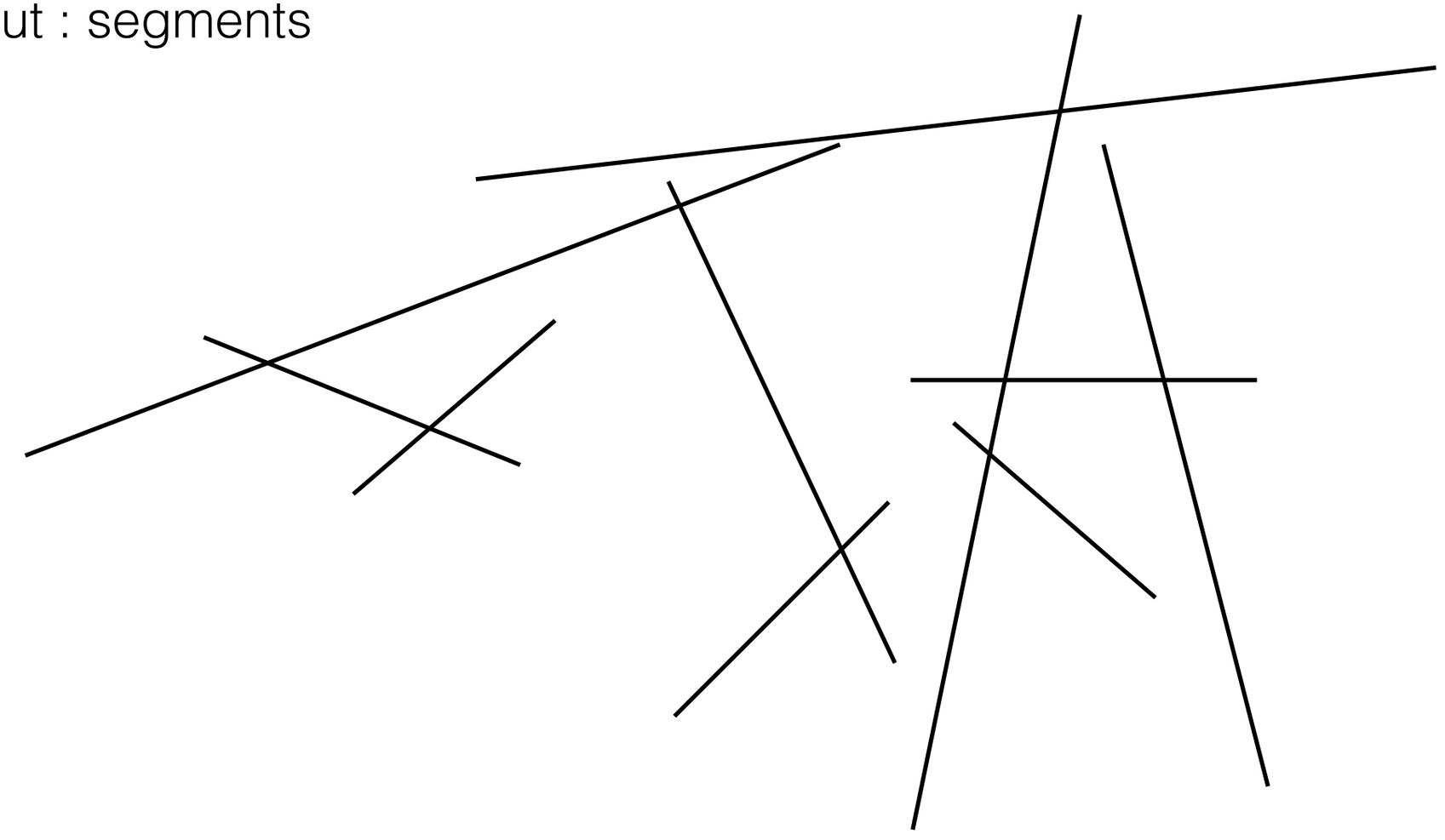
- Input : segments



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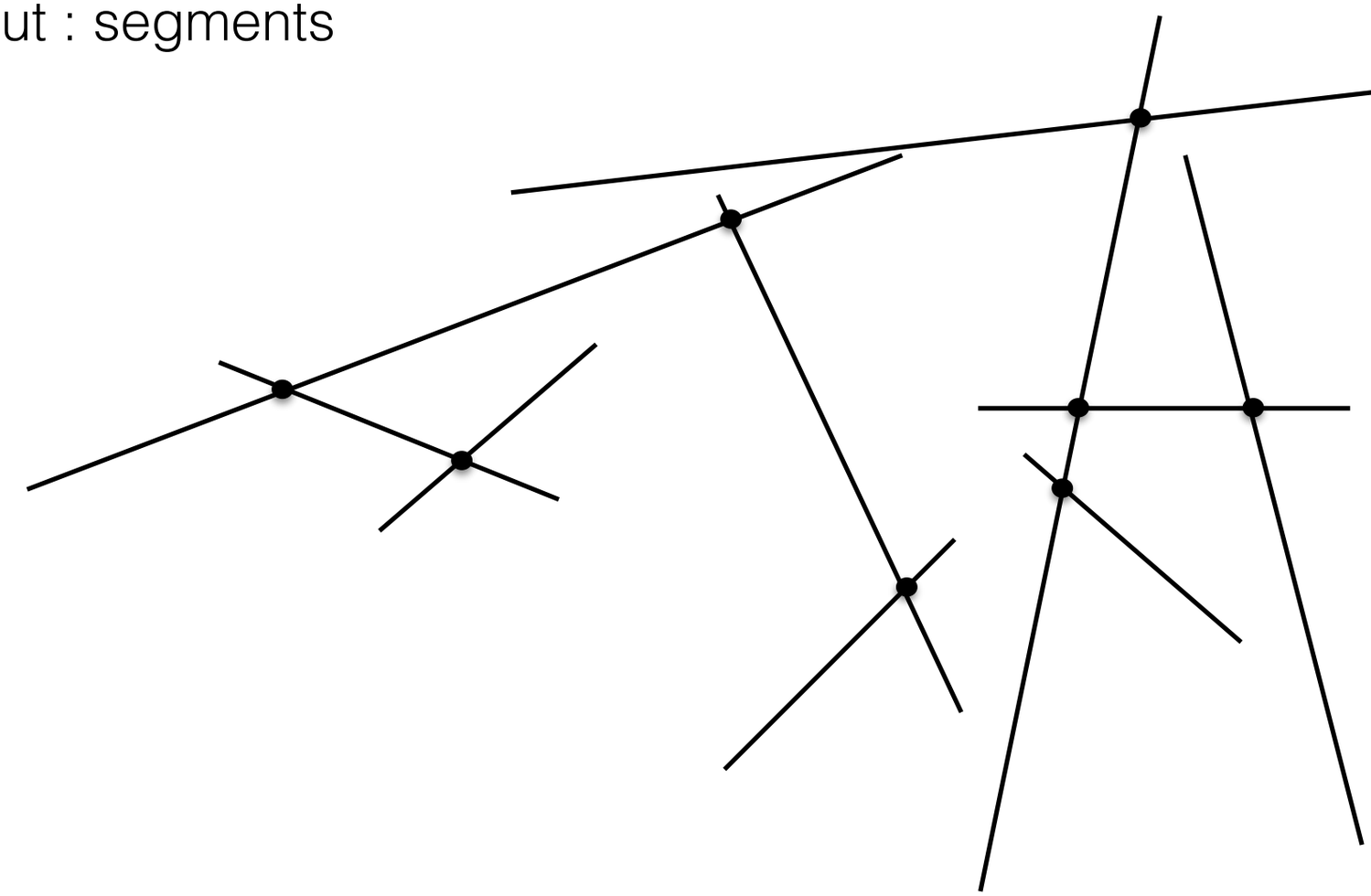
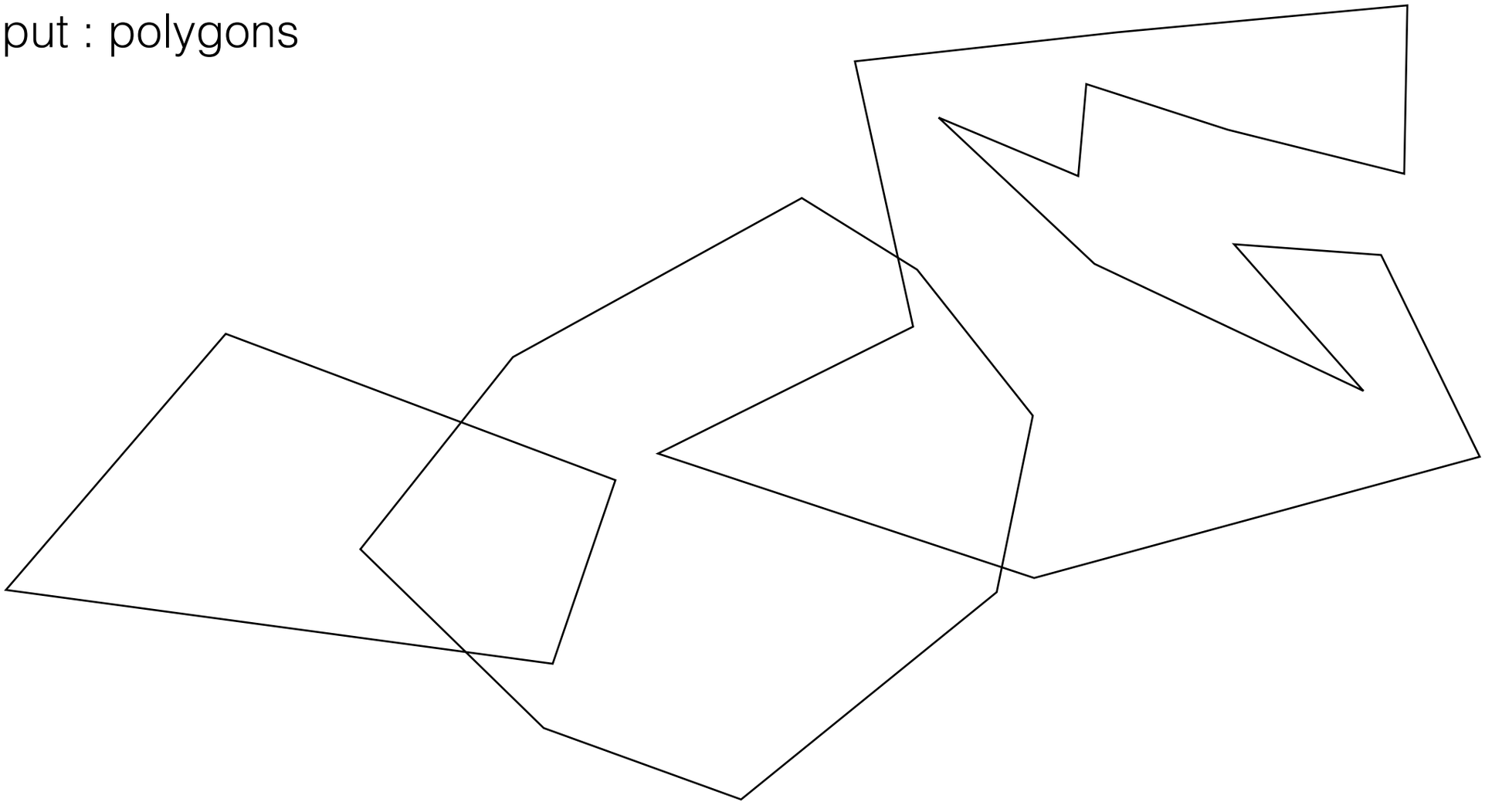


Line segment intersection

Given a set of line segments in 2D, find (report) all their pairwise intersections.

- Input : polygons



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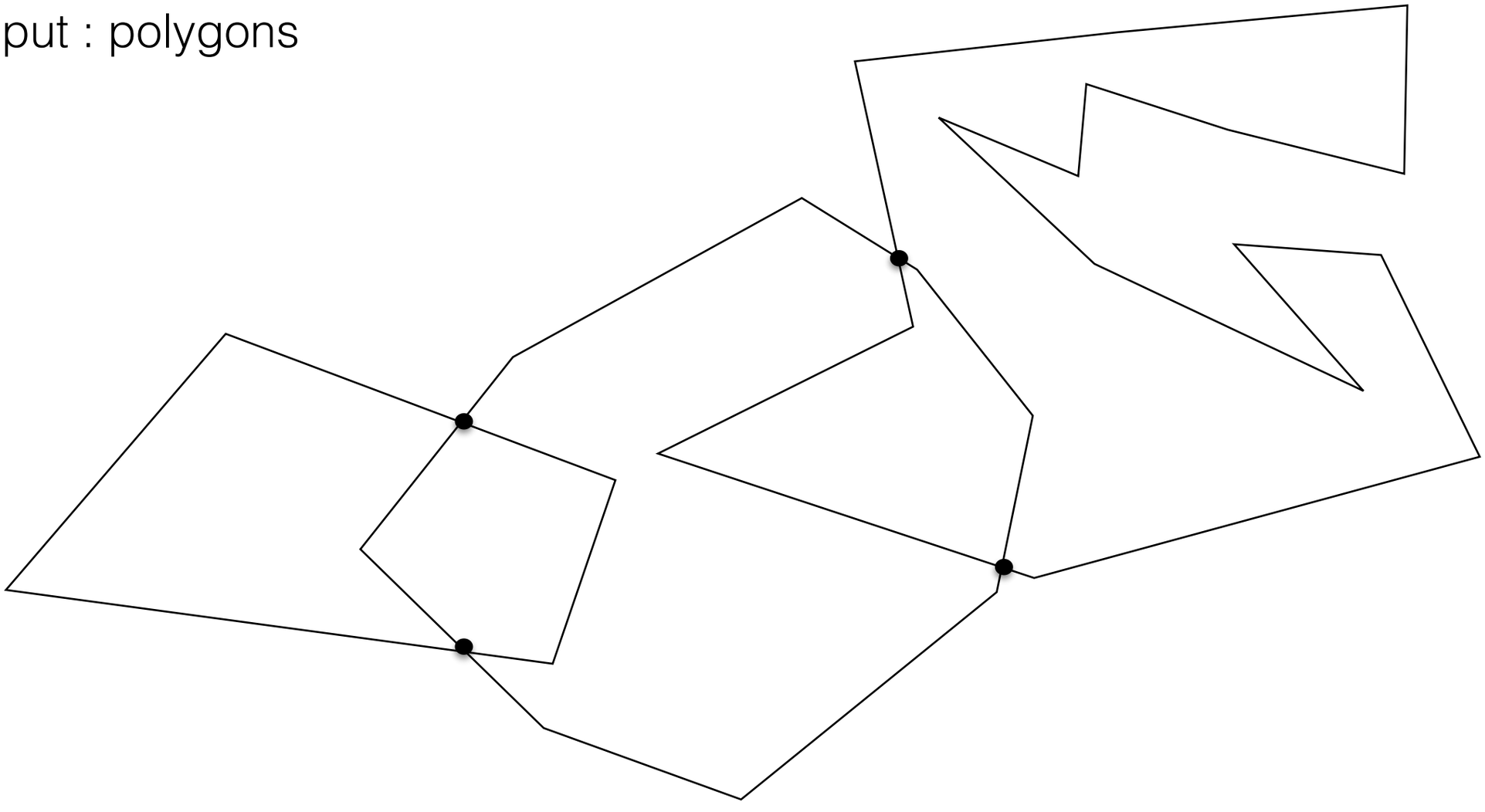
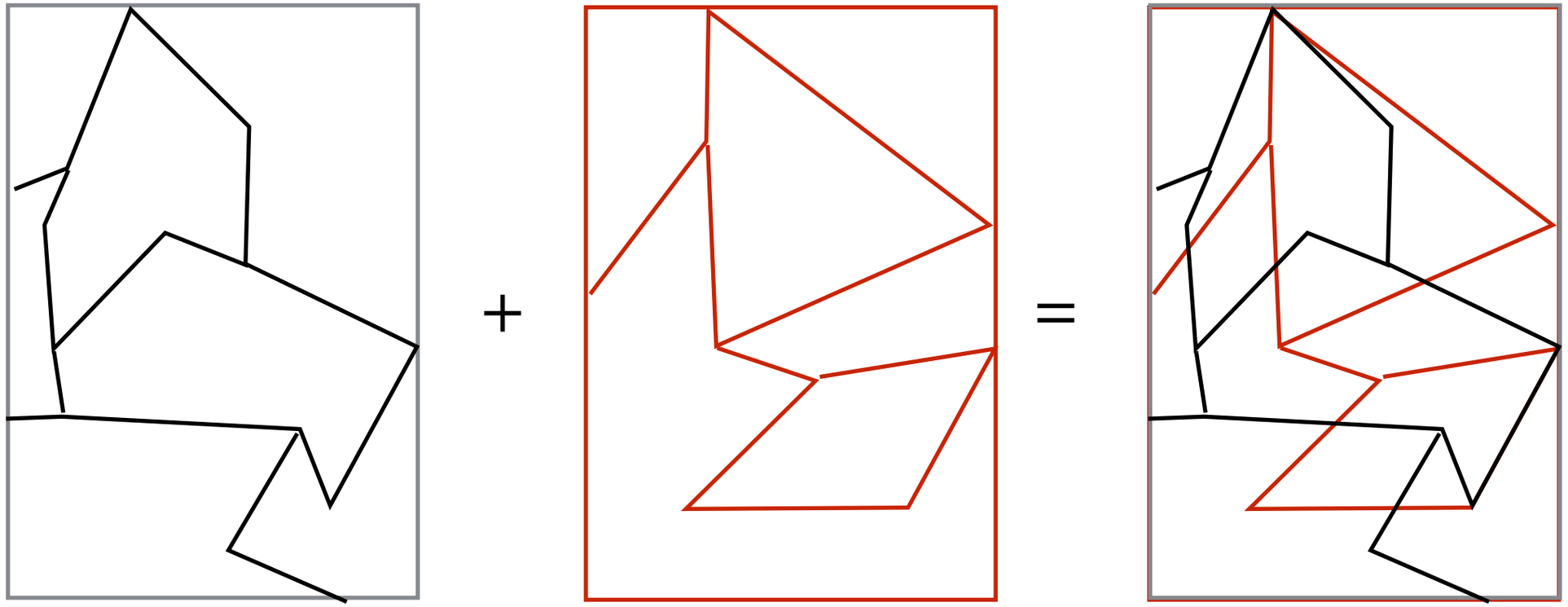
- Input : polygons



Line segment intersection

Given a set of line segments in 2D, find (report) all their pairwise intersections.

- Input : planar maps



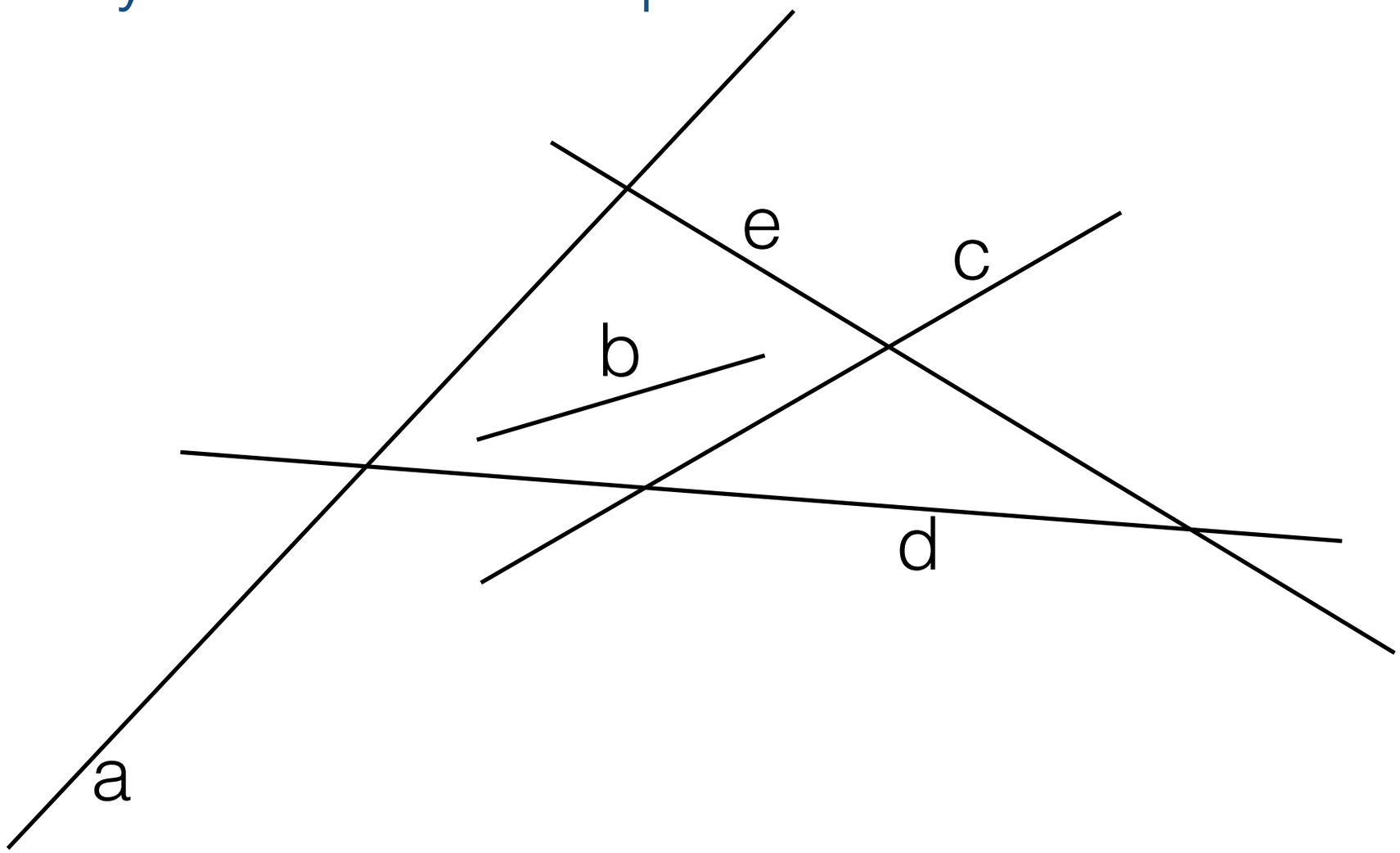
Line segment intersection

- n: size of the input (number of segments)
- k: size of output (number of intersections)

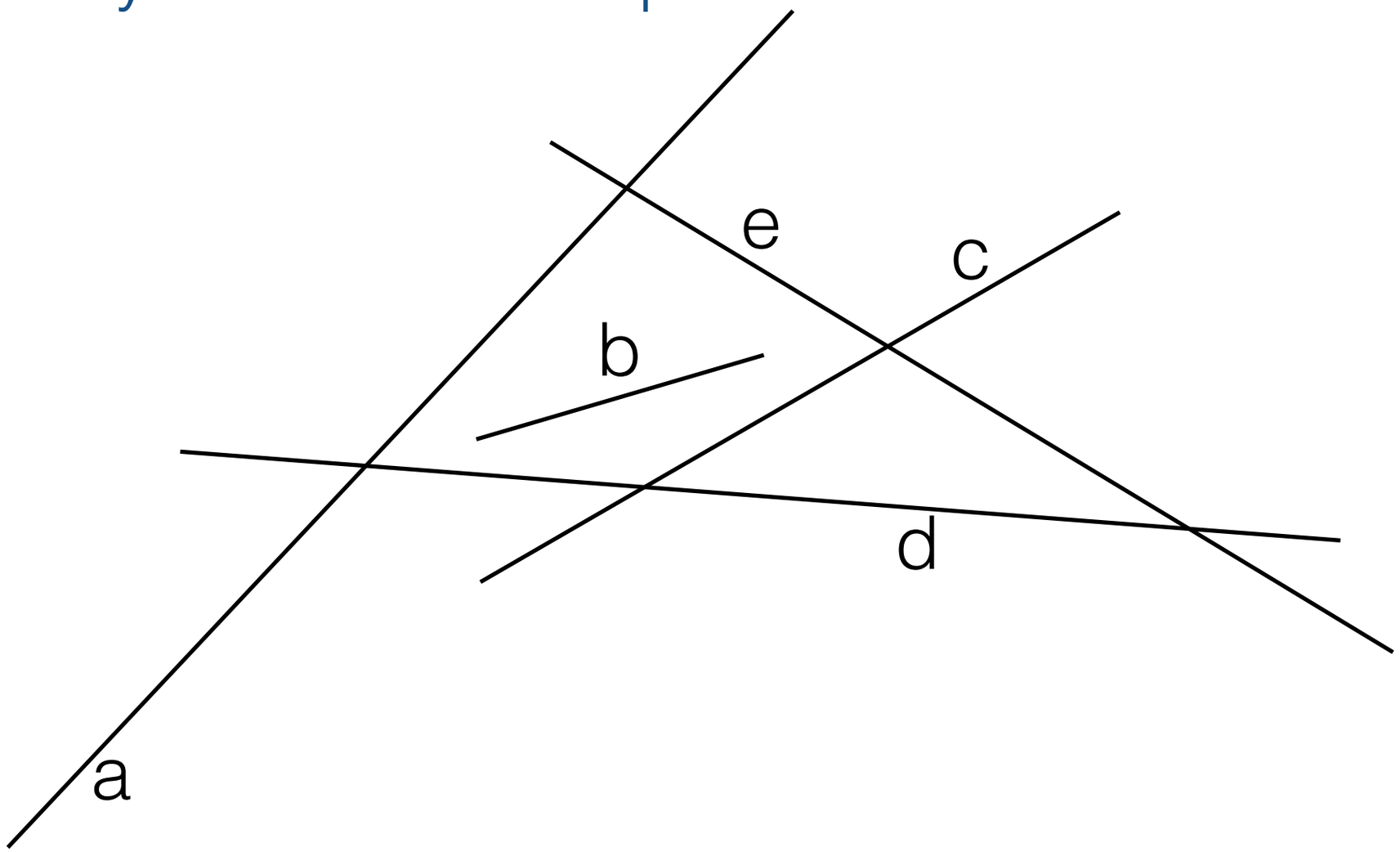
Given a set of line segments in 2D, find (report) all their pairwise intersections.

- Our goal is to extend the line sweep idea
- We'll get an overall bound of O(n lg n + k lg n) which improves on the naive $O(n^2)$ when k is small
- The algorithm was developed by Jon Bentley and Thomas Ottmann in 1979
- Simple and practical

Bentley-Ottmann sweep

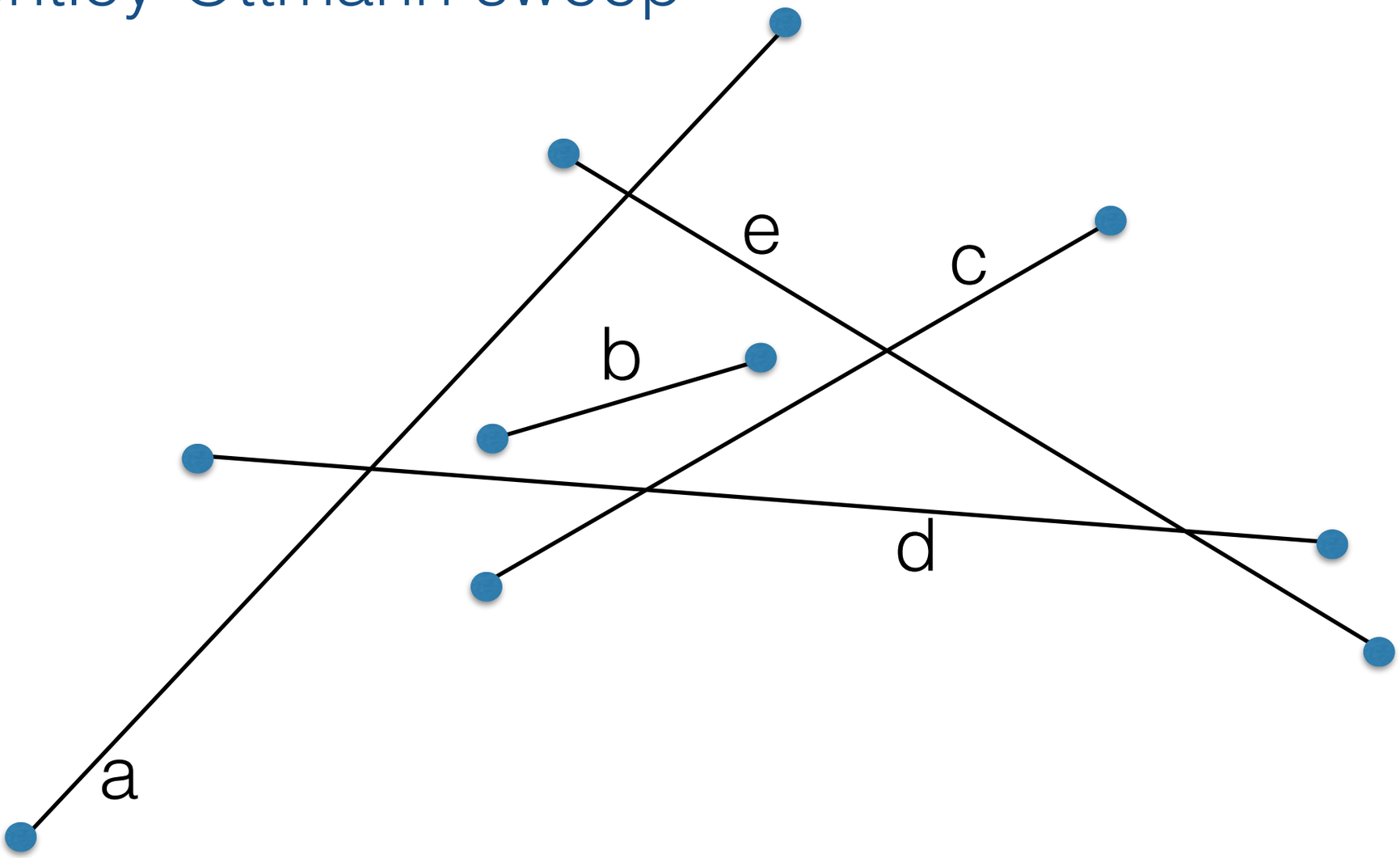


Bentley-Ottmann sweep



- Let X be the set of all x-coords of segments

Bentley-Ottmann sweep



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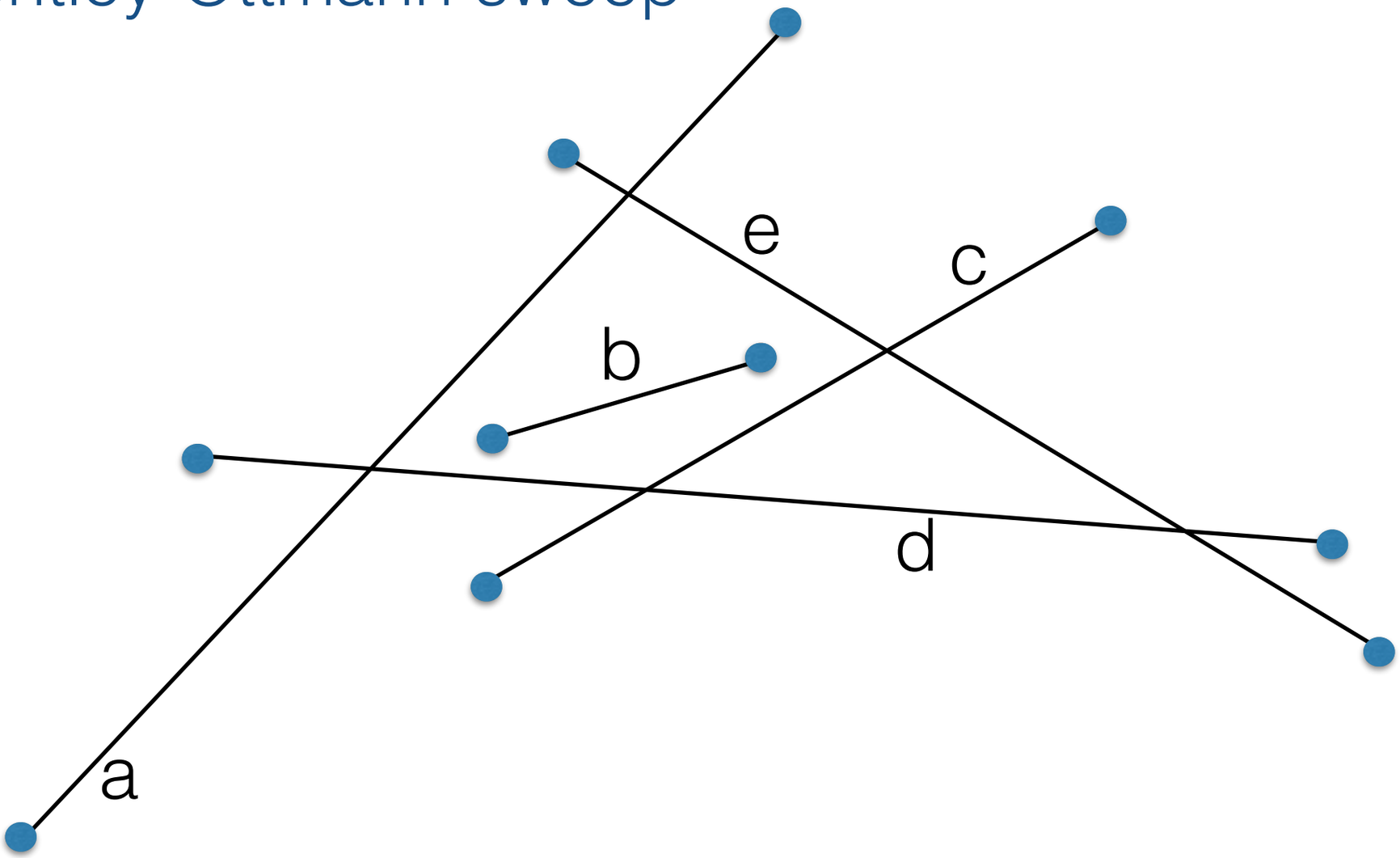
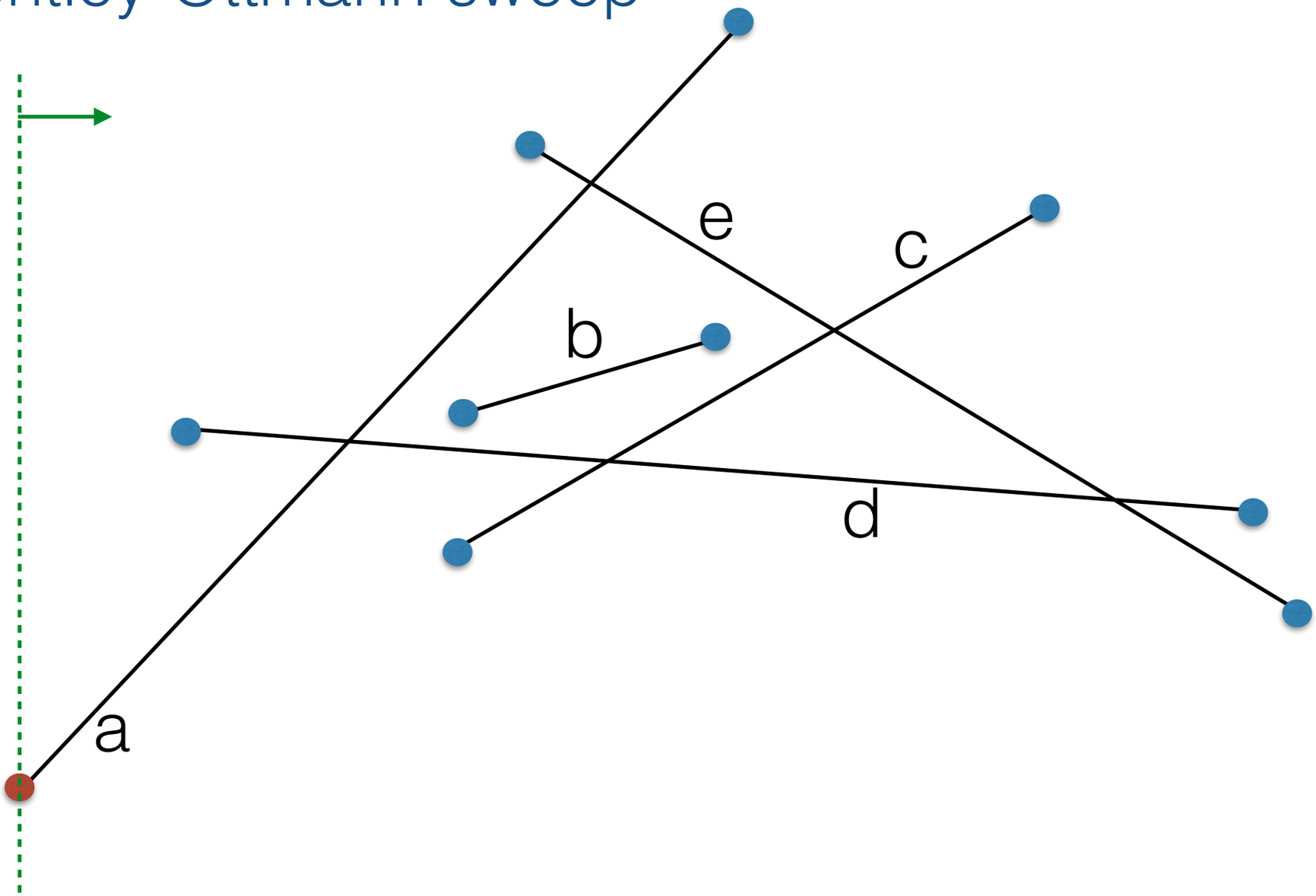


- Let X be the set of all x-coords of segments
- Traverse the events in X in order

Bentley-Ottmann sweep



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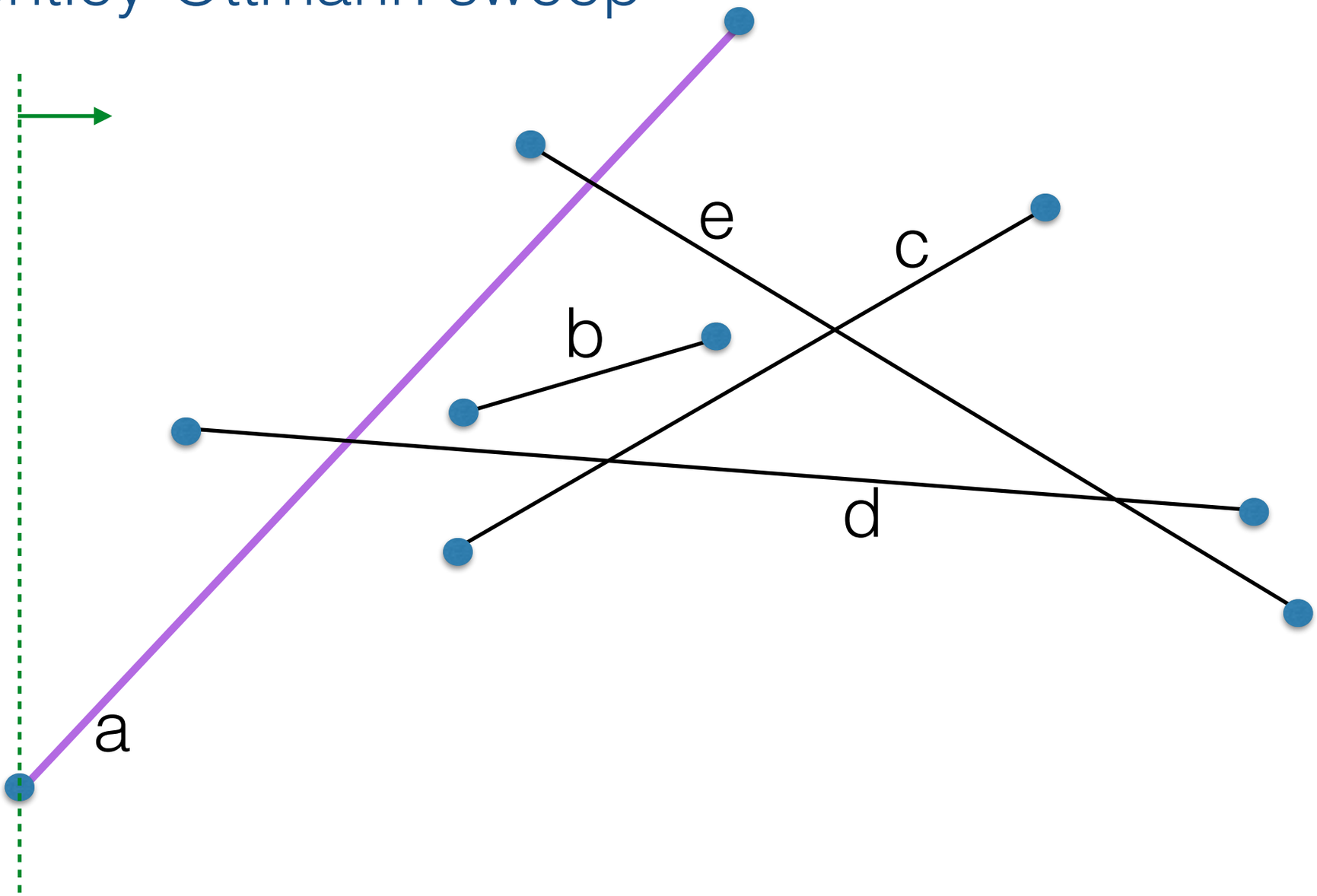

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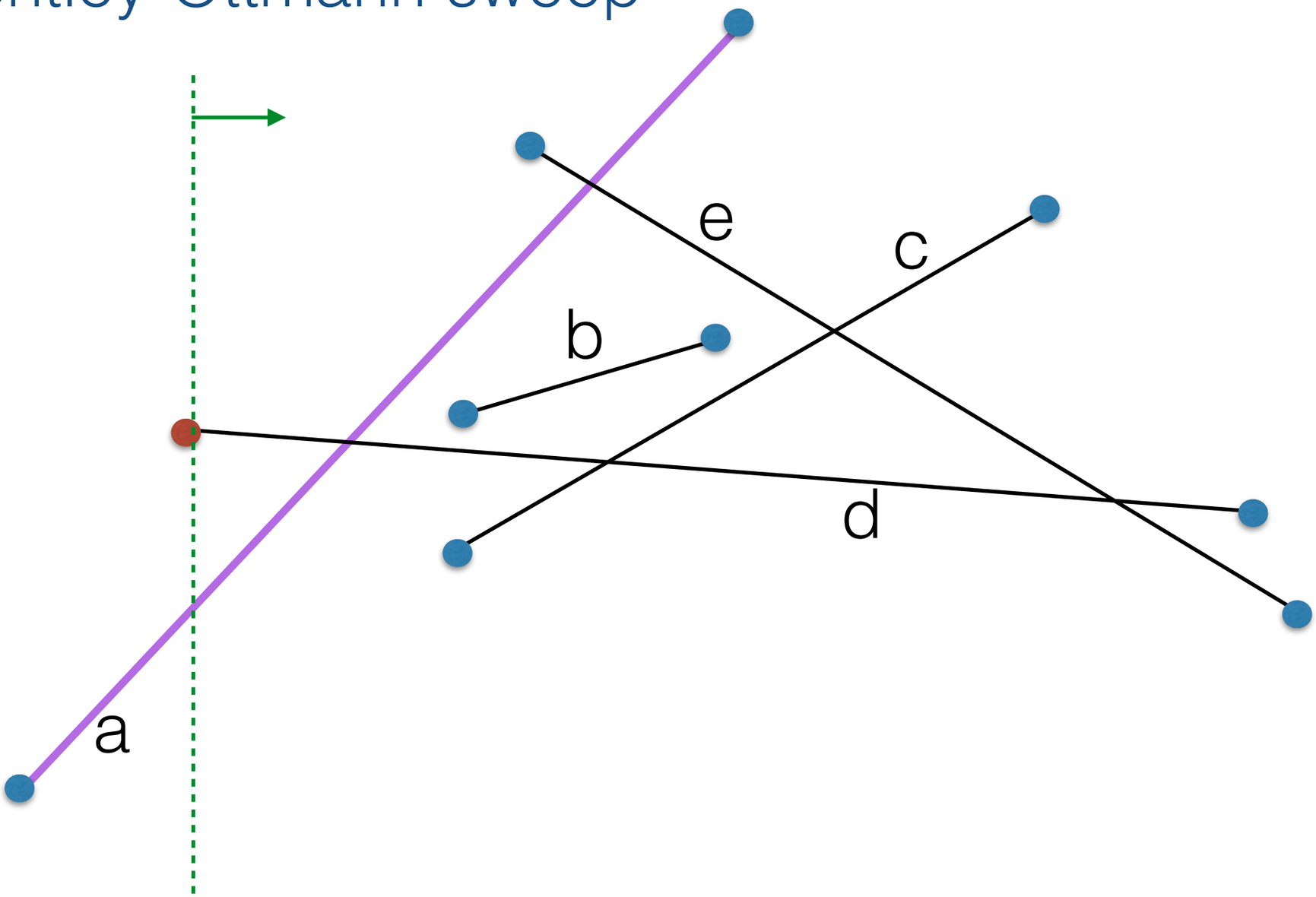

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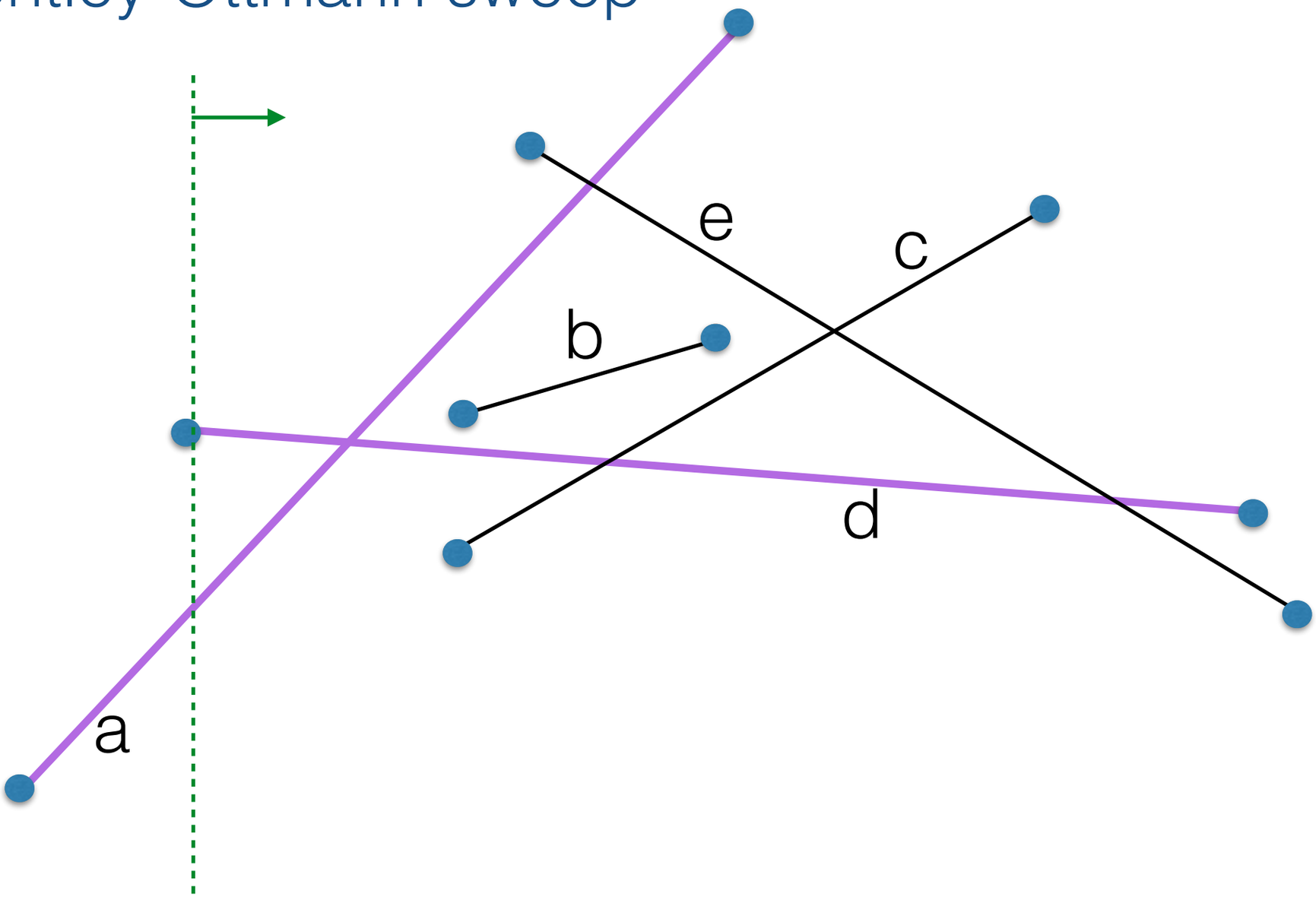

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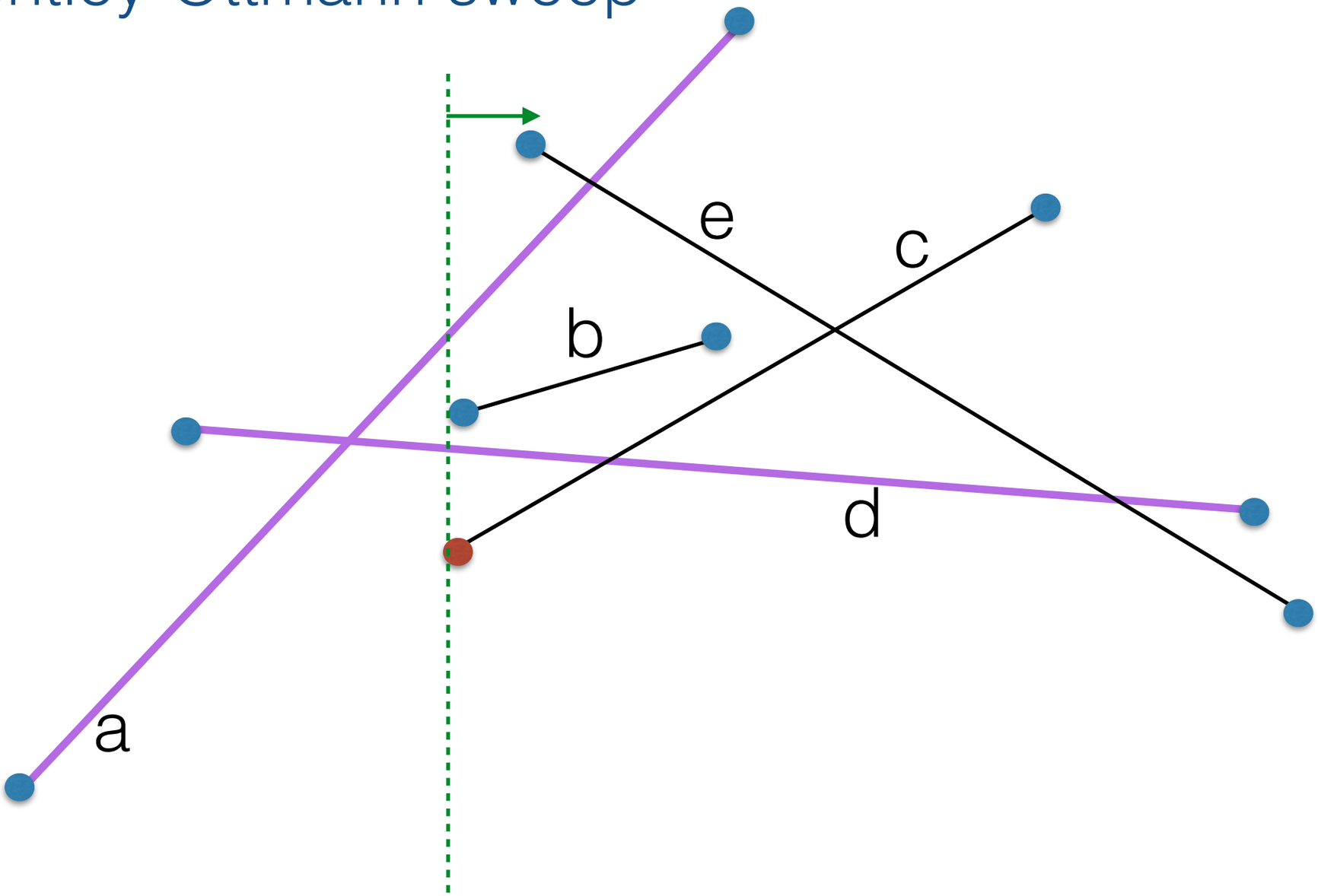
Bentley-Ottmann sweep



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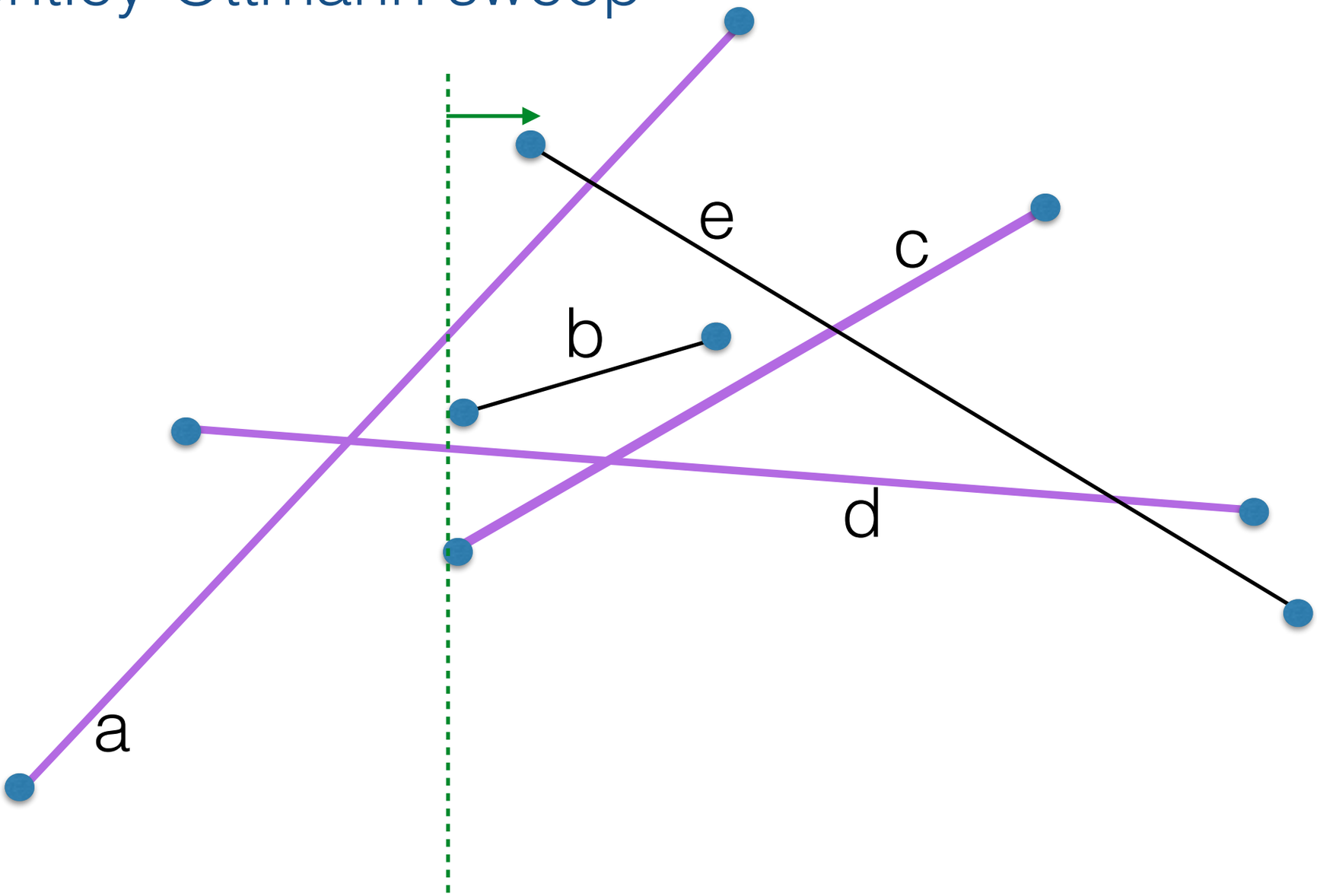

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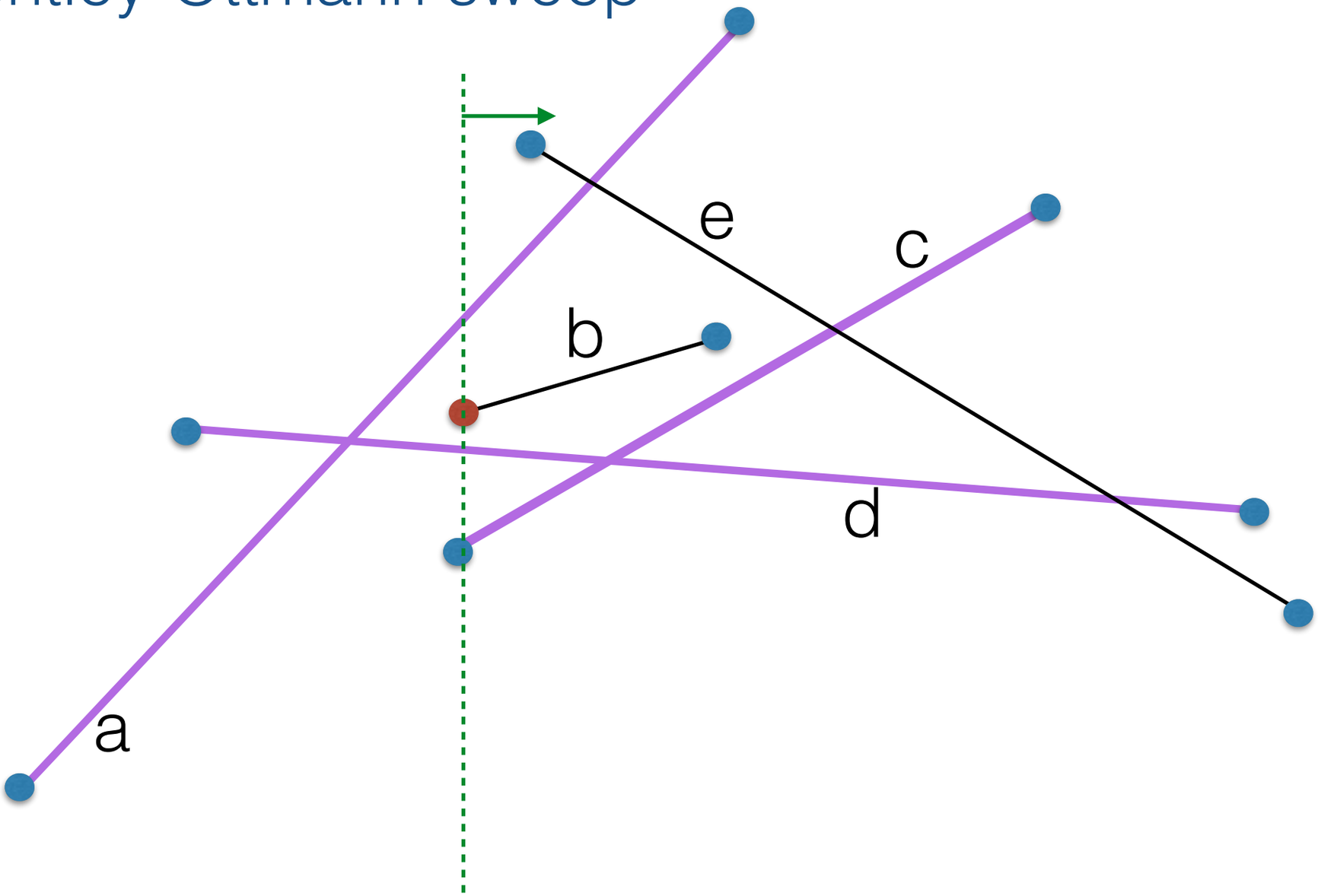

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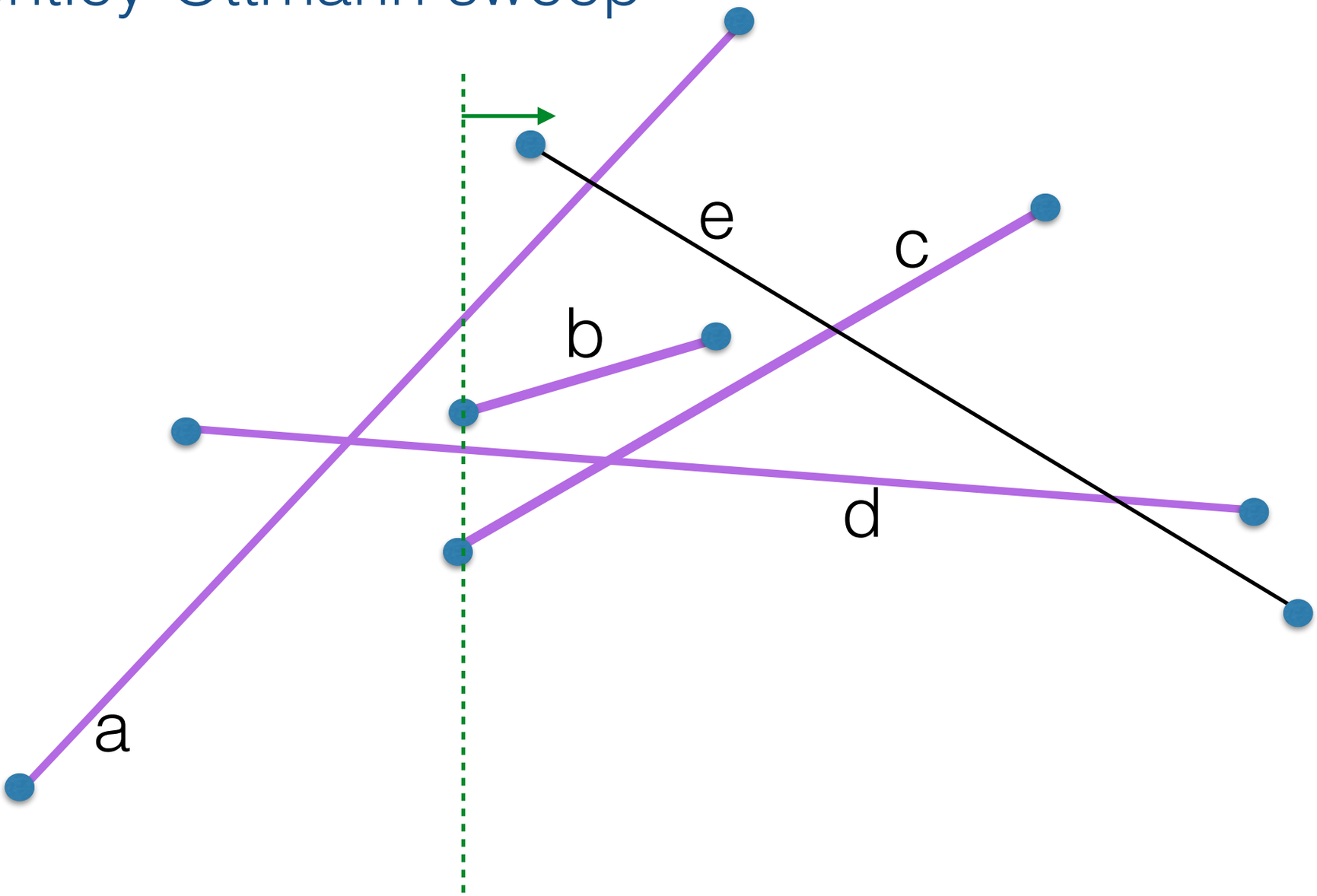
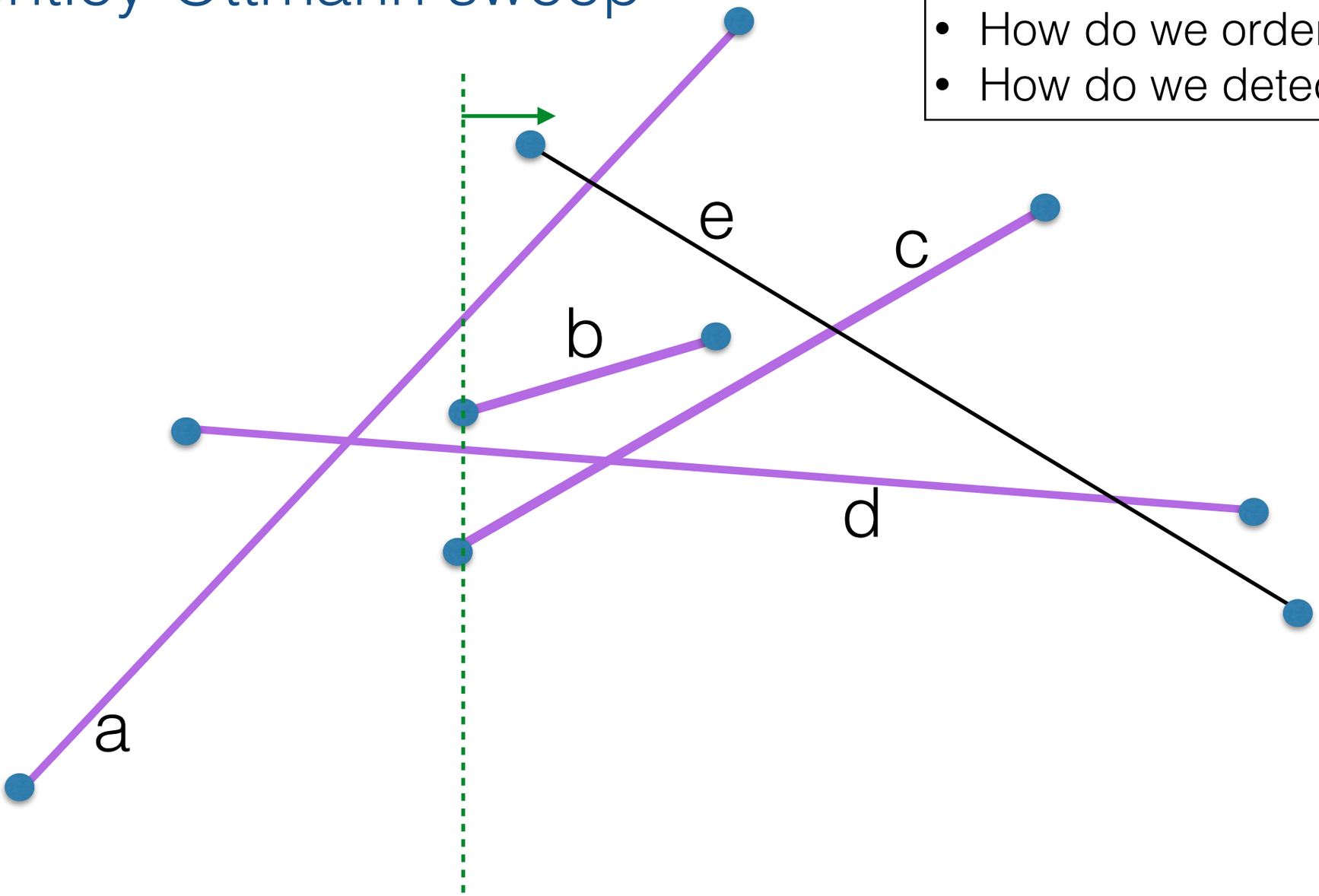
Bentley-Ottmann sweep



- Let X be the set of all x-coords of segments
- Traverse the events in X in order

Bentley-Ottmann sweep

- 4 segments are active
- How do we order these segments?
- How do we detect intersections?



- Let X be the set of all x-coords of segments
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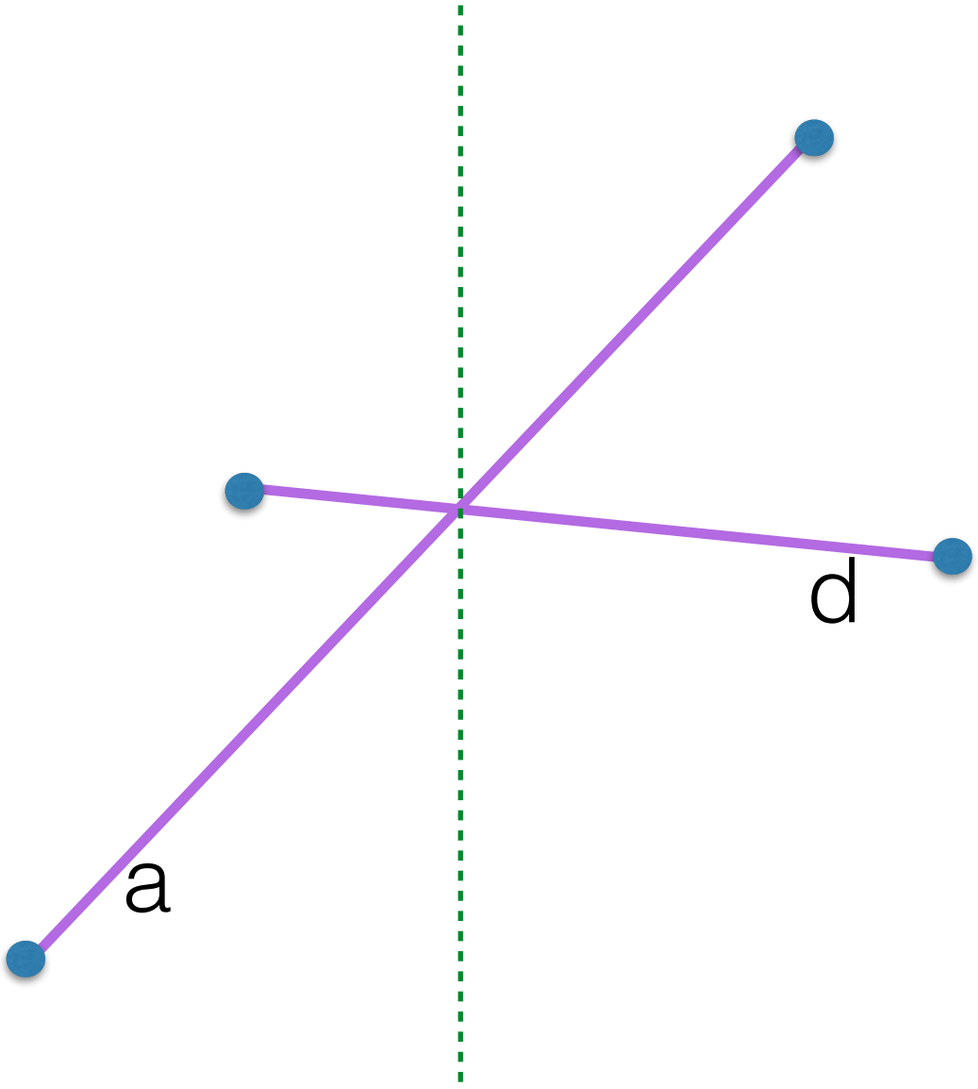
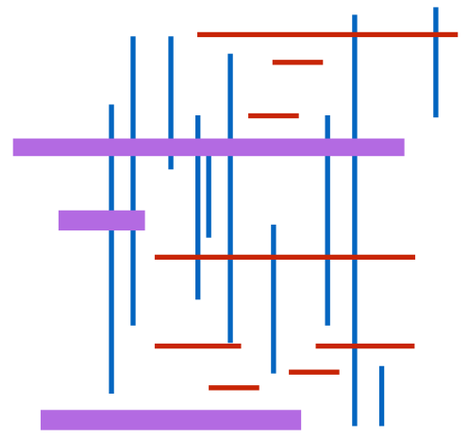
How do we order these segments?



a below d

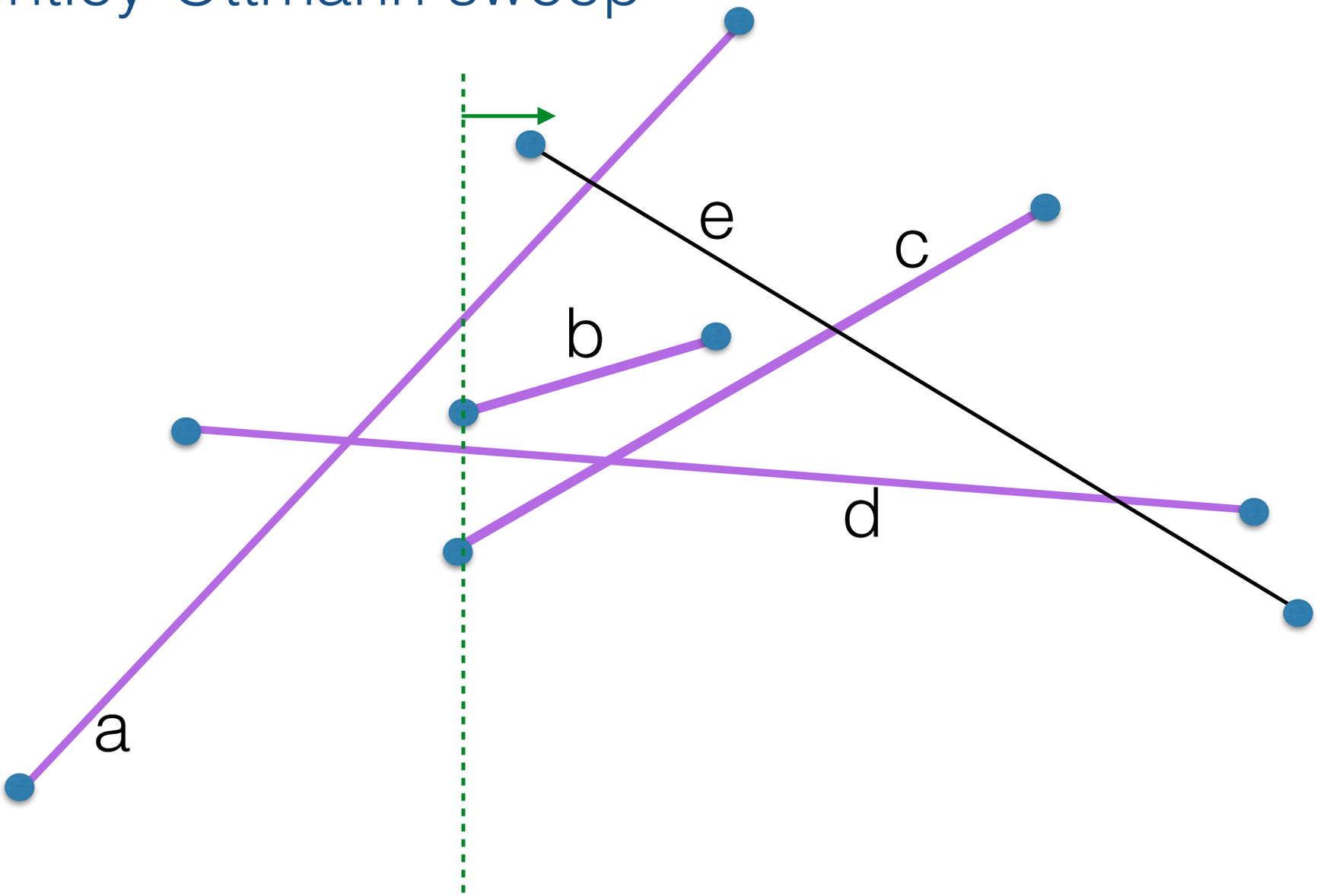
a above d

general segments

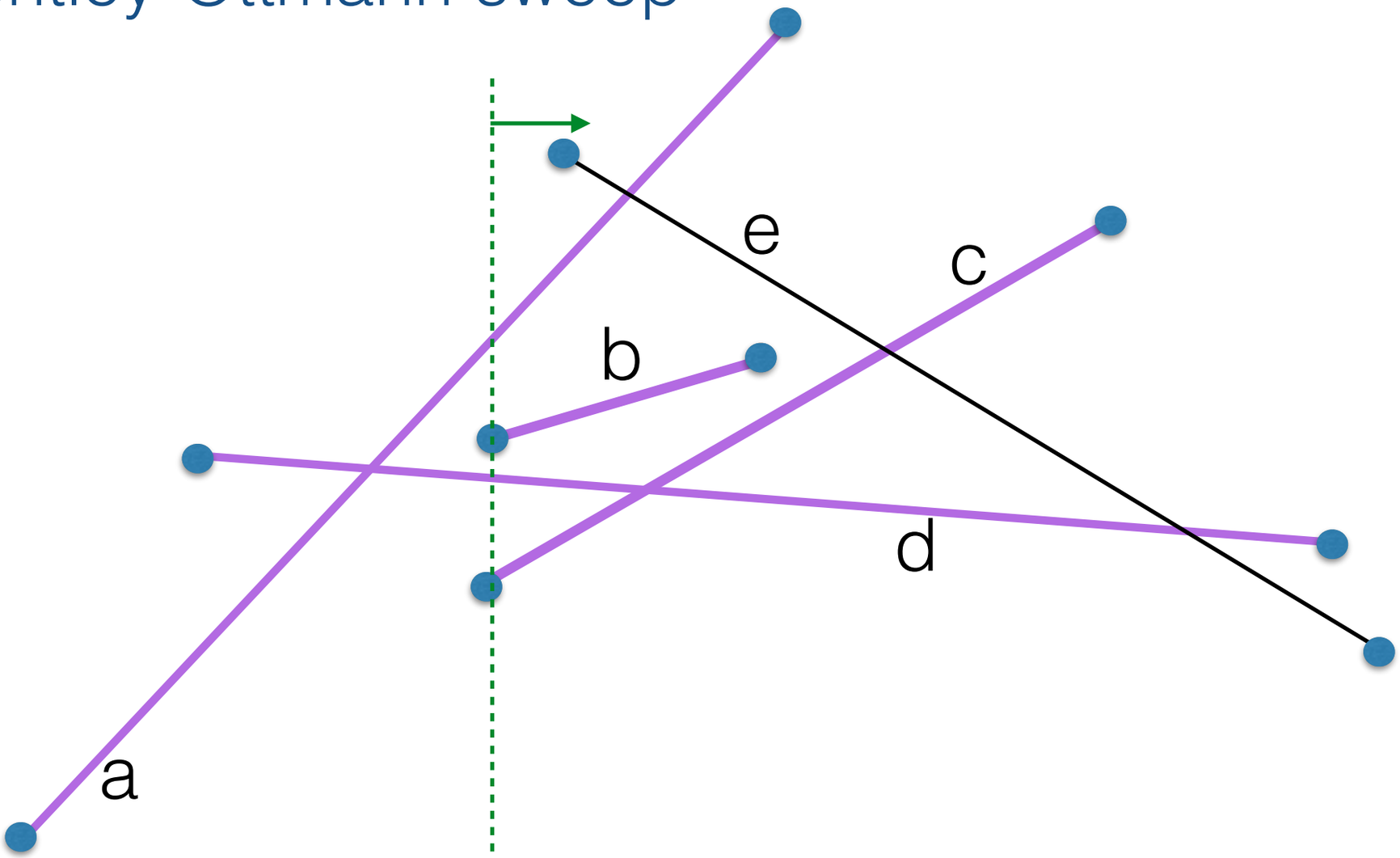


orthogonal segments

Bentley-Ottmann sweep

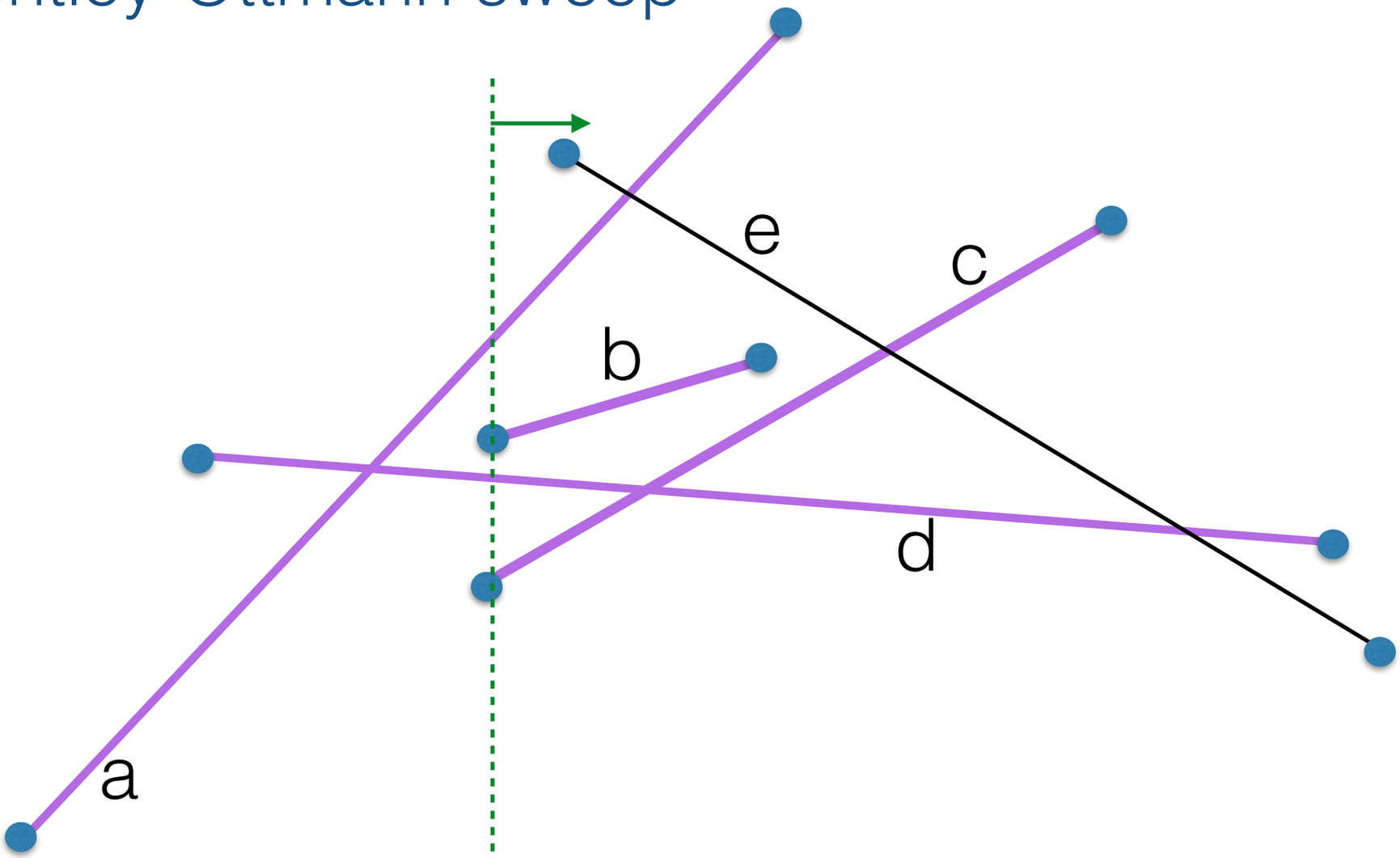


Bentley-Ottmann sweep



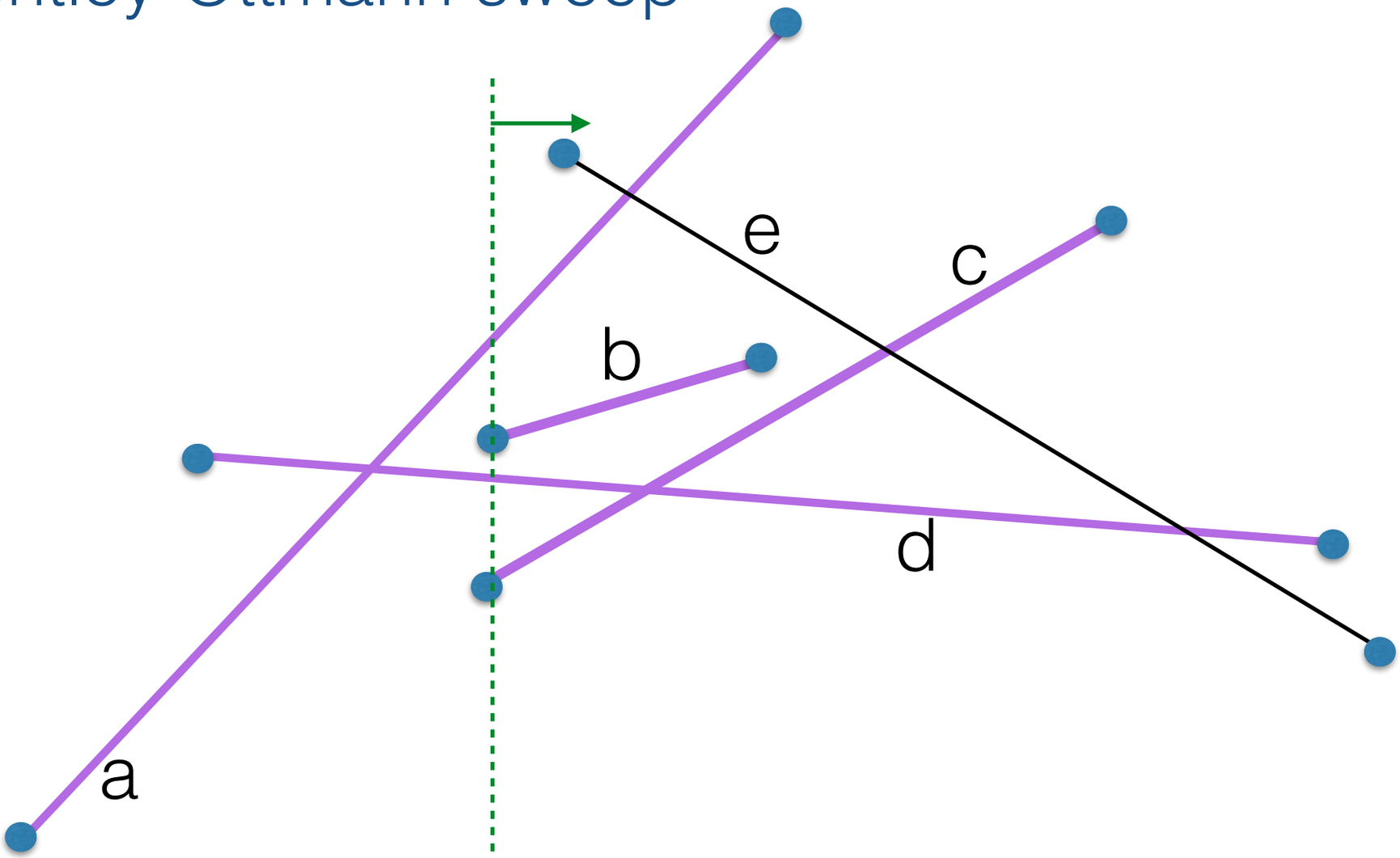
- How do we order the active segments?

Bentley-Ottmann sweep



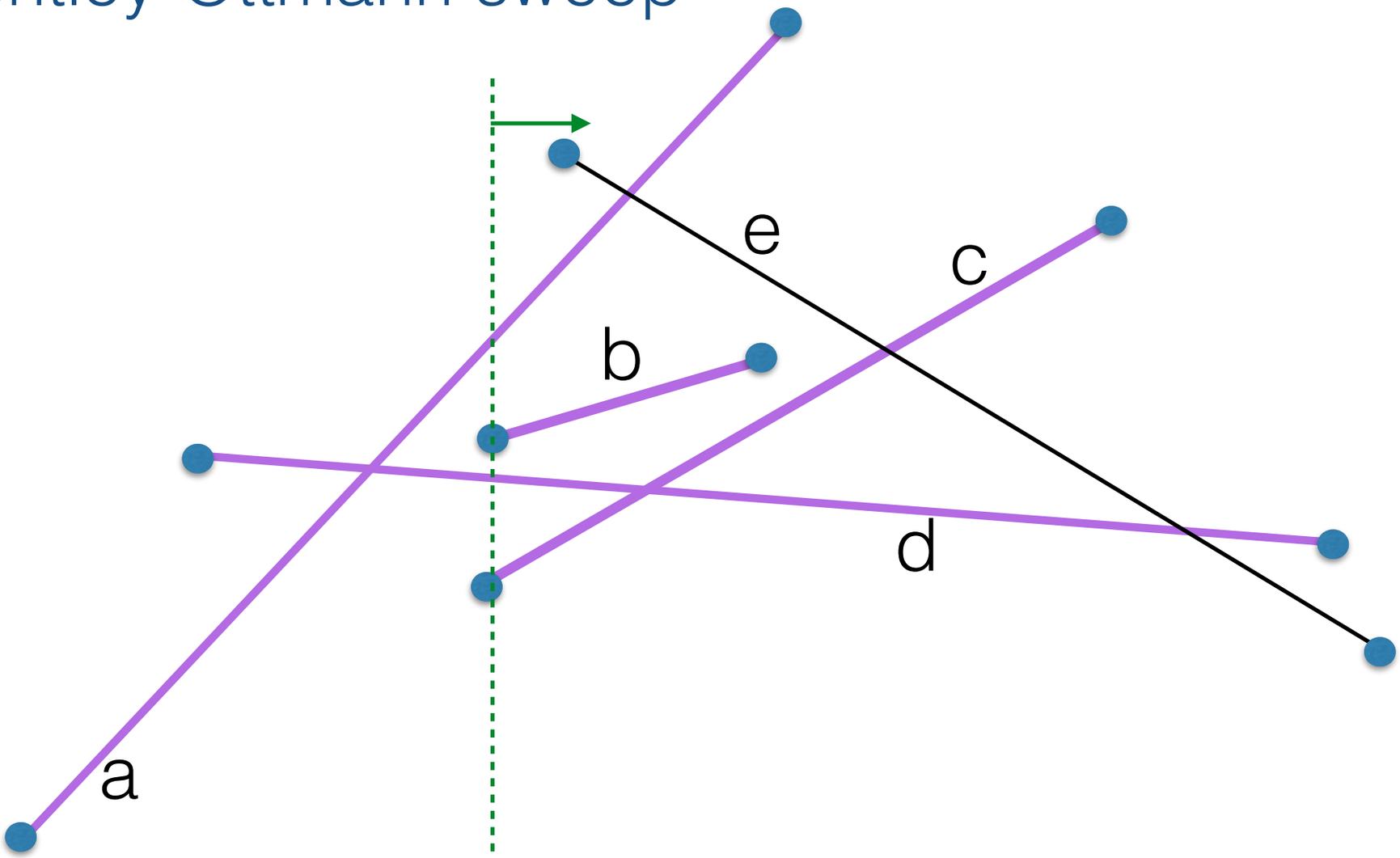
- How do we order the active segments?
  - Use above-below order

Bentley-Ottmann sweep



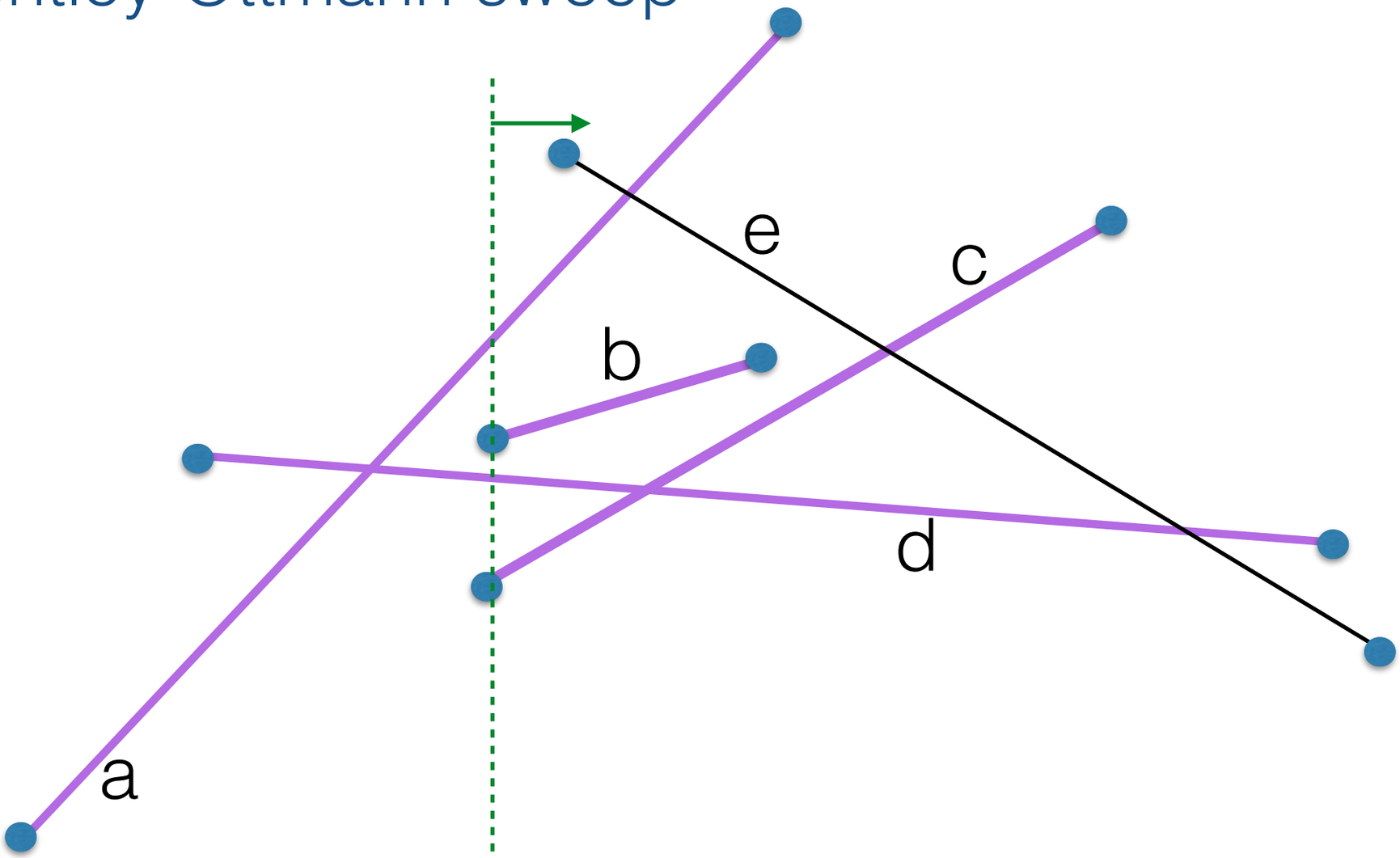
- How do we order the active segments?
  - Use above-below order
  - Order will flip at intersection point

Bentley-Ottmann sweep



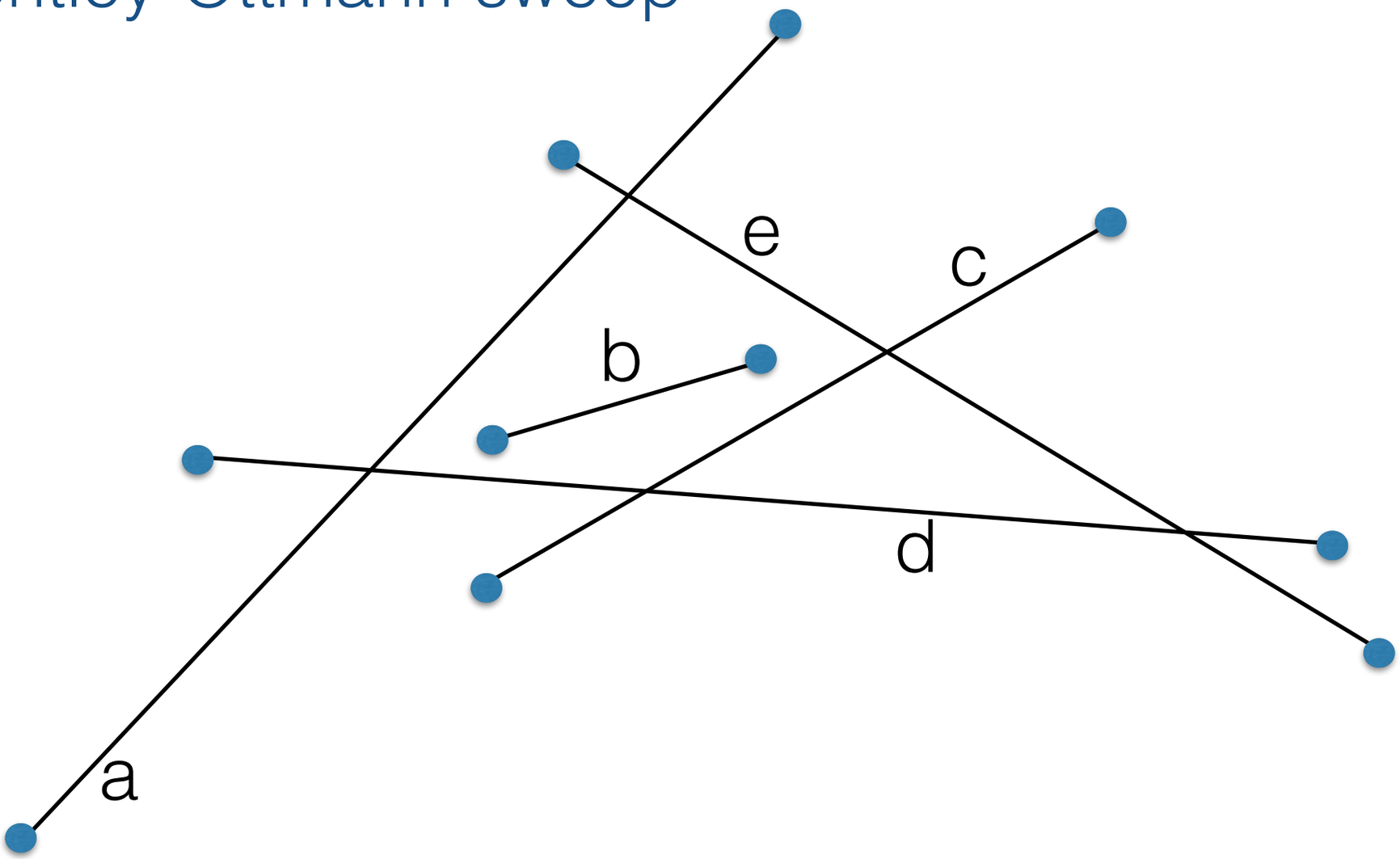
- How do we order the active segments?
  - Use above-below order
  - Order will flip at intersection point
- How do we detect intersections?

Bentley-Ottmann sweep



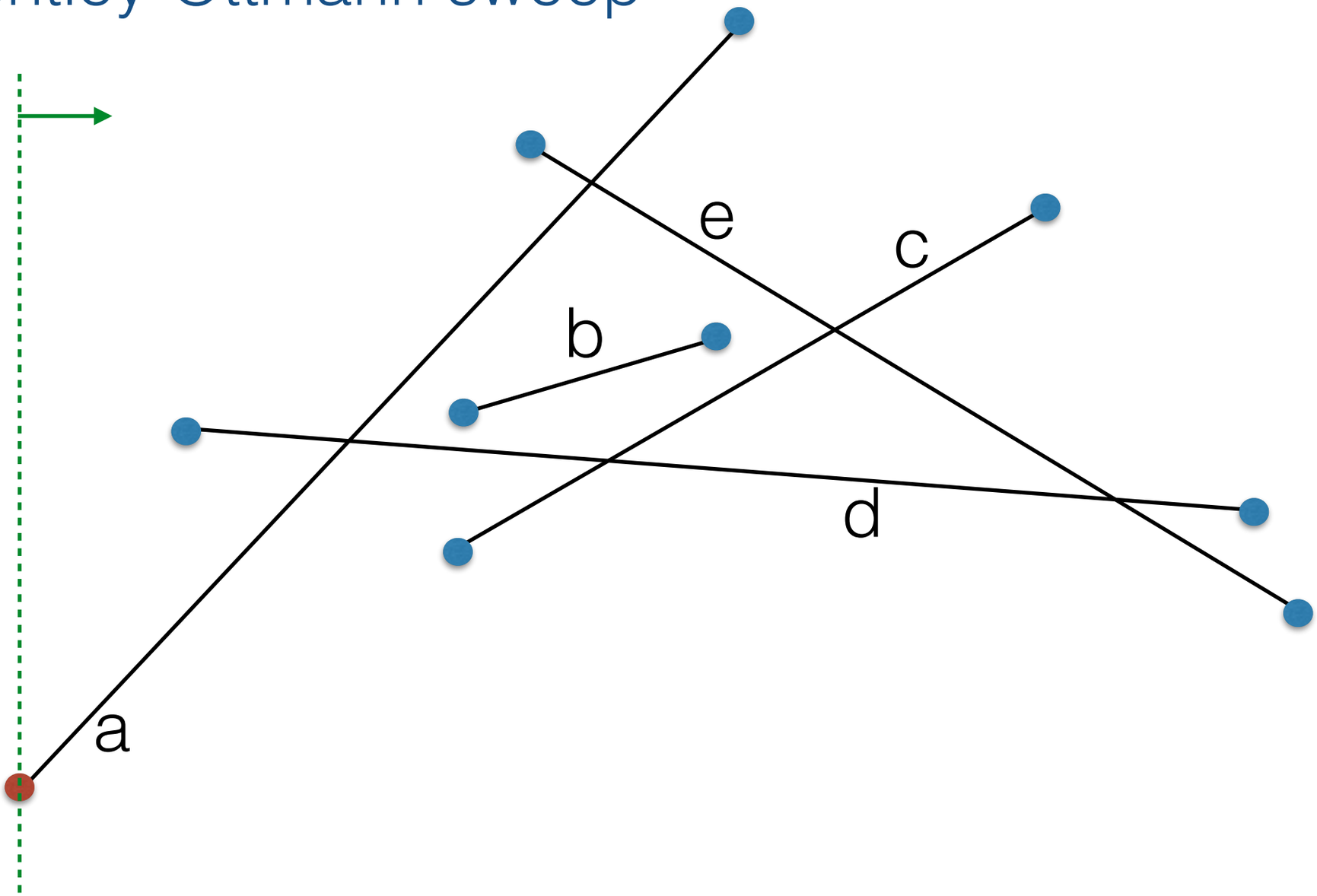
- How do we order the active segments?
  - Use above-below order
  - Order will flip at intersection point
- How do we detect intersections?
  - Segments that intersect are consecutive in above-below order just before they intersect

Bentley-Ottmann sweep

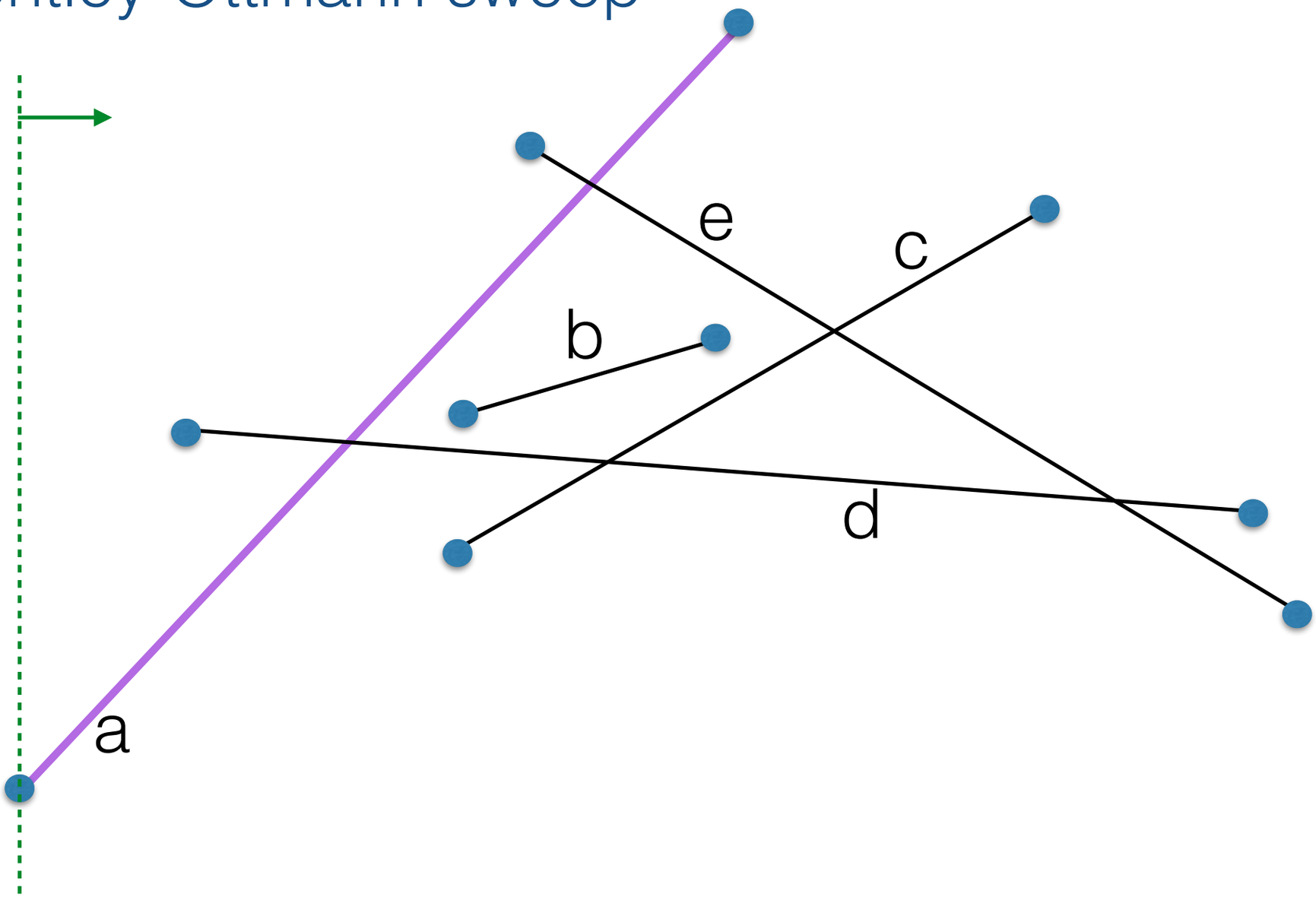


- Let X be the set of all x-coords of segments
- Initialize AS = {}
- Traverse events in order

Bentley-Ottmann sweep

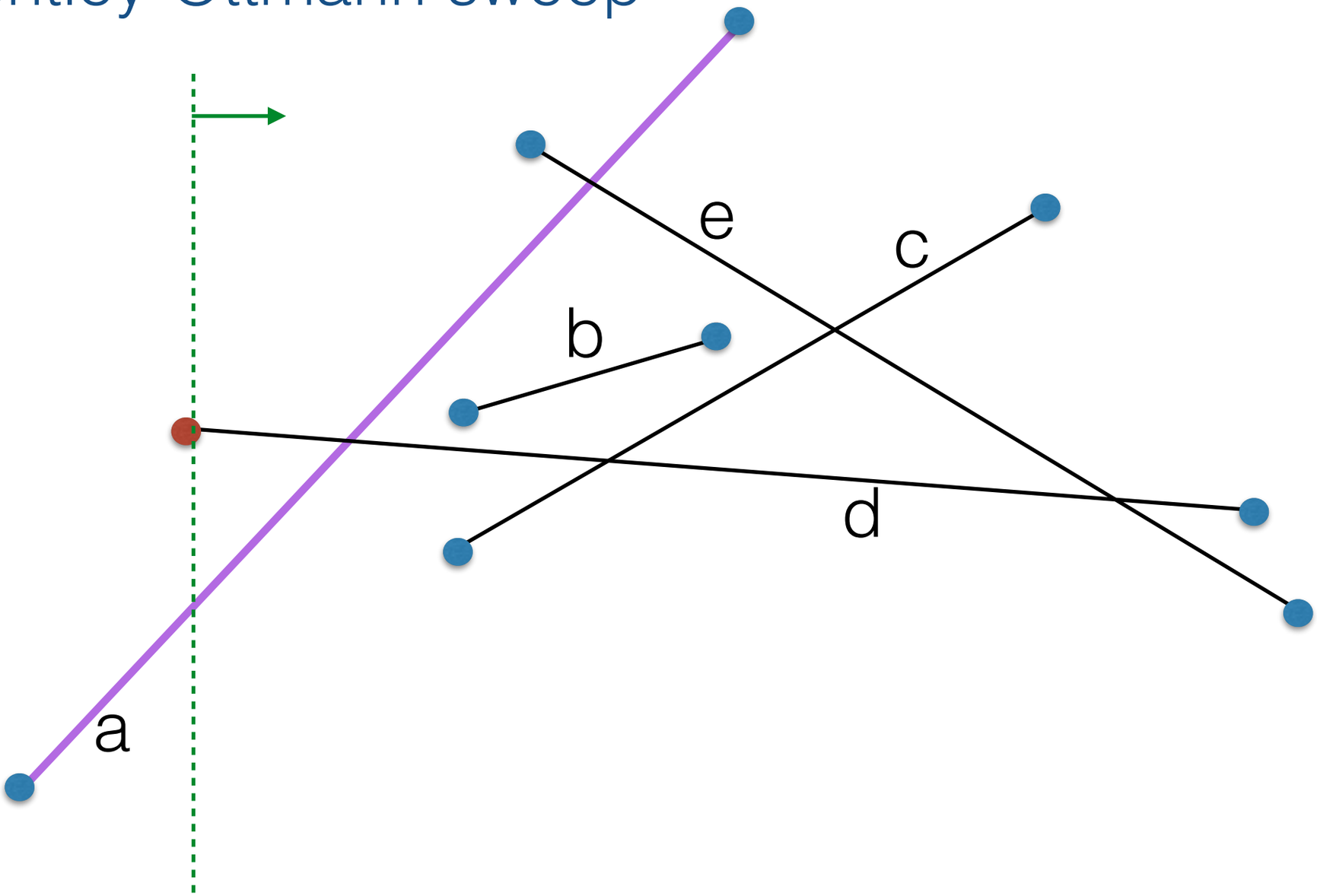


Bentley-Ottmann sweep

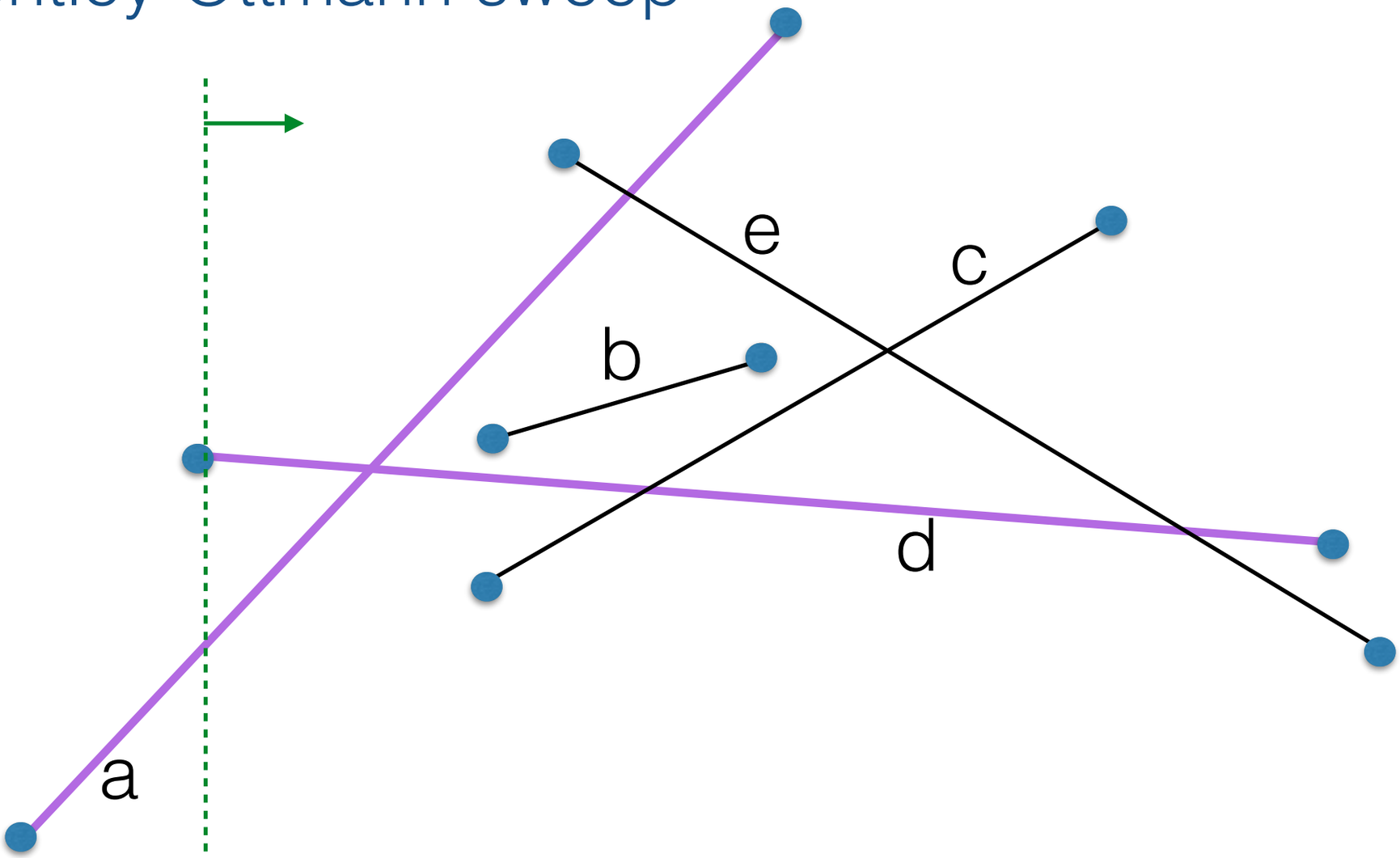


- this event is start of segment a:
  - insert a in AS: a

Bentley-Ottmann sweep

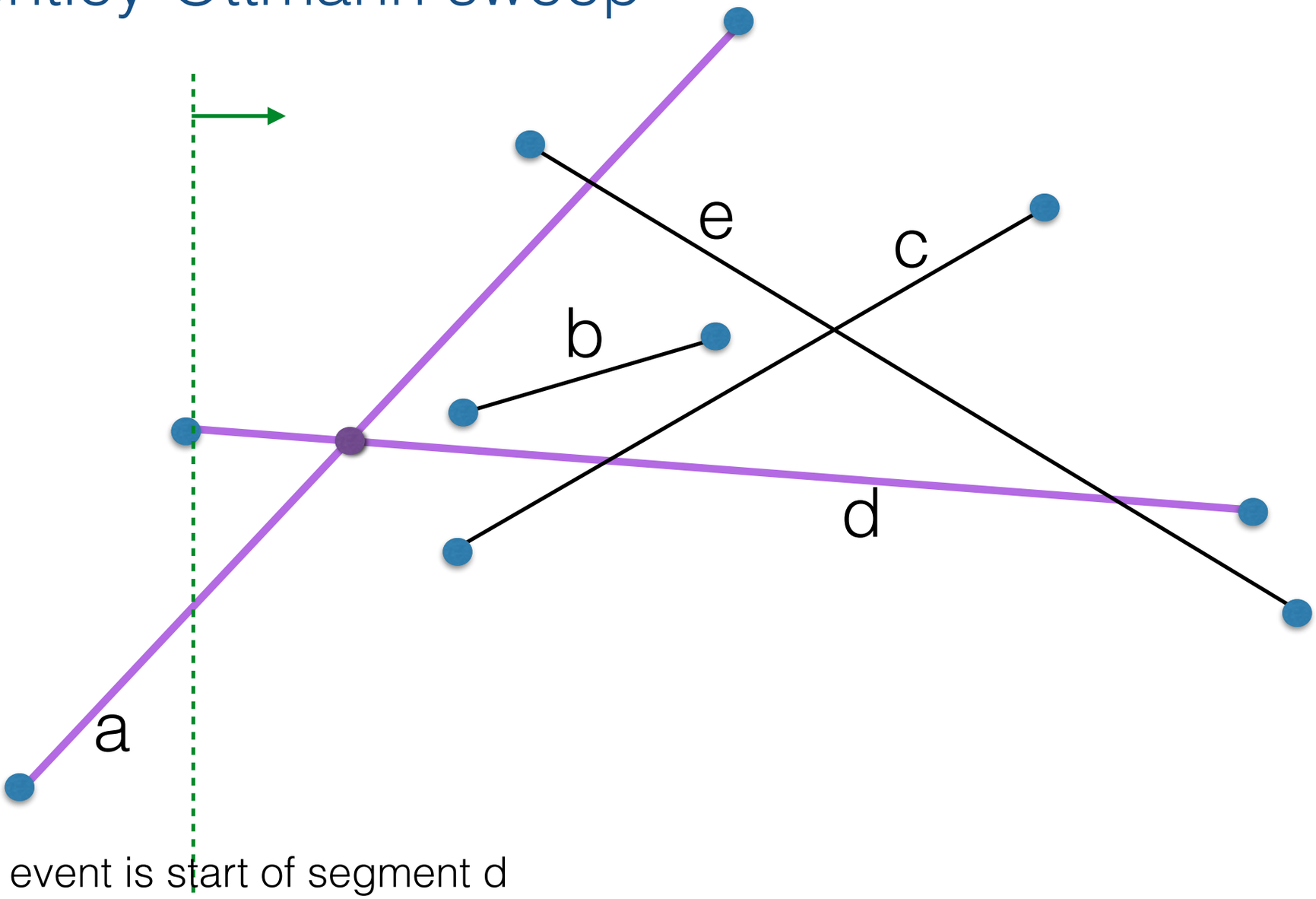


Bentley-Ottmann sweep



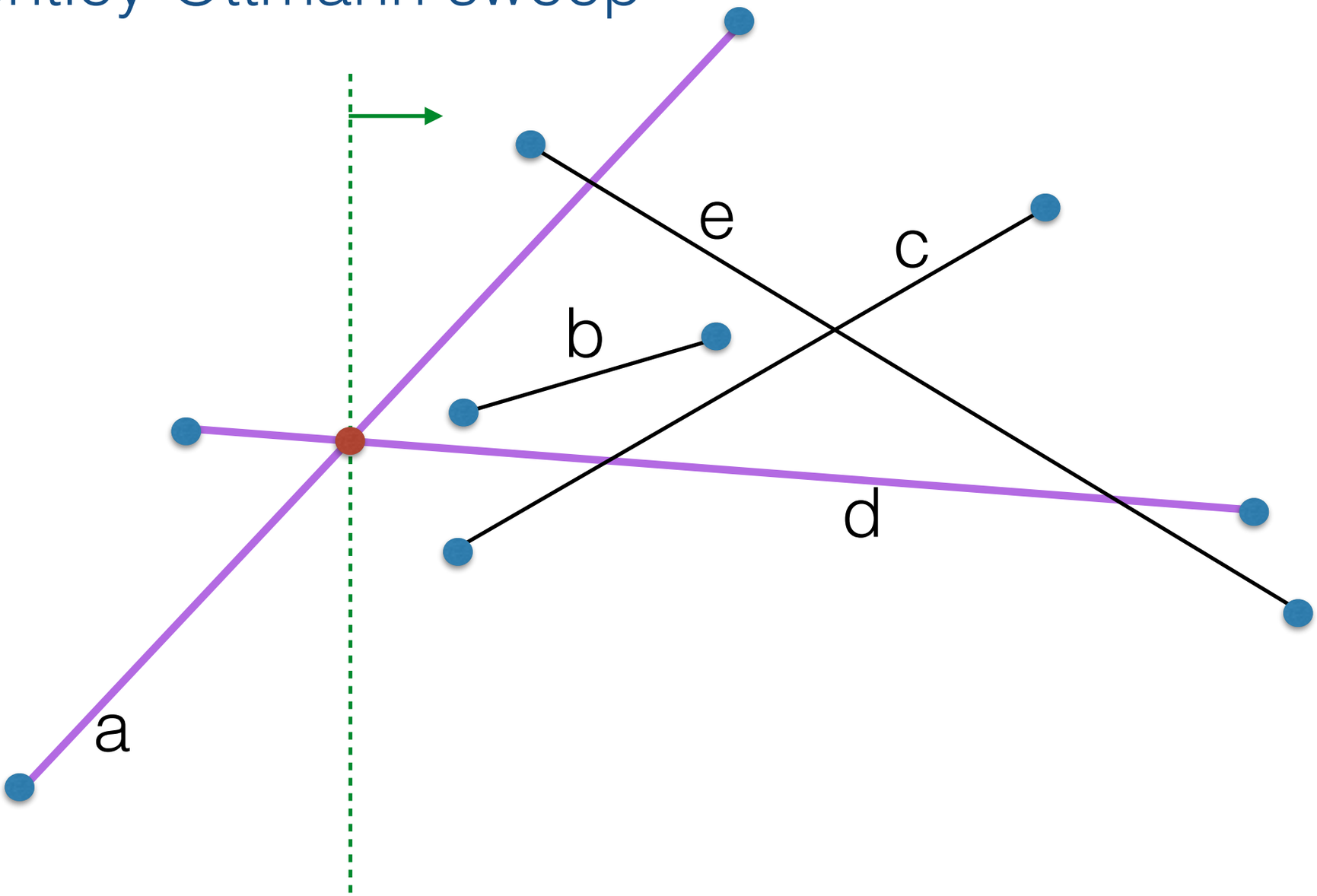
- this event is start of segment d
  - insert d in AS: a < d
  - check if (d,a) intersect to the right of the line; they do; report point and insert it in the list of future events

Bentley-Ottmann sweep

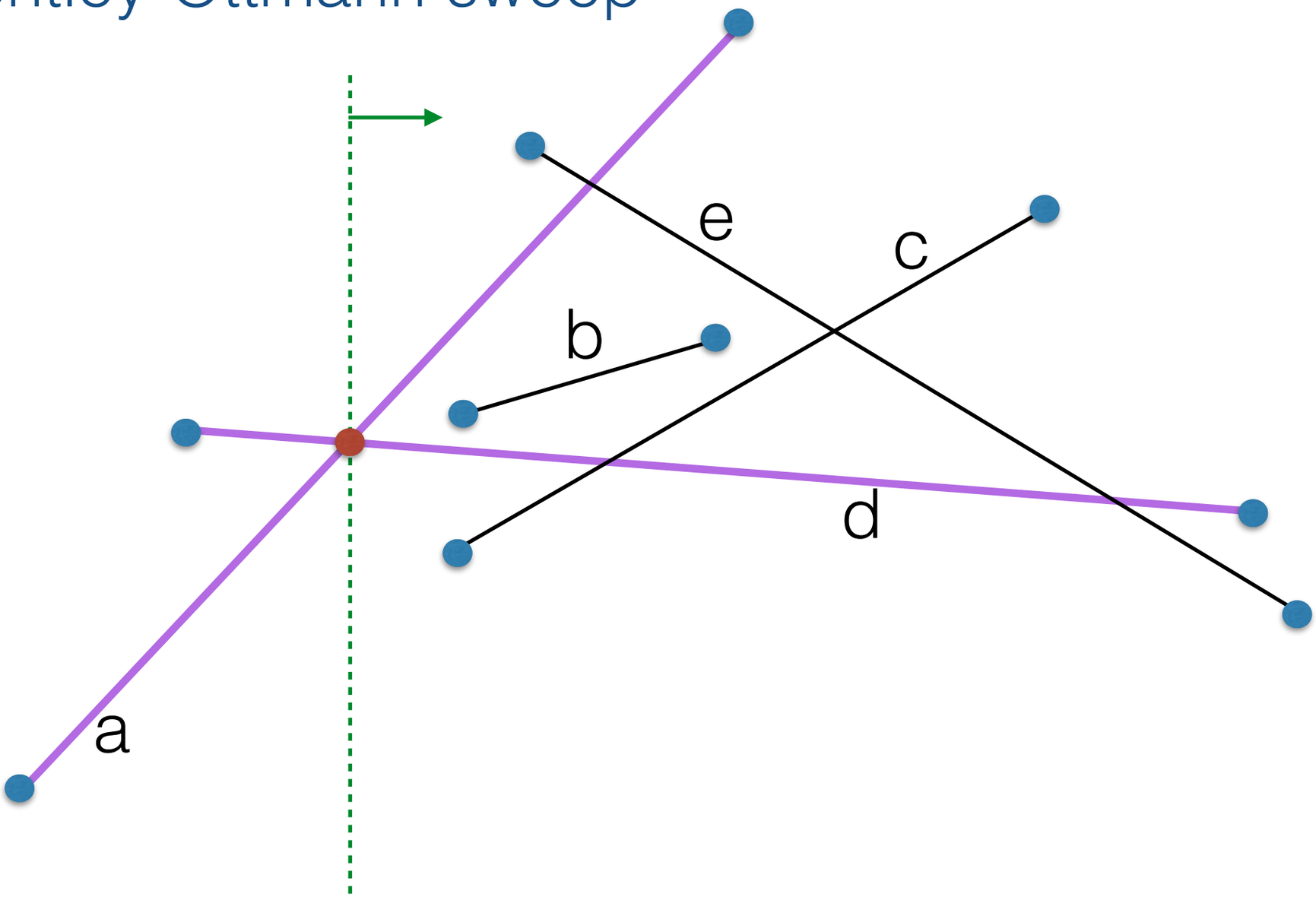


- this event is start of segment d
  - insert d in AS, d is above a (a < d)
  - check if (d,a) intersect to the right of the line; they do; report point and insert it in the list of future events

Bentley-Ottmann sweep

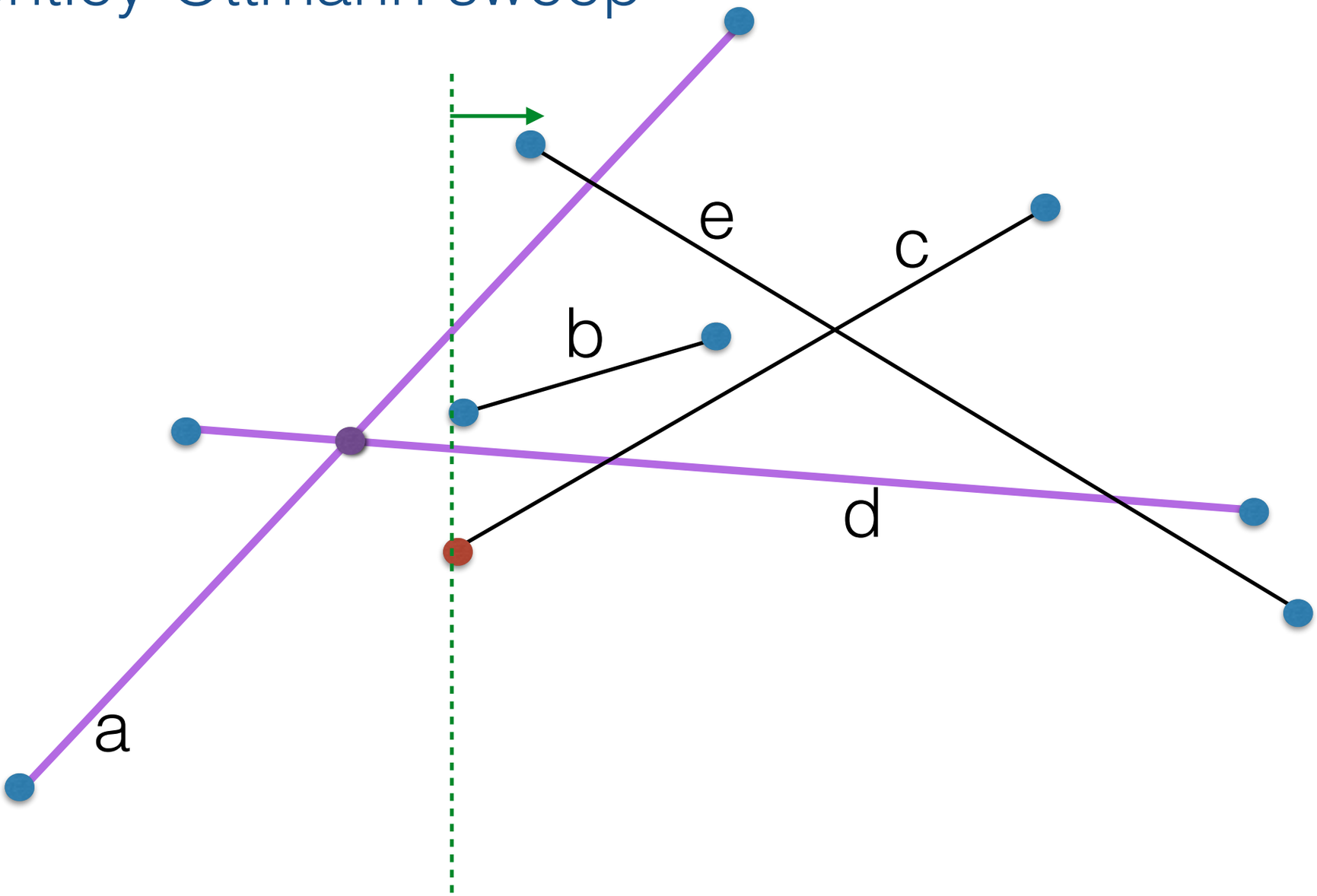


Bentley-Ottmann sweep

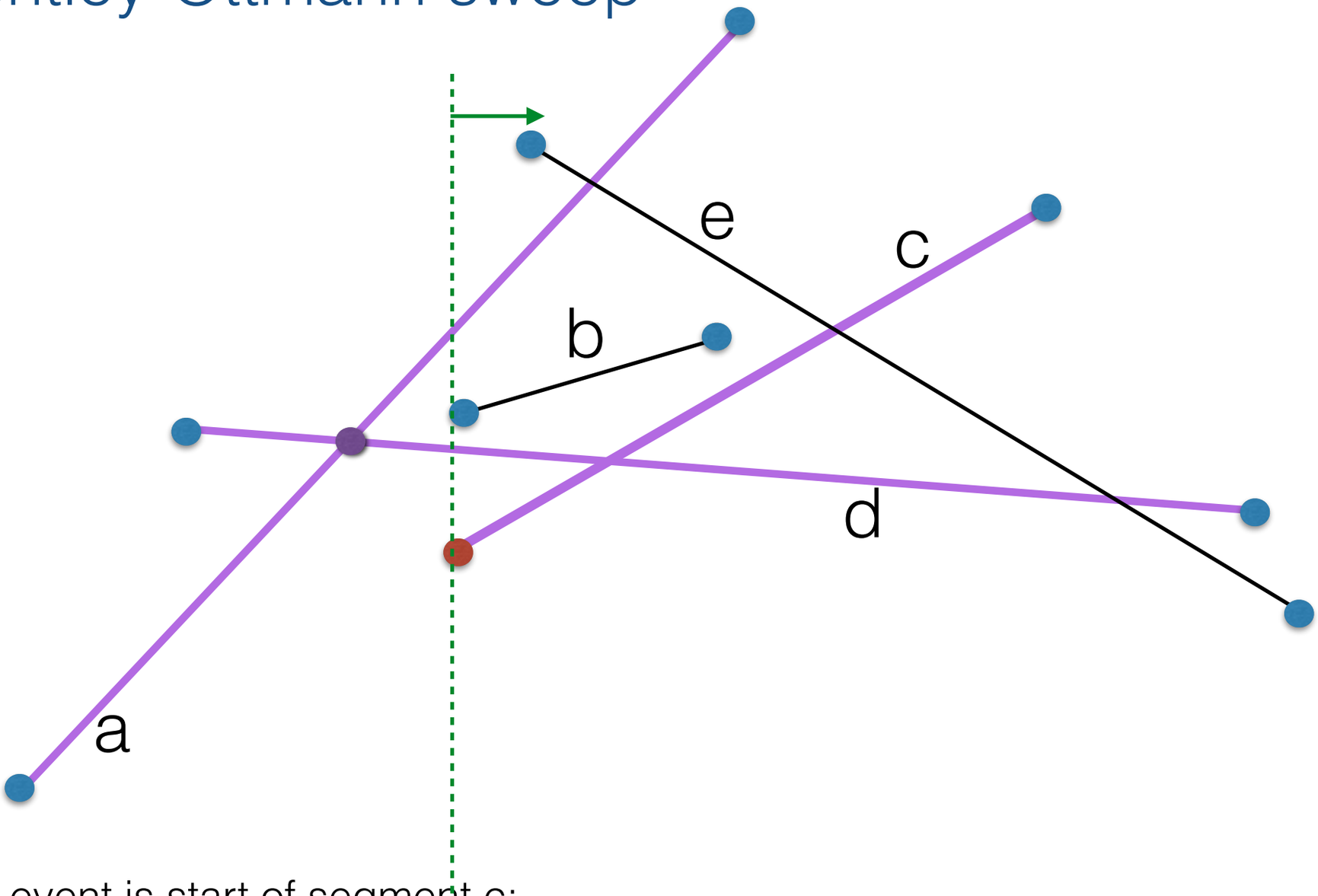


- this event is an intersection point of (a,d):
  - flip a and d is AS: a is now above d (d < a)

Bentley-Ottmann sweep

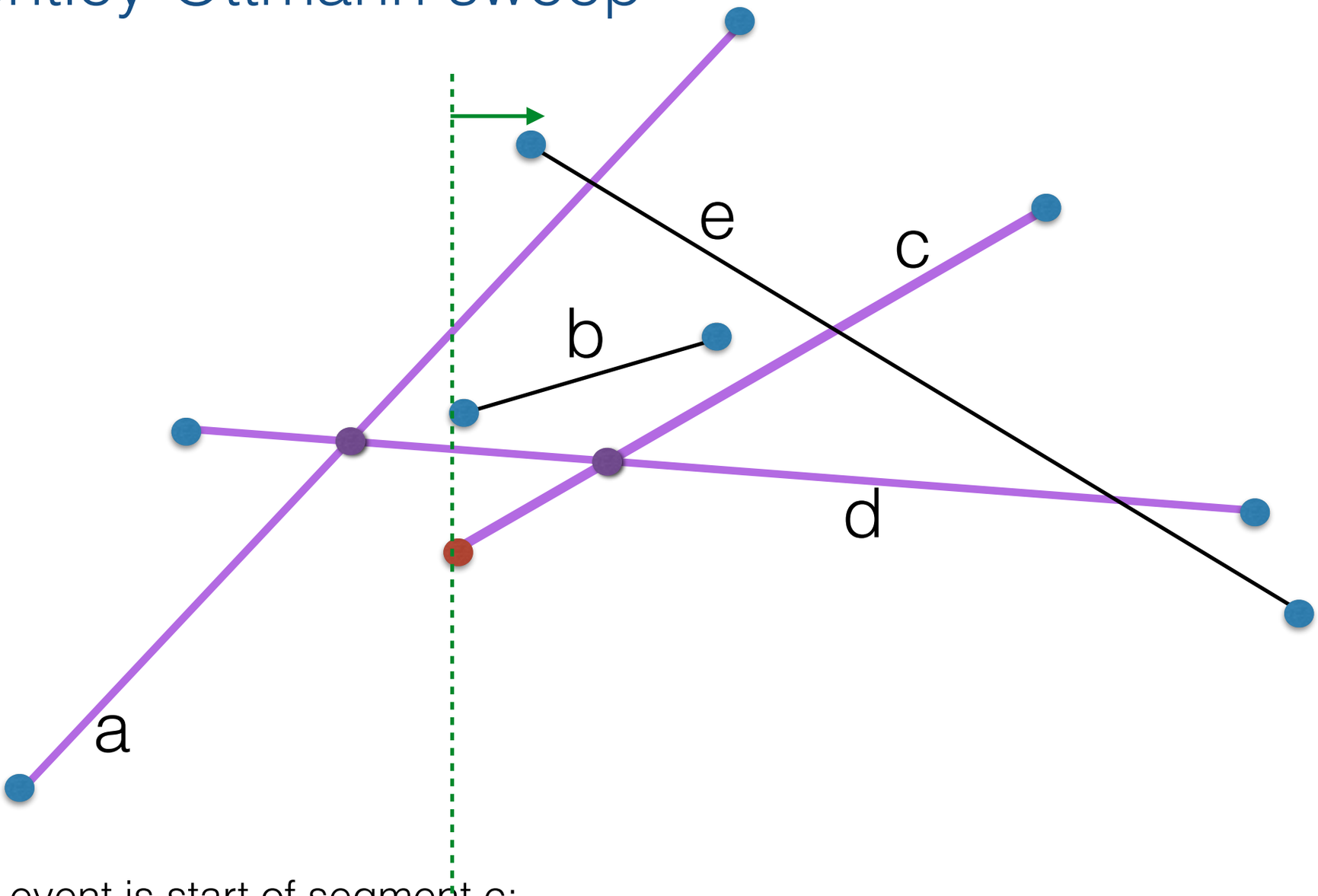


Bentley-Ottmann sweep



- this event is start of segment c:
  - insert c in AS; c is below d (c < d < a)
  - check c with its above and below neighbors for intersection to the right of the sweep line; this detects the intersection point of c and d; report it and insert it as future event

Bentley-Ottmann sweep



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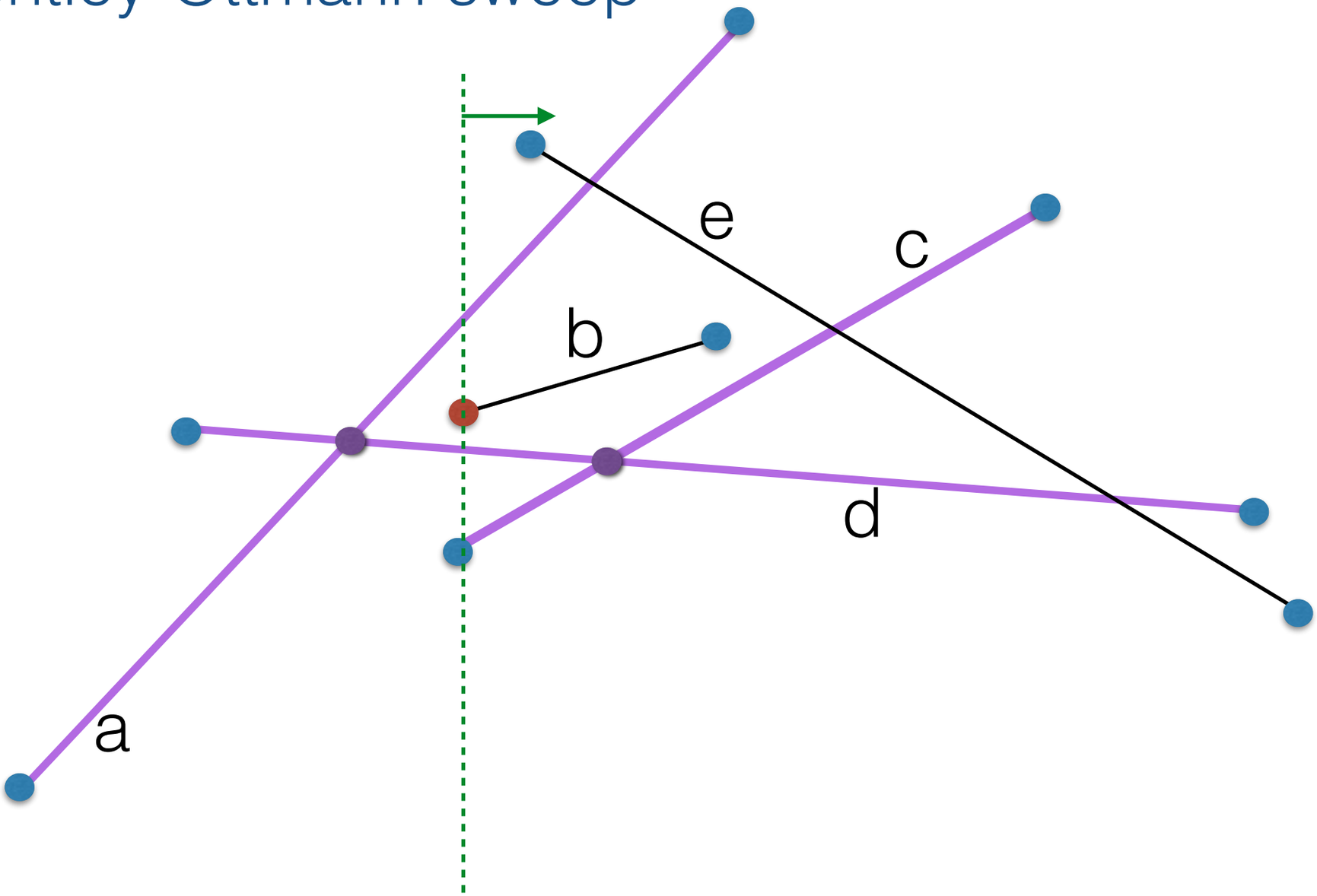
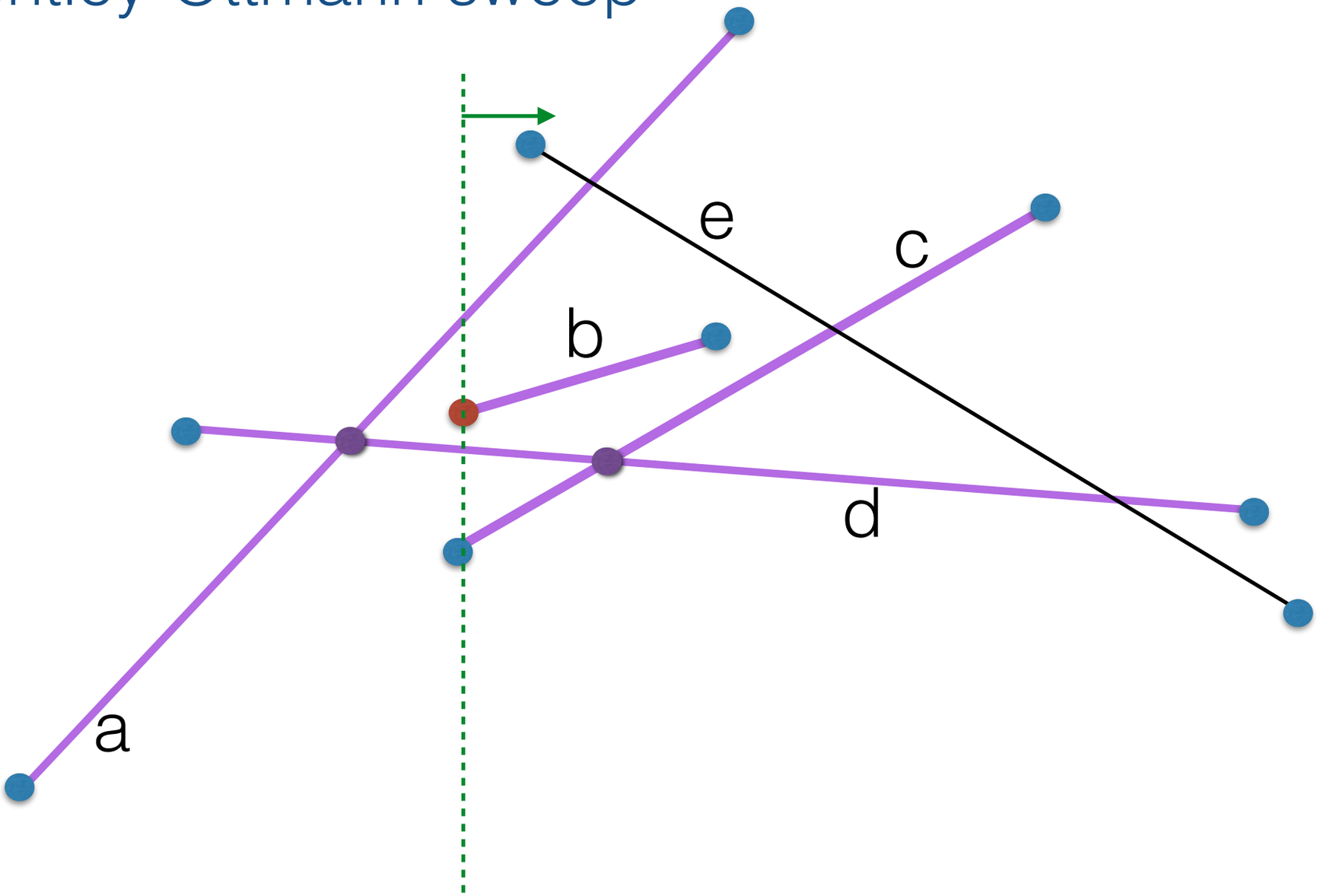


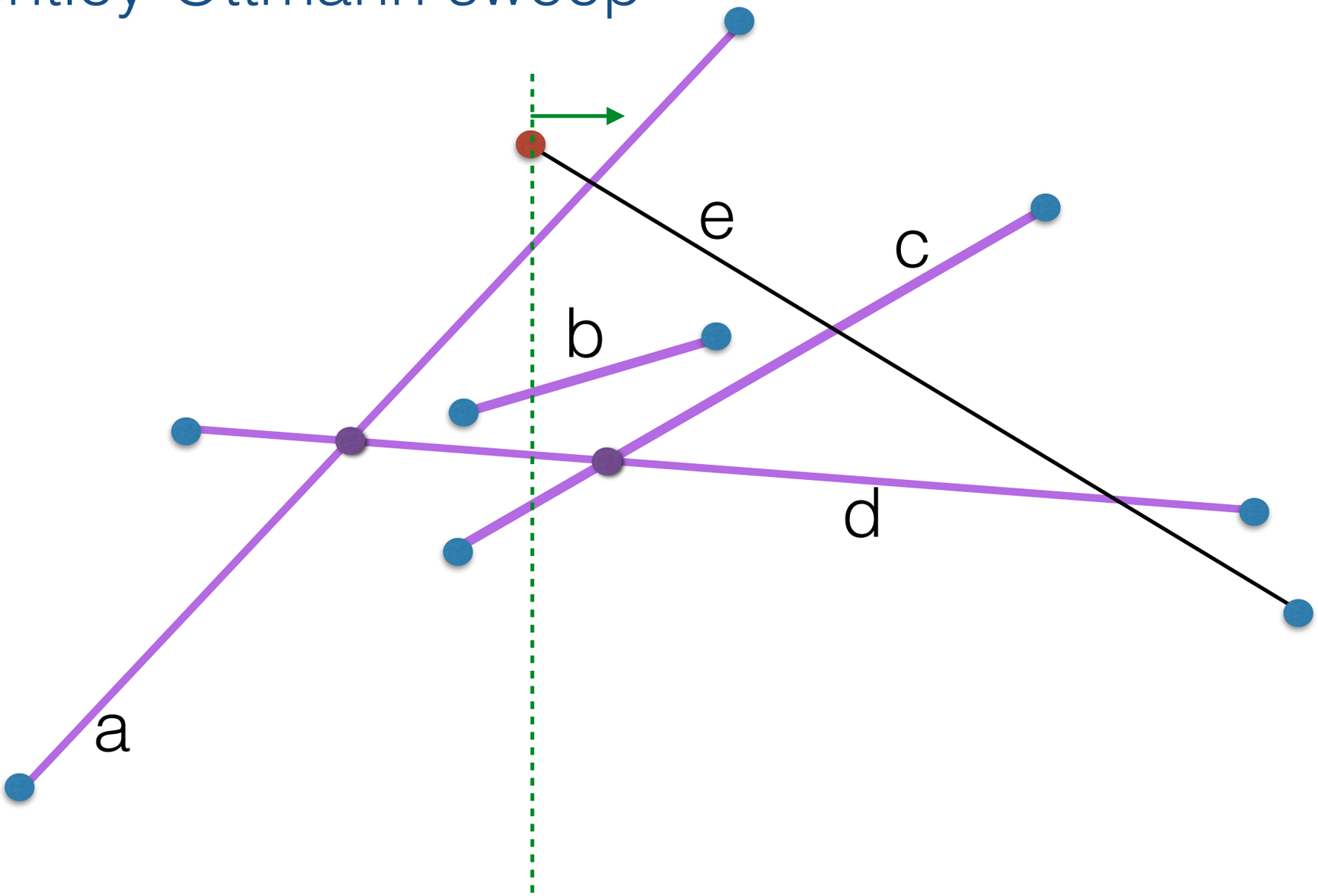
- this event is start of segment b:

Bentley-Ottmann sweep



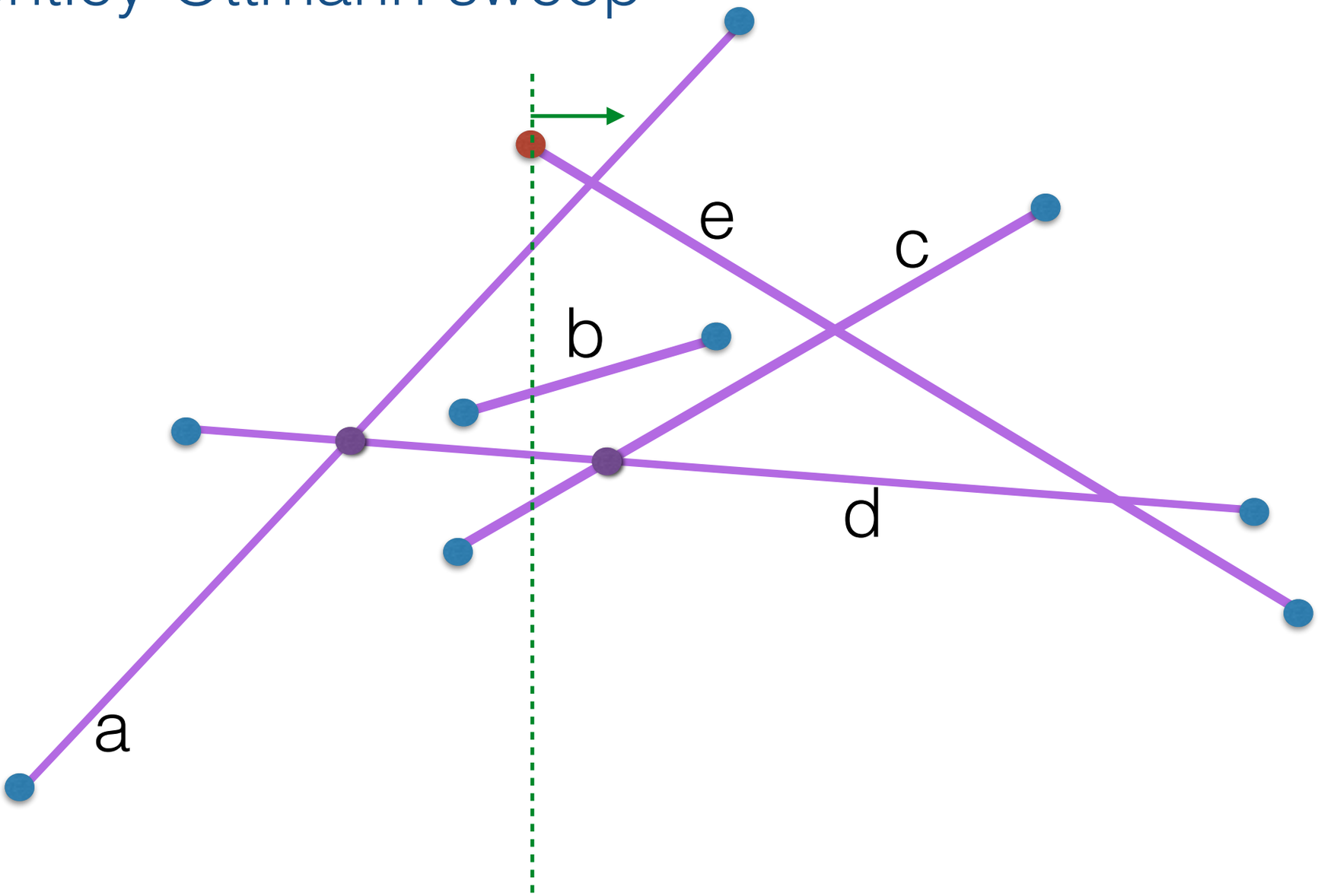
- this event is start of segment b:
  - insert b in AS; c < d < b < a
  - check b with its above and below neighbors for intersection to the right of the sweep line; (d,b) don't intersect; (b.a) don't intersect

Bentley-Ottmann sweep



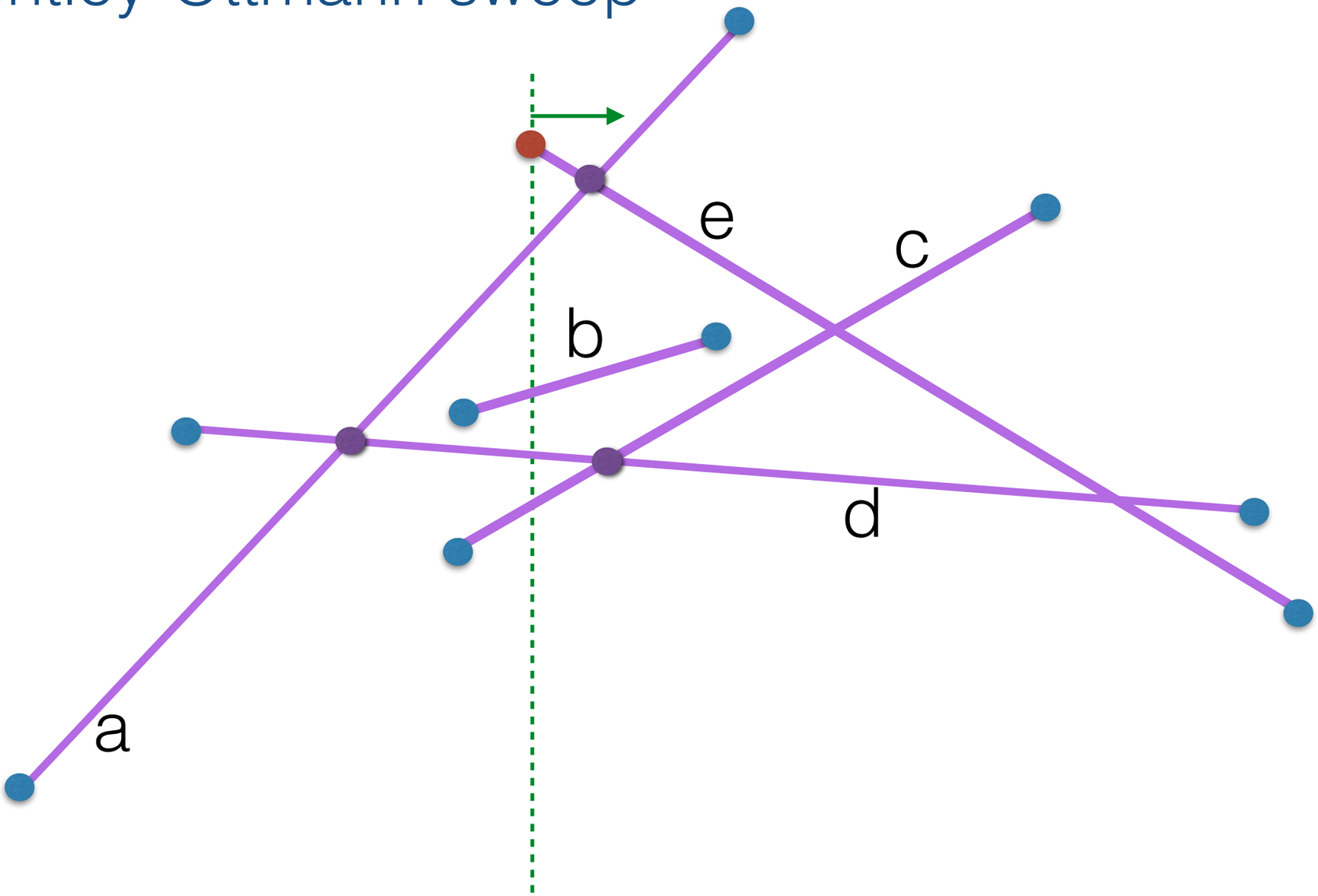
- this event is start of segment e:

Bentley-Ottmann sweep



- this event is start of segment e:
  - insert e in AS: c < d < b < a < e
  - check e with its above and below neighbors for intersection to the right of the sweep line; this detects intersection point of (a,e); report it and insert it as future event

Bentley-Ottmann sweep



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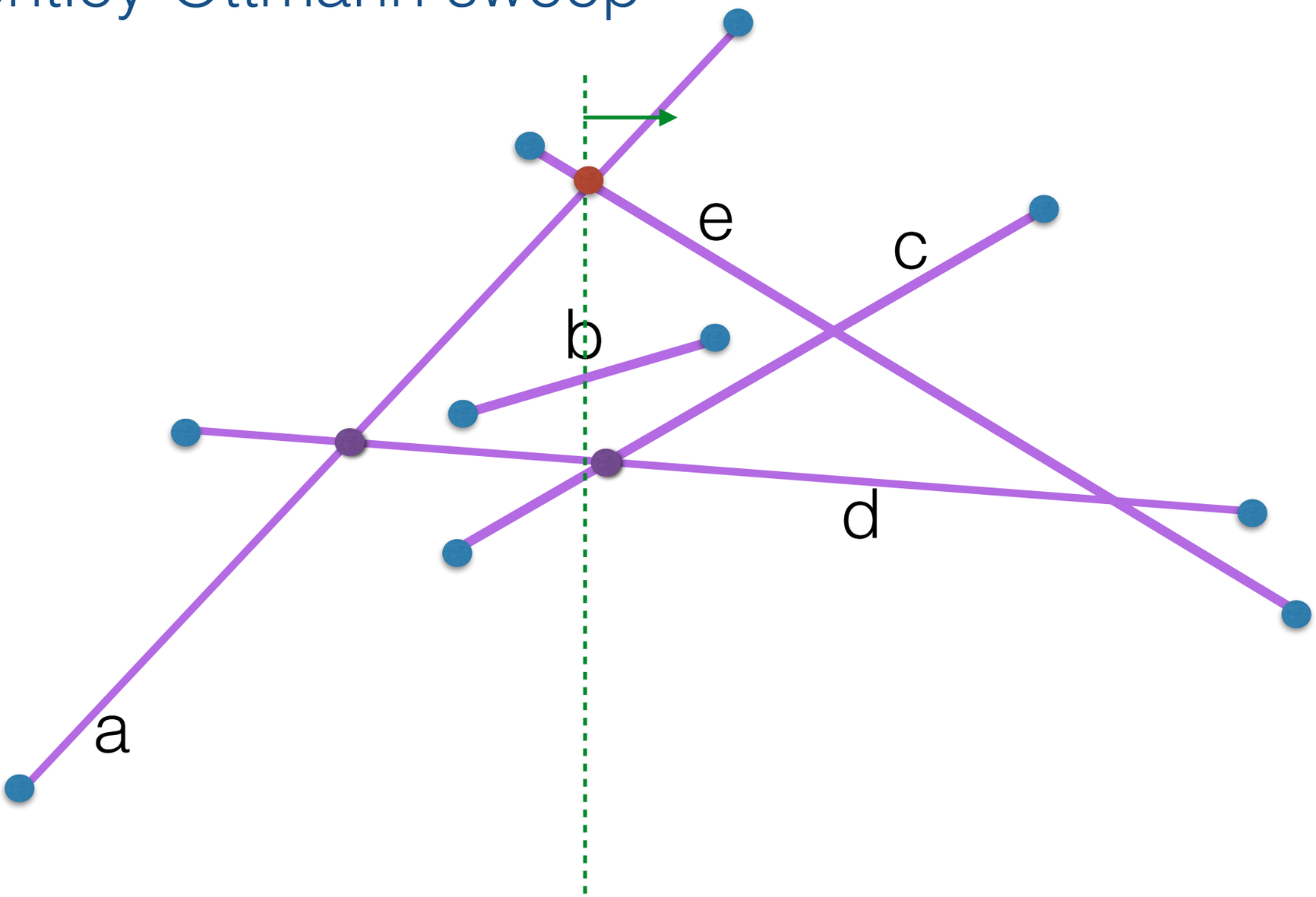
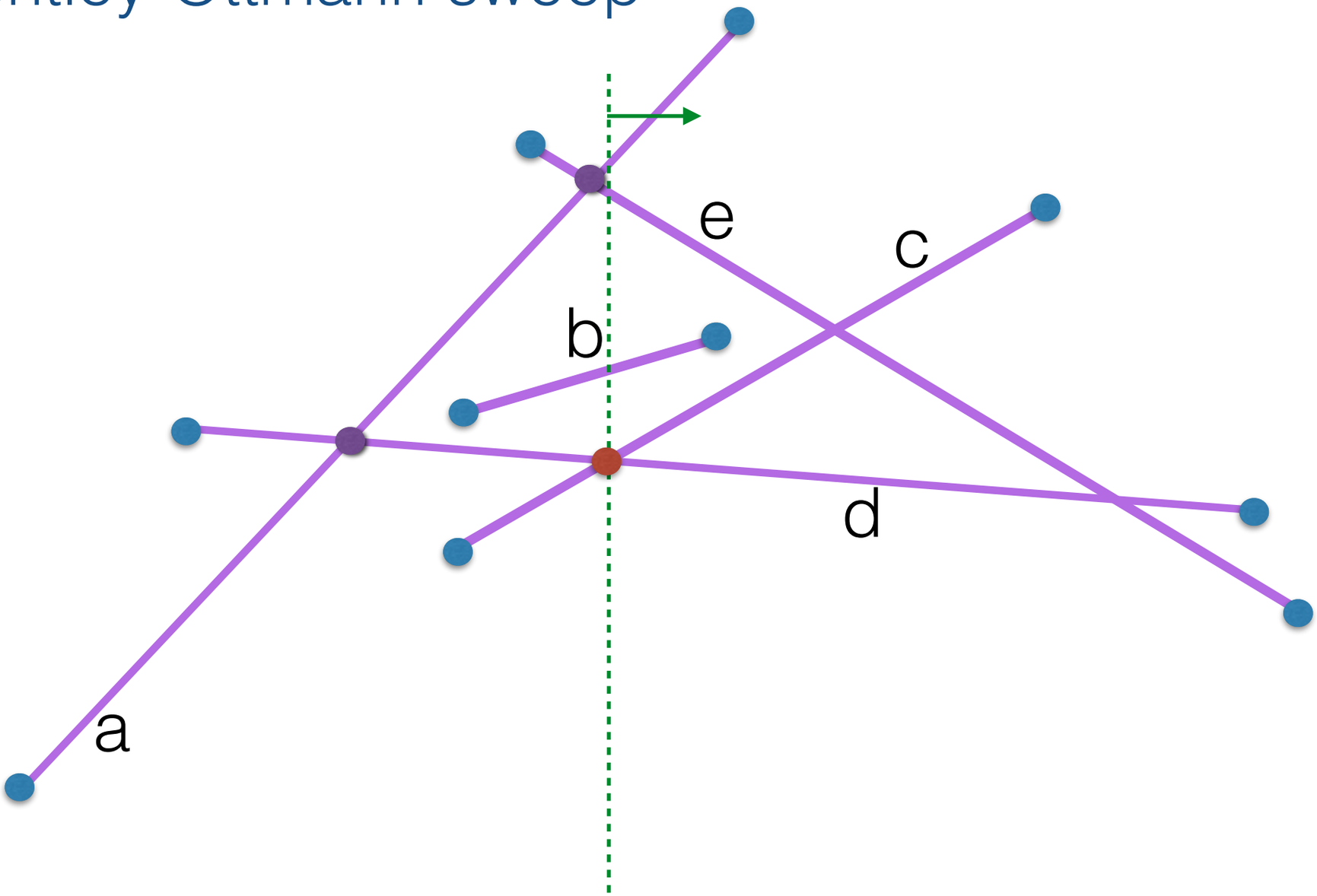


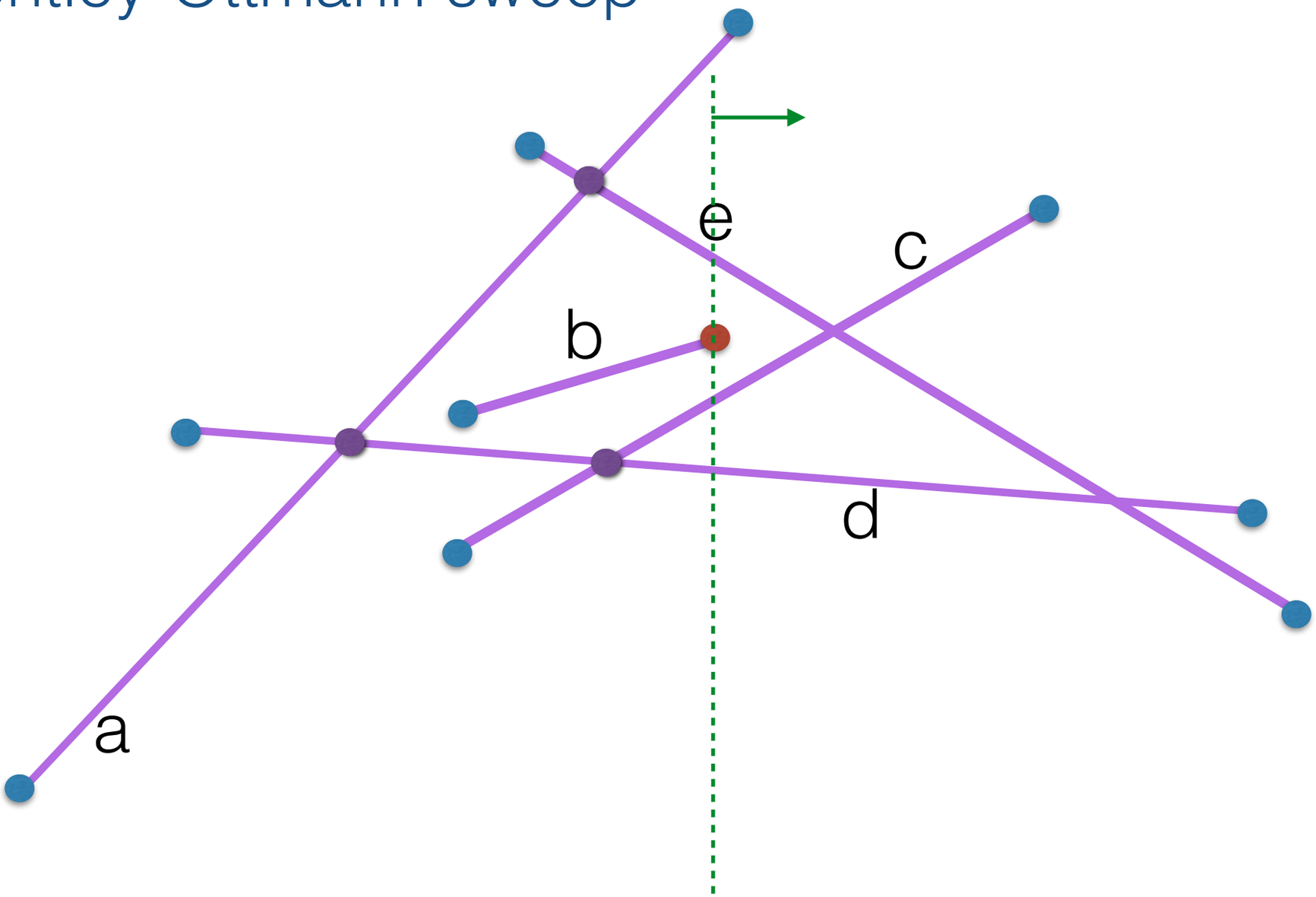
- this event is intersection of (a,e):
  - flip a and e: c < d < b < e < a
  - check new neighbors (e,b) for intersection to the right of the sweep line; (e,b) don’t intersect

Bentley-Ottmann sweep



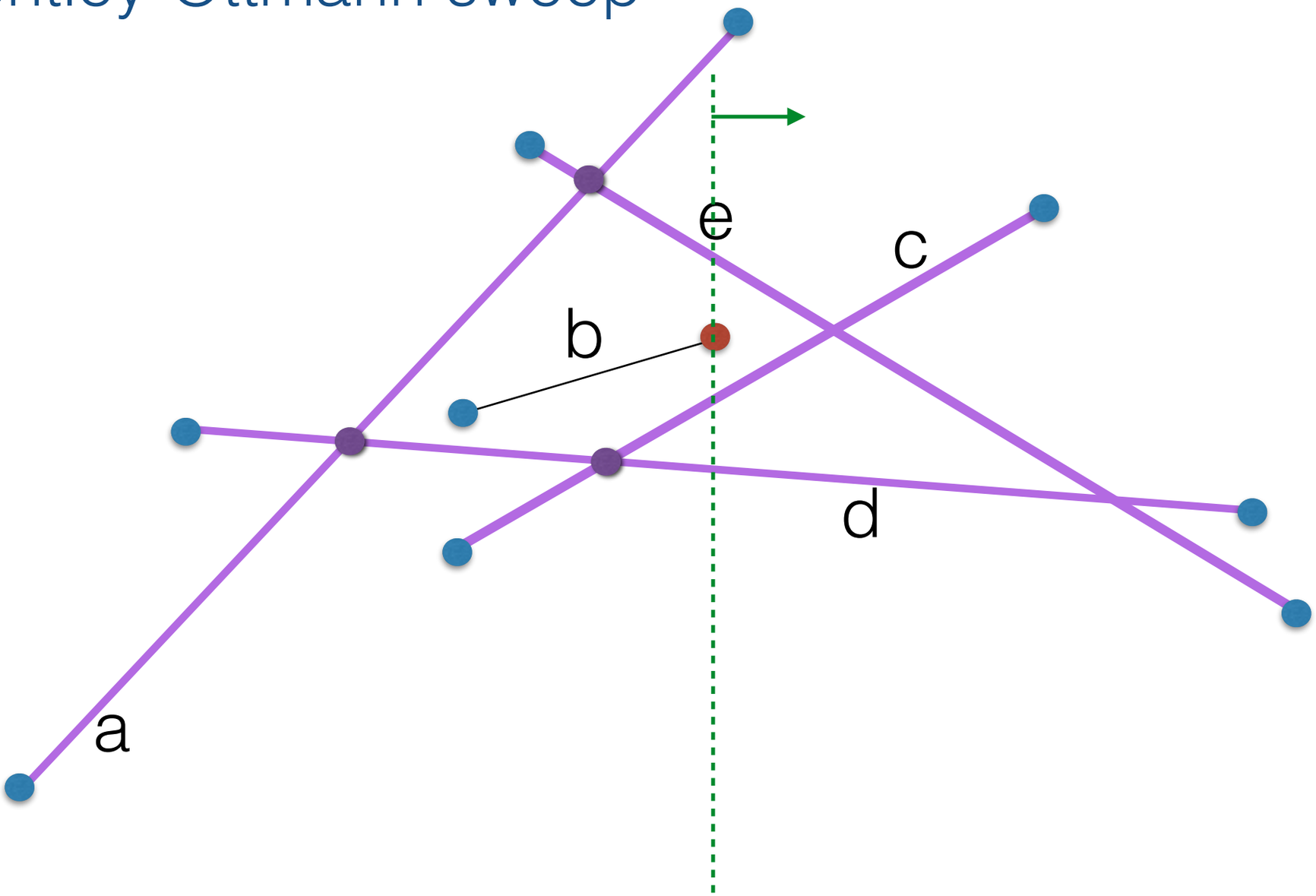
- this event is intersection of (c,d):
  - flip c and d: d < c < b < e < a
  - check new neighbors (c,b) for intersection to the right of the sweep line; (c,b) don’t intersect

Bentley-Ottmann sweep



- this event is end of segment b:
  - delete b from AS: d < c < e < a
  - check new neighbors (c,e) for intersection to the right of the sweep line; this detects the intersection point of (c,e); report it and insert it as future event

Bentley-Ottmann sweep



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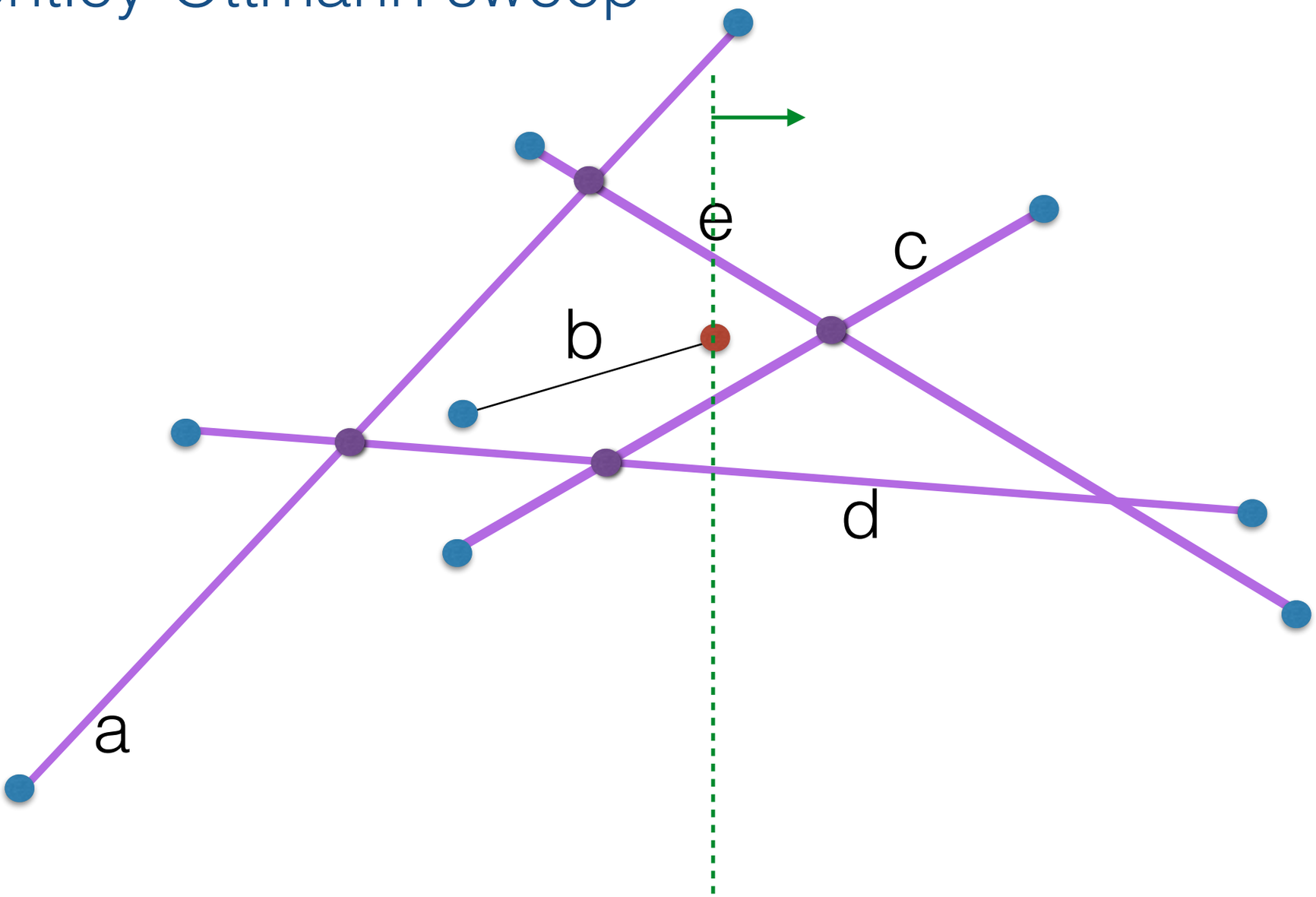
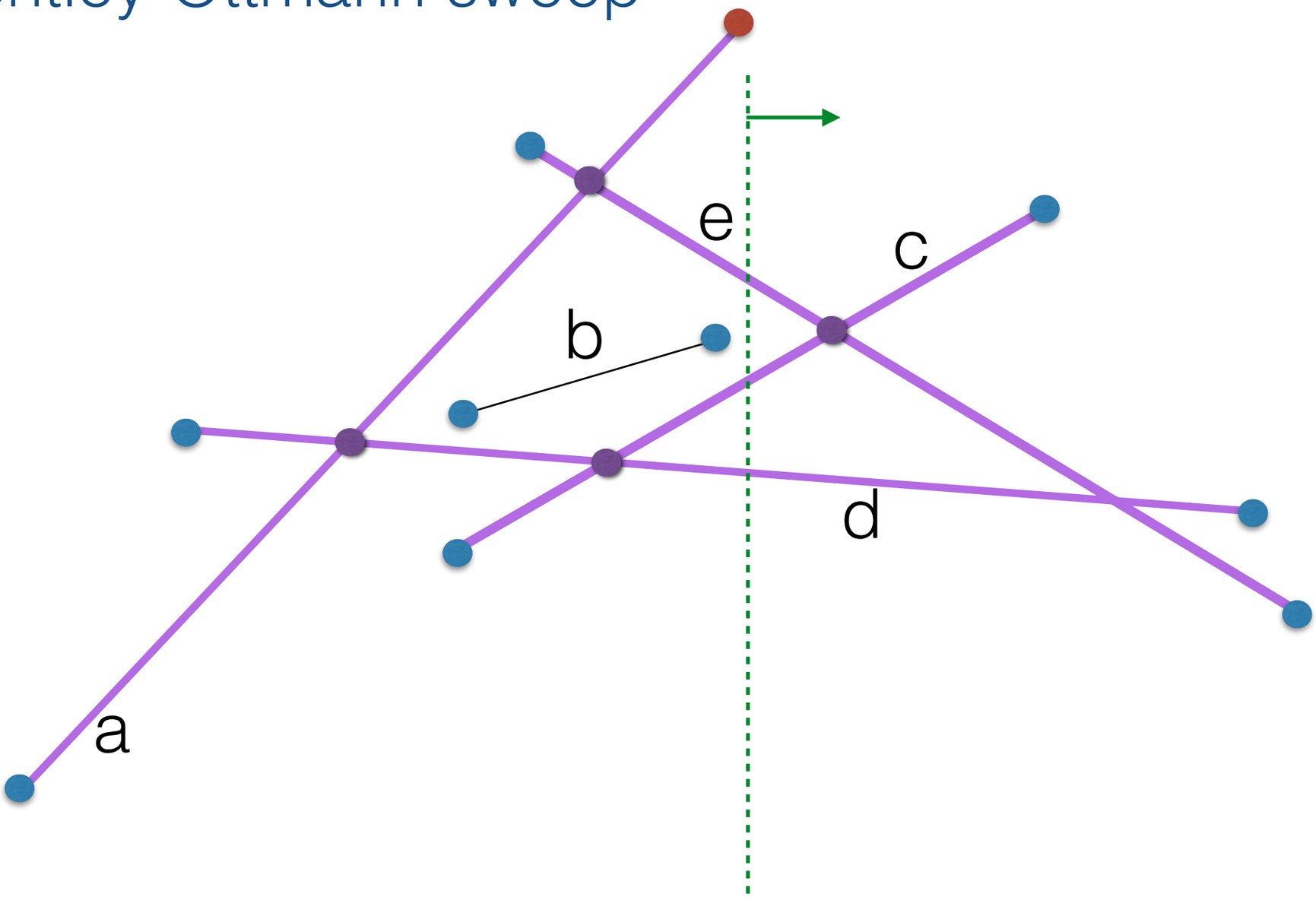


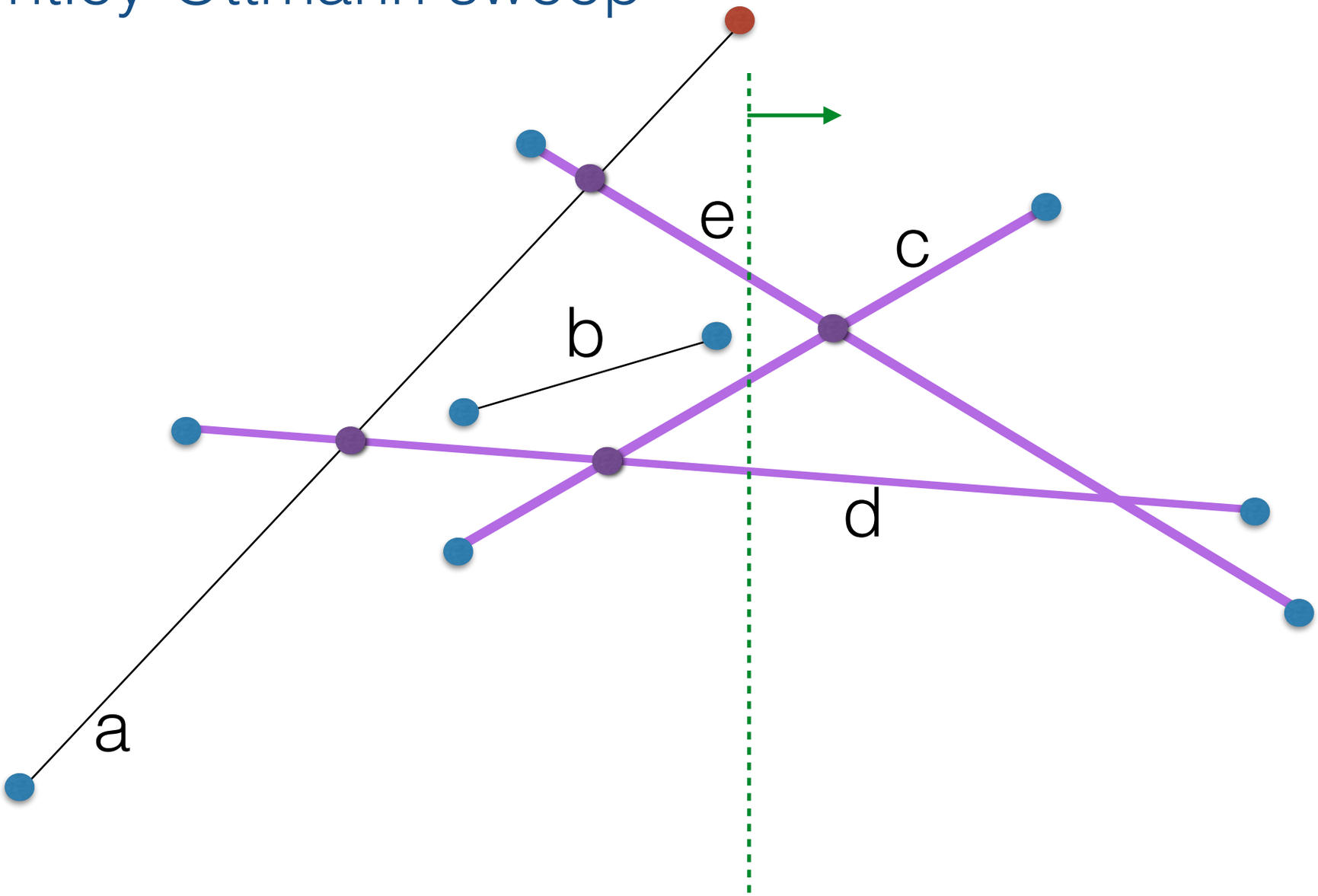
- this event is end of segment b:
  - delete b from AS: d < c < e < a
  - check new neighbors (c,e) for intersection to the right of the sweep line; this detects the intersection point of (c,e); report it and insert it as future event

Bentley-Ottmann sweep



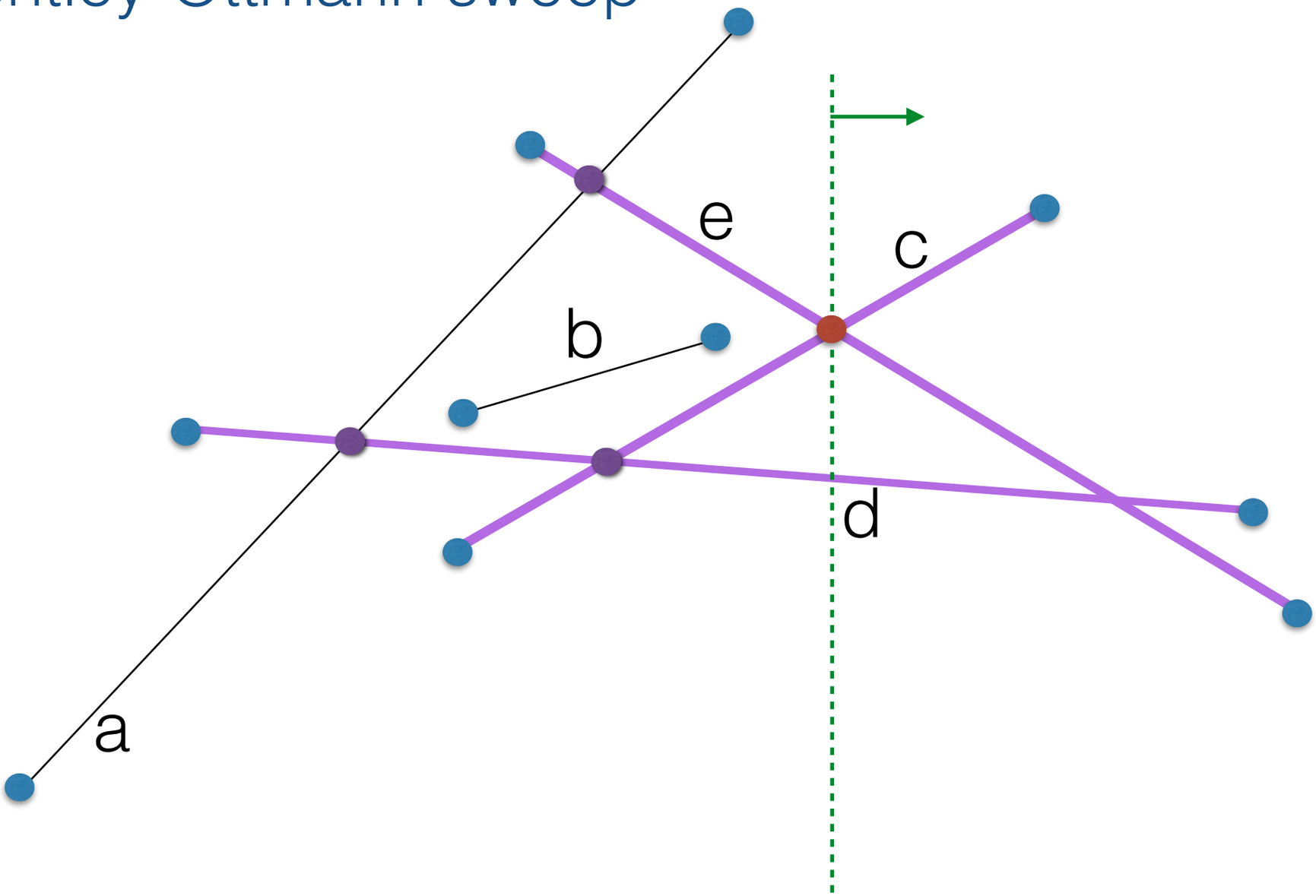
- this event is end of segment a:
  - delete a from AS: d < c < e
  - no new neighbors

Bentley-Ottmann sweep



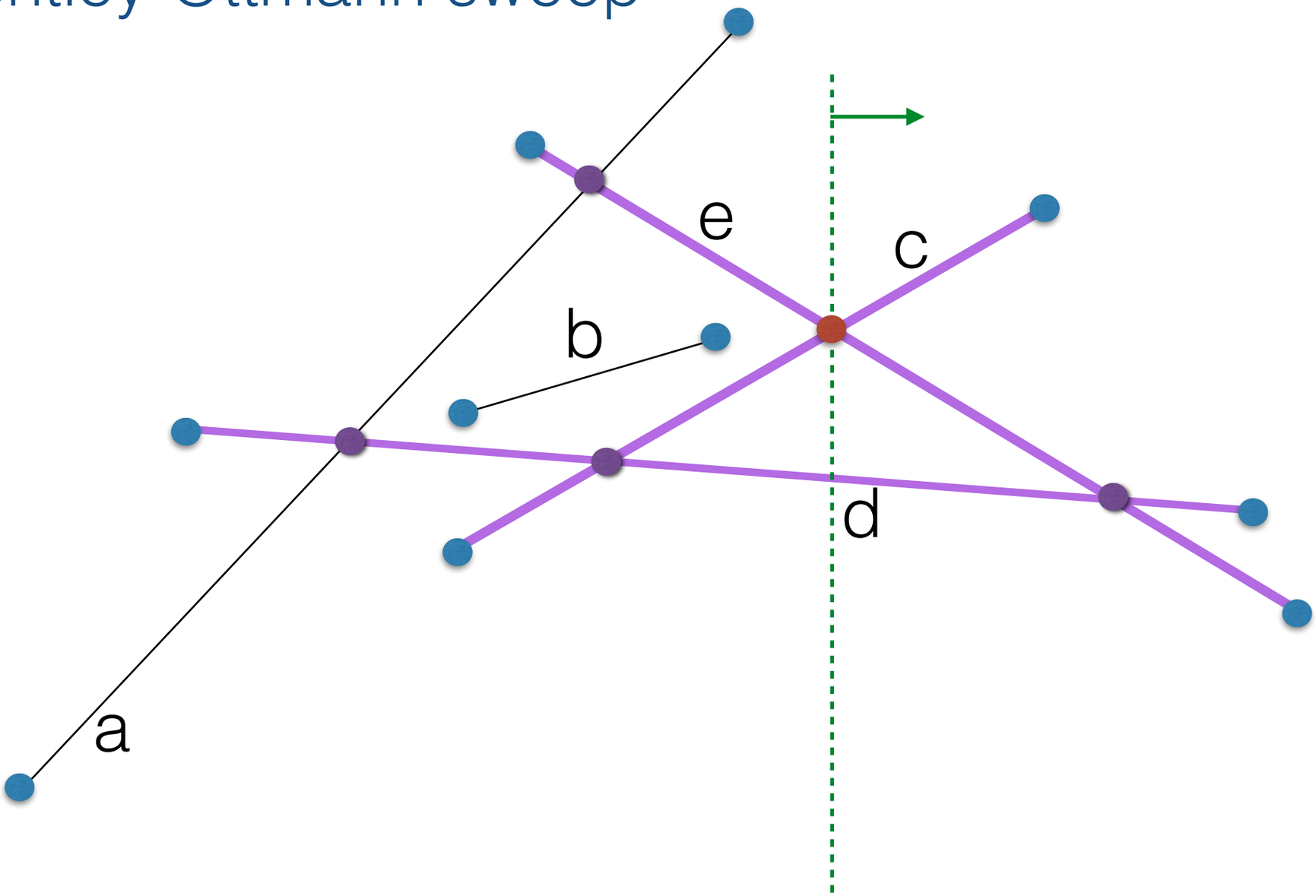
- this event is end of segment a:
  - delete a from AS: d < c < e
  - no new neighbors

Bentley-Ottmann sweep



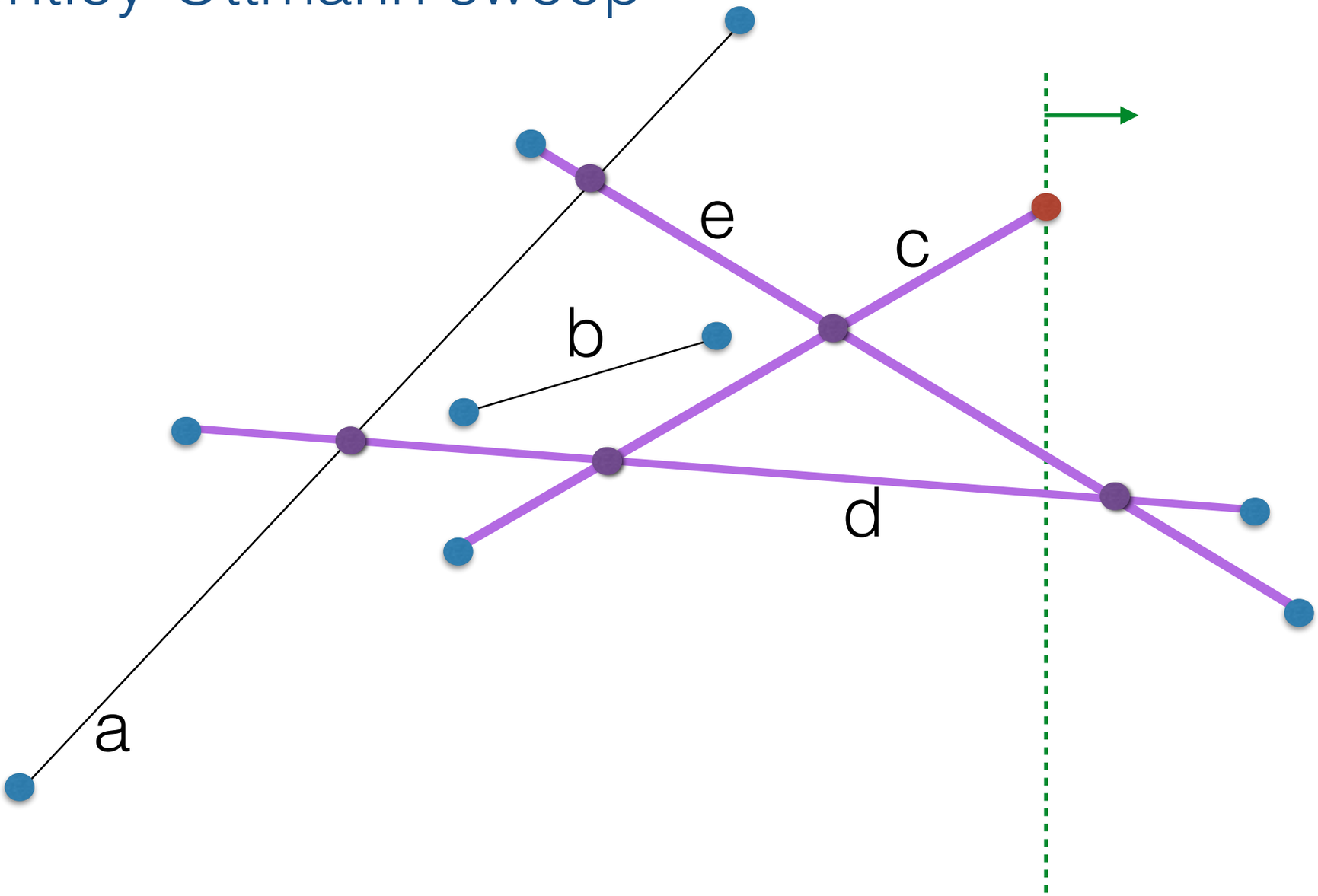
- this event is the intersection of (c,e):
  - flip c,e in AS: d < e < c
  - check new neighbors (d,e) for intersection to the right of the sweep line; this detects the intersection of (d,e); report it and insert it as future event

Bentley-Ottmann sweep



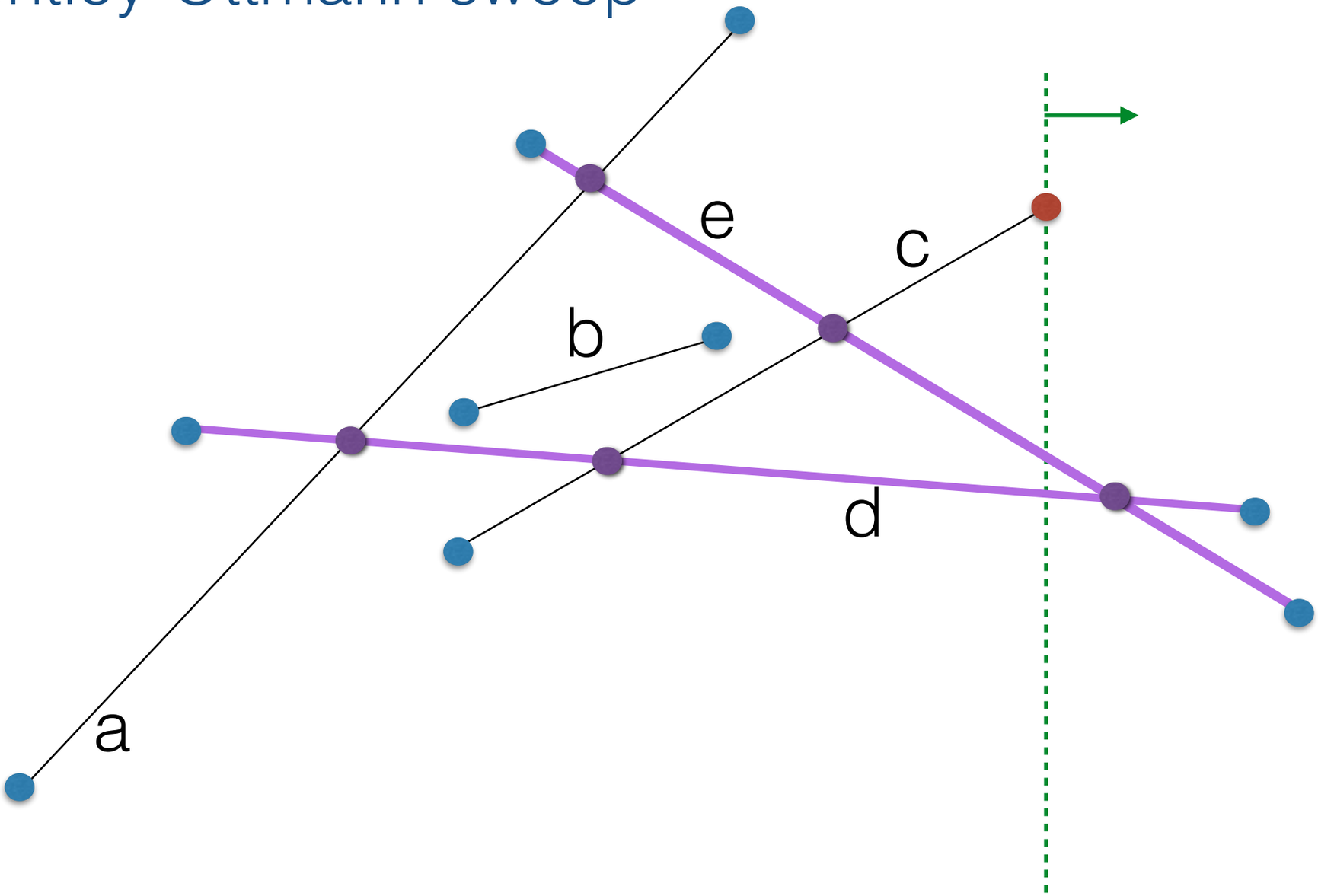
- this event is the intersection of (c,e):
  - flip c,e in AS: d < e < c
  - check new neighbors (d,e) for intersection to the right of the sweep line; this detects the intersection of (d,e); report it and insert it as future event

Bentley-Ottmann sweep



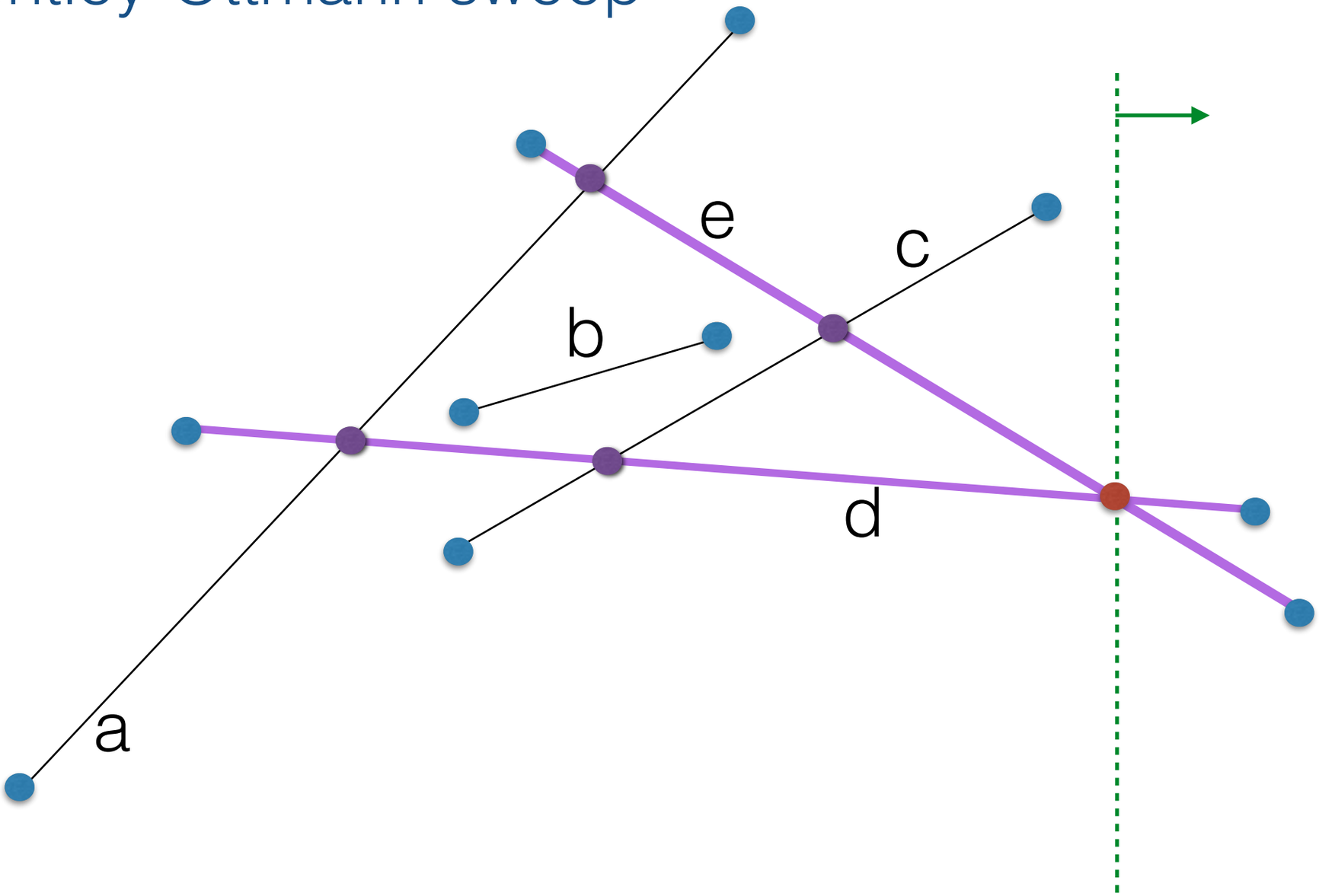
- this event is end of segment c:
  - delete c in AS: d < e
  - no new neighbors

Bentley-Ottmann sweep



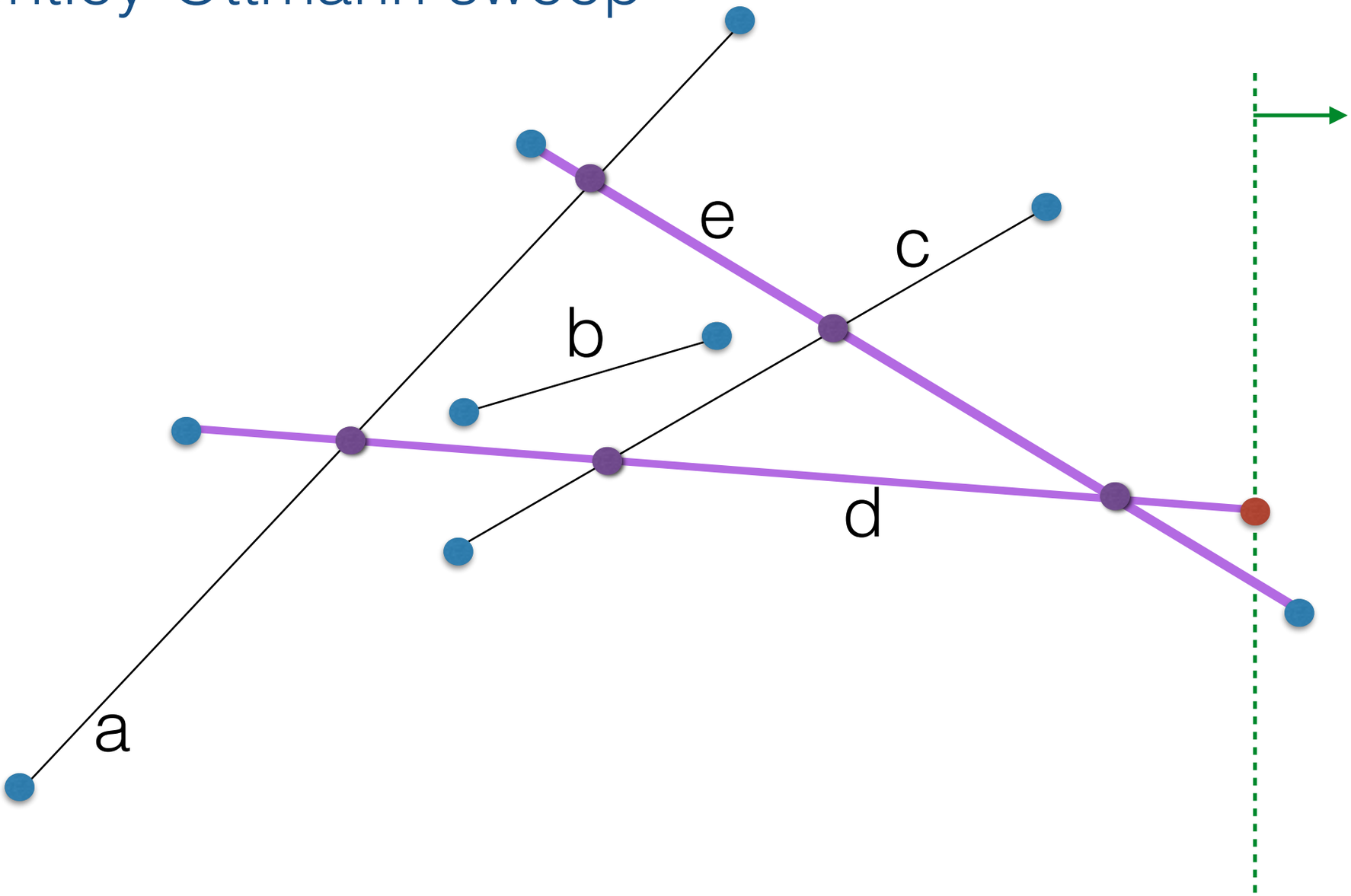
- this event is end of segment c:
  - delete c in AS: d < e
  - no new neighbors

Bentley-Ottmann sweep



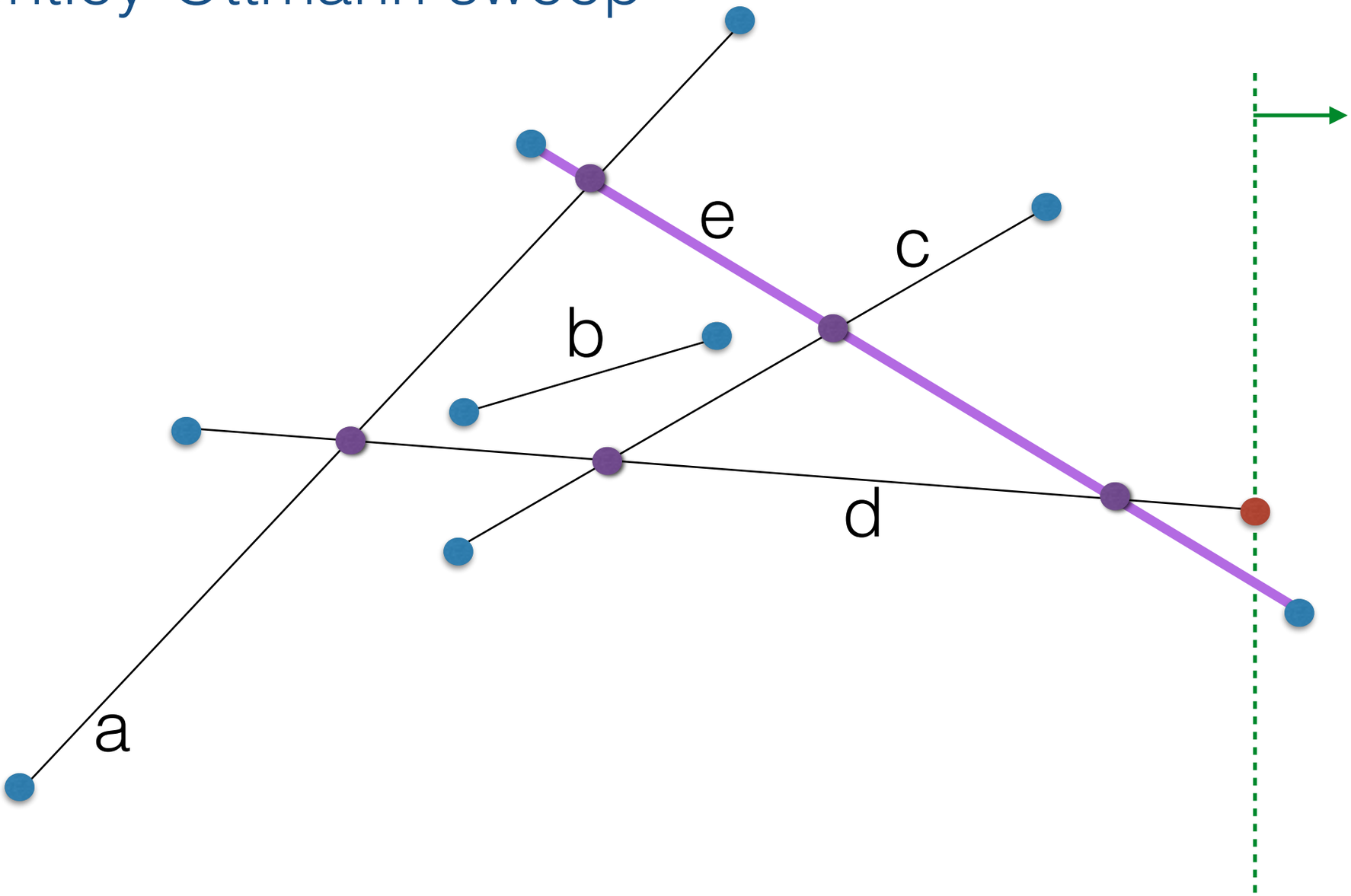
- this event is the intersection of (d,e):
  - flip d,e in AS: e < d
  - no new neighbors

Bentley-Ottmann sweep



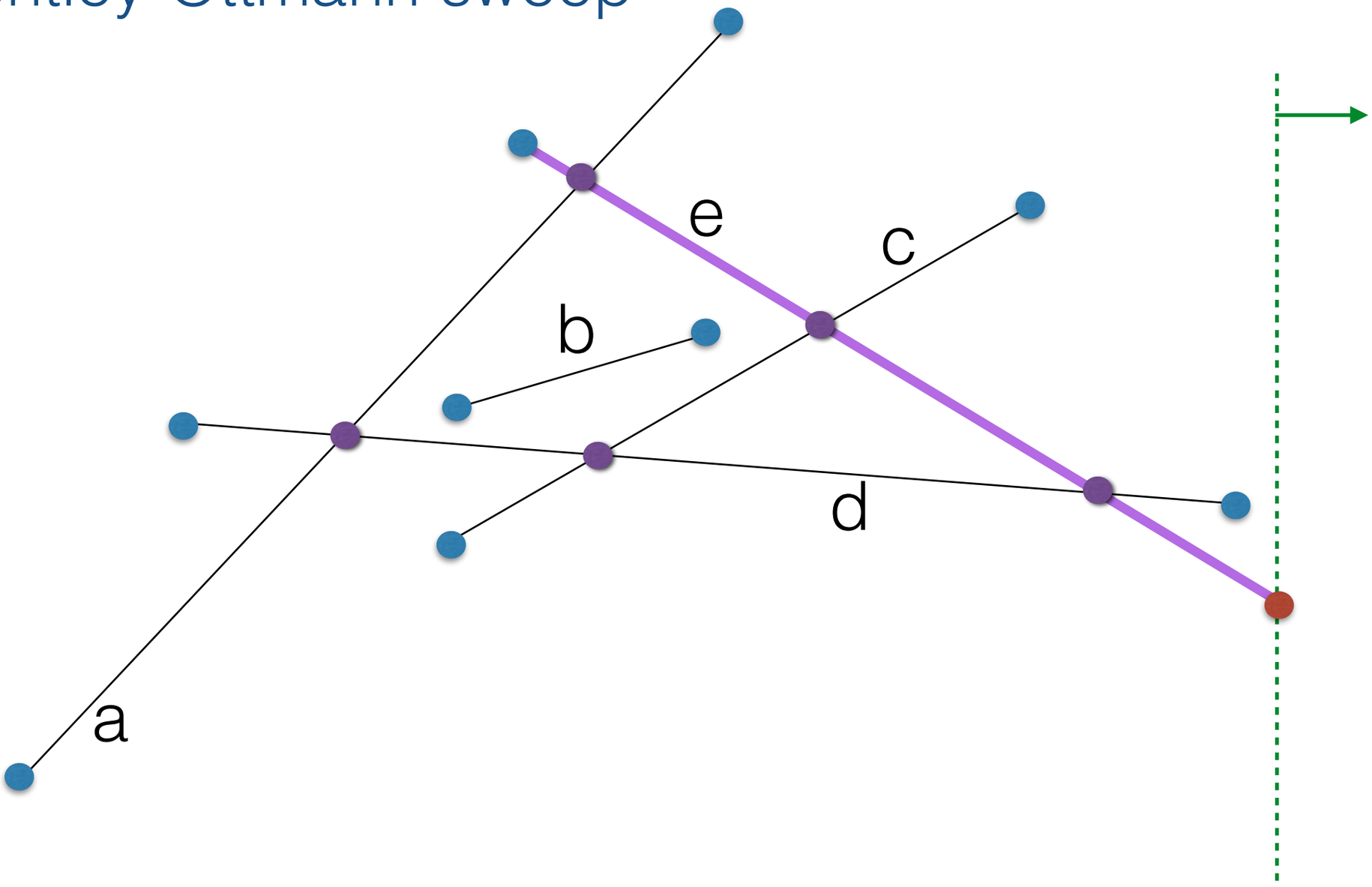
- this event is the end of d:
  - delete d in AS: e
  - no new neighbors

Bentley-Ottmann sweep



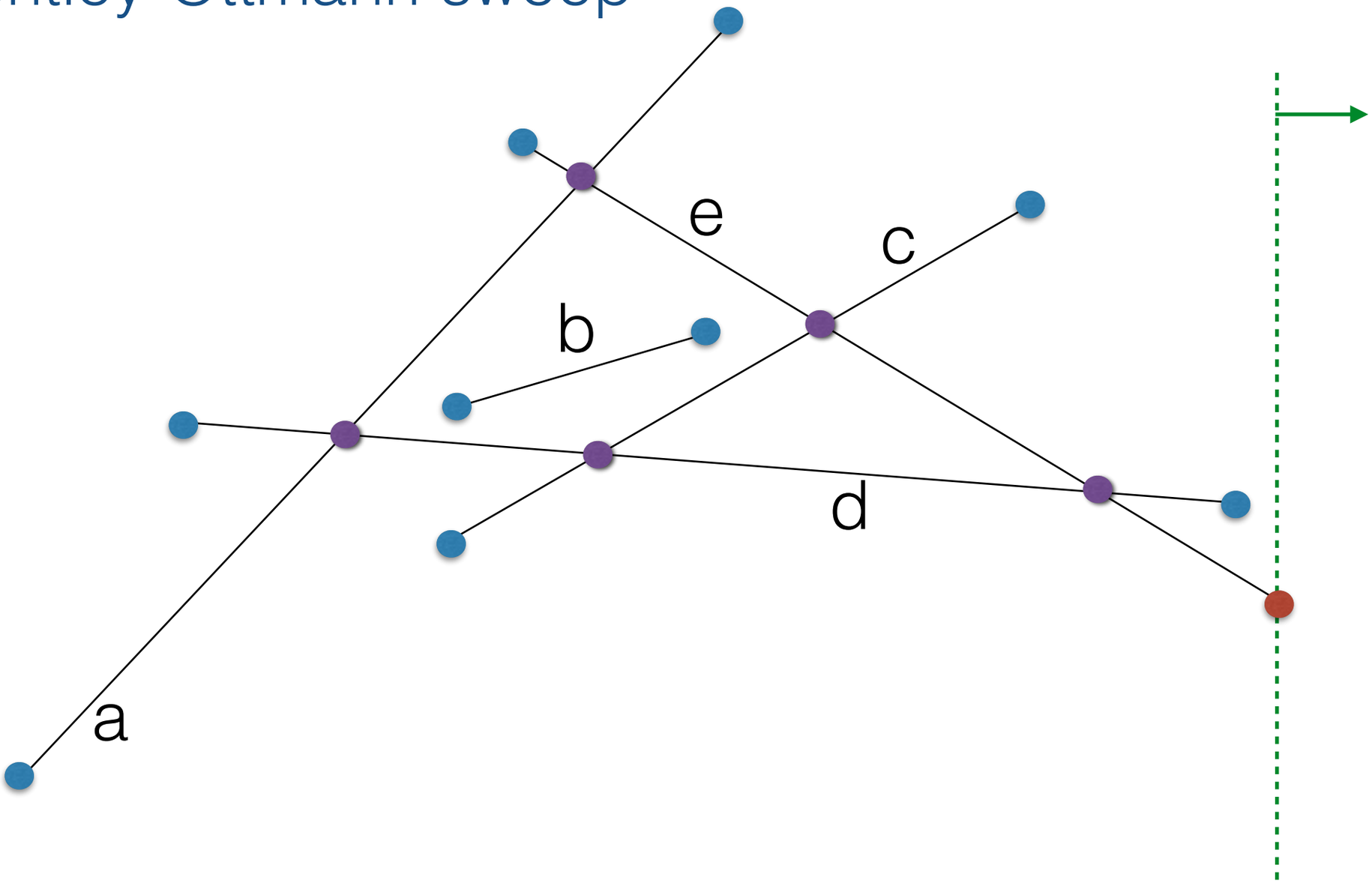
- this event is the end of d:
  - delete d in AS: e
  - no new neighbors

Bentley-Ottmann sweep



- this event is the end of e:
  - delete e in AS:
  - no new neighbors

Bentley-Ottmann sweep



- this event is the end of e:
  - delete e in AS:
  - no new neighbors

Bentley-Ottmann sweep

- Simplifying assumptions
  - no vertical segments
  - no two segments intersect at their endpoints
  - no three (or more) segments have a common intersection
  - all endpoints (of segments) and all intersection points have different x-coordinates
  - no segments overlap

- These assumptions are not realistic for real data..
- But, they don't provide insight into the plane sweep technique, so we omit them
- Real data challenges
  - dealing with degenerate cases like
  - dealing with finite precision arithmetic and precision problems

Bentley-Ottmann sweep

- Invariants:

  - For any position of the sweep line SL, the Active Structure AS contains all active segments (ie segments that start before SL and end after SL)
    - AS is sorted by their y-coordinates of their intersection with SL

  - For any position of the sweep line SL, the event list EL contains segment endpoints to the right of SL, and also the intersections to the right of SL of active segments that were/are neighbors in SL
    - the EL is sorted by x-coordinate

  - For any position of the sweep line SL, all pairs of intersecting dead segments have been reported.

Algorithm Bentley-Ottmann (S)

//S is a set of n line segments in the plane

- initialize AS= {}
- sort 2n endpoints of all segments in S by x-coord and store them in EventList
- while EventList not empty

    - let e be the next event from EventList; delete it from EL

    //sweep line moves to x=e.x

    - if e is left endpoint of a segment l

        //l becomes active

        - insert l in AS in the right place
        - check if l intersects with l->prev and l->succ in AS to the right of the sweep line; if they do, insert their intersection point in the EventList

        //optional: since l.prev and l.succ are not neighbors anymore, we check if they intersect and if they do, delete that intersection point from the EventList

    - if e is the right endpoint of a segment
        - …

    - if e is the intersection of two segments
        - …
- end.

Bentley-Ottmann sweep

Questions

- AS
  - What operations do we do on AS?
  - What data structure should we use for AS?

- EventList
  - Note that we know a priori the 2n events corresponding to start and end-points of segments, but EventList is not static; the events corresponding to intersection points are generated on the fly
  - What operations do we do on EventList?
  - What data structure should we use for EventList?

Bentley-Ottmann sweep

Bentley-Ottmann sweep

Running time

Bentley-Ottmann sweep

Running time

- AS

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

- EventList

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

- EventList
  - What is the size of EventList?

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

- EventList
  - What is the size of EventList?
    - O(n+k)

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

- EventList
  - What is the size of EventList?
    - O(n+k)
  - How many operations?

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

- EventList
  - What is the size of EventList?
    - O(n+k)
  - How many operations?
    - O(n+k)

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
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    - O((n+k)lg n)

- EventList
  - What is the size of EventList?
    - O(n+k)
  - How many operations?
    - O(n+k)
  - Overall time?

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
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    - O((n+k)lg n)

- EventList
  - What is the size of EventList?
    - O(n+k)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

Bentley-Ottmann sweep

Bentley-Ottmann sweep runs in $O((n+k) \lg n)$ time.

Bentley-Ottmann sweep

Running time

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

Running time

- AS

Bentley-Ottmann sweep runs in $O((n+k) \lg n)$ time.

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
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Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)
- EventList

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)
- EventList
  - What is the size of EventList?

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

- EventList
  - What is the size of EventList?
    - O(n+k)

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

- EventList
  - What is the size of EventList?
    - O(n+k)
  - How many operations?

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

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  - What is the size of AS?
    - O(n)
  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

- EventList
  - What is the size of EventList?
    - O(n+k)
  - How many operations?
    - O(n+k)

Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

Running time

- AS
  - What is the size of AS?
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  - How many operations?
    - O(n+k)
  - Overall time?
    - O((n+k)lg n)

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  - What is the size of EventList?
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  - How many operations?
    - O(n+k)
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Bentley-Ottmann sweep runs in O((n+k) lg n) time.

Bentley-Ottmann sweep

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    - O(n+k)
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Bentley-Ottmann sweep runs in O((n+k) lg n) time.